On Certain Projective Motion in an N- Birecurrent Finsler Space

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Abstract

In the present paper, the necessary and sufficient conditions for this projective motion to be affine motion are obtained. Projective motion is studied in birecurrent Finsler space. Several results by authors extended to Finsler spaces of recurrent curvature by R. B. Misra, N. Kishore, and P. N. Pandey [6], A. Kumar, H. S. Shulka and R. P. Tripathi [2], S. P. Singh ([9], [10]) and others. C. K. Misra and D. D. S. Yadav [3] and S. P. Singh [11] discussed the affine motion in birecurrent non - Riemannian space.

Keywords: birecurrent Finsler space, affine motion and projective motion.

1.Introduction

Let us consider an n-dimensional affine connected Finsler space F_n with appositively homogeneous metric function $F(x,y)$ of degree one in y^i.

The fundamental metric tensor g_{ij} of F_n is given by

(1.1) $$g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y).$$

The tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in y^i and symmetric in i and j.

The covariant derivative of any vector field X^i with respect to x^j is given by [13]

(1.2) $$\mathcal{B}_j X^i = \partial_j X^i - (\dot{\partial}_k X^i)\Pi^k_{rj} y^r + X^k \Pi^i_{kj},$$

where

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{jkr}.$$

The normal projective connection Π^i_{jk} and the connection parameters G^i_{jk} are positively homogeneous of degree zero in y^i and skew –symmetric.[13]

The normal projective curvature tensor N^i_{jkh} is given by

(1.3) $$N^i_{jkh} = 2\{\dot{\partial}_j \Pi^i_{[kh]} + \Pi^i_{rj[h}\Pi^r_{k]s} y^s + \Pi^i_{r[h}\Pi^r_{k]j}\},$$

where [kh] represents skew – symmetric part. The derivatives $\dot{\partial}_j \Pi^i_{kh}$ denoted by Π^i_{jkh} is given by

$$\Pi^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + y^i G^r_{jkhr}),$$

are symmetric in k and h only and are positively homogeneous of degree -1 in y^i and the tensor satisfies the following identity

(1.4) $\Pi^i_{jkh} y^j = 0.$

The normal projective curvature tensor N^i_{jkh} is skew -symmetric in its last two indices, i.e.

(1.5) $N^i_{jkh} = -N^i_{jhk}.$

Also, this tensor satisfies the following identity [12]

(1.6) $N^i_{jkh} + N^i_{khj} + N^i_{hjk} = 0.$

The normal projective curvature tensor N^i_{jkh} is related with Berwald curvature tensor H^i_{jkh} by

(1.7) $N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1} y^i \dot{\partial}_j H^r_{rkh},$

where the curvature tenser H^i_{jkh} is positively homogeneous of degree zero in y^i and skew - symmetric in its last two indices and given by

(1.8) $H^i_{jkh} := 2\{\partial_{[h}G^i_{k]j} + G^r_{j[k}G^i_{h]r} - G^i_{rj[k}G^r_{h]}\}$.

The curvature tenser H^i_{jkh} satisfies the following identities

(1.9) $H^i_{jkh}y^j = H^i_{kh} = N^i_{jkh}y^j$.

The commutation formulae, involving the above curvature tenser, are given by

(1.10) $2\,\mathcal{B}_{[l}\mathcal{B}_{m]}T^i_j = T^r_j N^i_{lmr} - T^i_r N^r_{lmj} - (\dot{\partial}_r T^i_j)N^r_{lms}y^s$.

In particular, the Berwald covariant derivative of the vector y^i vanishes identically, i.e.

(1.11) $\mathcal{B}_k y^i = 0$.

Definition 1.1. The normal projective curvature tensor N^i_{jkh} satisfies the relation

(1.12) $\mathcal{B}_l N^i_{jkh} = \lambda_l N^i_{jkh}, \quad N^i_{jkh} \neq 0,$

where λ_l *is a non – zero recurrence vector field*, the space is called *recurrent Finsler space* ([4], [11], [12]).

Transvecting (1.12) by y^j, using (1.9) and (1.11), we get

(1.13) $\mathcal{B}_l H^i_{kh} = \lambda_l H^i_{kh}$.

Definition 1. 2. The normal projective curvature tensor N^i_{jkh} satisfies the relation

(1.14) $\mathcal{B}_m\mathcal{B}_l N^i_{jkh} = a_{lm}N^i_{jkh}, \quad N^i_{jkh} \neq 0,$

where a_{lm} *is a non – zero recurrence tensor field*, the space is called *birecurrent Finsler space* [1].

Transvecting (1.14) by y^j, using (1.9) and (1.11), we get

(1.15) $\mathcal{B}_m\mathcal{B}_l H^i_{kh} = a_{lm}H^i_{kh}$.

Let us consider an infinitesimal transformation

(1.16) $\bar{x}^i = x^i + \epsilon v^i(x^j),$

where ε is *an infinitesimal constant* and $v^i(x^j)$ is called *contravariant vector filed* independent of y^i.Also, this transformation gives rise to a process of differentiation called *Lie- differentiation*.

Let X^i be an arbitrary contravariant vector filed. Its Lie- derivative with respect to the above infinitesimal transformation is given by ([7], [8], [13])

(1.17) $L_v X^i = v^r\mathcal{B}_r X^i - X^r\mathcal{B}_r v^i + (\dot{\partial}_r X^i)\mathcal{B}_s v^r y^s,$

where the symbol L_v stands for the Lie- differentiation.

In view of (1.16) the Lie-derivatives of y^i and v^i with respect to above infinitesimal transformation vanish, i.e.

(1.18) $a)\ L_v y^i = 0$

and

$b)\ L_v v^i = 0.$

Lie-derivative of an arbitrary tensor T^i_j, with respect to the above infinitesimal transformation, is given by

(1.19) $L_v T^i_j = v^r\mathcal{B}_r T^i_j - T^r_j\mathcal{B}_r v^i + T^i_r\mathcal{B}_j v^r + (\dot{\partial}_r T^i_j)\mathcal{B}_s v^r y^s$.

Lie-derivative of the normal projective connection parameters Π^i_{jk} is given by [13]

(1.20) $L_v\Pi^i_{jk} = \mathcal{B}_j\mathcal{B}_k v^i + N^i_{rjk}v^r + \Pi^i_{rjk}y^s\mathcal{B}_s v^r$.

The commutation formulae for the operators $\mathcal{B}_k$, $\dot{\partial}_j$ and L_v are given by ([5],[11])

(1.21) $(L_v\mathcal{B}_k - \mathcal{B}_k L_v)X^i = X^h L_v\Pi^i_{kh} - (\dot{\partial}_r X^r)L_v\Pi^i_{kh}y^h,$

where X^i is *a contravariant vector filed*.
The infinitesimal transformation (1.16) defines a motion, affine motion or projective motion if it preserves the distance between two points, parallelism of pair of vector or the geodesics, respectively. Necessary and sufficient conditions for the transformation (1.16) to be a motion, affine motion and projective motion are respectively given by [4]

(1.22) $L_v g_{ij} = 0,$

(1.23) $L_v \Pi_{kh}^i = 0$

and

(1.24) $L_v \Pi_{jk}^i = \delta_j^i P_k + \delta_k^i P_j,$

where P_j is defined as

(1.25) $P_j = \dot{\partial}_j P,$

P being a scalar, positively homogeneous of degree one in y^i.
Transvecting (1.25) by y^j, using homogeneity of P_j [9], we get

(1.26) $P_j y^j = P.$

Let an infinitesimal transformation (1.16) be generated by a vector filed$v^i(x^j)$.
The vector filed$v^i(x^j)$ is called *contra*, *concurrent* and *special concircular* according as it satisfies

(1.27) $a)\ \mathcal{B}_k v^i = 0,$

$b)\ \mathcal{B}_k v^i = c\delta_k^i,$ c being a constant

and

$c)\ \mathcal{B}_k v^i = \rho\delta_k^i,$ ρ is not a constant,

respectively. The affine motion generated by the above vectors is called *contra affine motion*, *concurrent affine motion* and *special concircular*, respectively.

2. Projective Motion in an N- Birecurrent Finsler Space

Definition 2. 1. A birecurrent Finsler space characterized by (1.14), in which infinitesimal transformation (1.16) defines a projective motion, is called *projective birecurrent Finsler space* briefly denoted by $NB - P\bar{F}_n$.
Lie-derivative of the normal projective curvature tensor N_{jkh}^i satisfies the relation

(2.1) $L_v N_{jkh}^i = 2\mathcal{B}_{[j} P_{k]} \delta_h^i - 2\delta_{[j}^i \mathcal{B}_{k]} P_h - 2P\Pi_{[jk]h}^i.$

Transvecting (2.1) by y^j, using (1.4), (1.9), (1.11), (1.18a) and (1.26), we get

(2.2) $L_v H_{kh}^i = \delta_h^i \mathcal{B}_k P - y^i \mathcal{B}_k P_h.$

In view of (1.5) and (1.11), applying Lie –derivative to (1.14) and observing (2.1), we get

(2.3) $L_v \mathcal{B}_m \mathcal{B}_l N_{jkh}^i = (L_v a_{lm}) N_{jkh}^i + a_{lm}\big(2\mathcal{B}_{[j} P_{k]} \delta_h^i - 2\delta_{[j}^i \mathcal{B}_{k]} P_h - 2P\Pi_{h[jk]}^i\big).$

Similarly, the Lie –derivative to (1.14) in view of (1.18), we get

(2.4) $L_v \mathcal{B}_m \mathcal{B}_l H_{kh}^i = (L_v a_{lm}) H_{kh}^i + a_{lm}\big(\delta_h^i \mathcal{B}_k P - y^i \mathcal{B}_k P_h\big).$

Thus, we conclude

Theorem 2.1.*In an $NB - P\bar{F}_n$, which admits a projective motion, the relations (2.3) and (2.4) hold. .*

When the projective motion becomes an affine motion, the condition $L_v \Pi_{kh}^i = 0$ is satisfied. If we apply this condition in (1.24), we get

(2.5) $\delta_j^i P_k + \delta_k^i P_j = 0.$

Contracting the indices i and k in (2.5), we get

(2.6) $(1 + n)P_j = 0,$

which implies

(2.7) $P_j = 0.$

Conversely, if (2.7) is true, the equation (1.16) reduces to $L_v \Pi_{kh}^i = 0.$ i.e. the necessary and sufficient condition for the infinitesimal transformation (1.16) which defines a projective motion to be an affine motion.

Using the equations (1.26) and (2.7) in (2.1) and (2.2), we get

(2.8) $L_v N_{jkh}^i = 0$

and

(2.9) $L_v H_{kh}^i = 0.$

Using the equations (1.26) and (2.7) in (2.3) and (2.4), we get

(2.10) $L_v \mathcal{B}_m \mathcal{B}_l N_{jkh}^i = (L_v a_{lm}) N_{jkh}^i$

and

(2.11) $L_v \mathcal{B}_m \mathcal{B}_l H_{kh}^i = (L_v a_{lm}) H_{kh}^i.$

Thus, we conclude

Theorem 2.2. *In an $NB - P\bar{F}_n$, if the a projective motion becomes an affine motion, then the relations (2.10) and (2.11) are necessarily true.*

Using the commutation formulae (1.10) for N_{jkh}^i, we get

(2.12) $2\, \mathcal{B}_{[l} \mathcal{B}_{m]} N_{jkh}^i = N_{jkh}^r N_{lmr}^i - N_{rkh}^i N_{lmj}^r - N_{jrh}^i N_{lmk}^r - (\dot{\partial}_r N_{jkh}^i) N_{lms}^r y^s$
$-N_{jkr}^i N_{lmh}^r.$

Applying Lie –derivative to both sides of (2.12) and using (2.8), we get

(2.13) $L_v \mathcal{B}_{[l} \mathcal{B}_{m]} N_{jkh}^i = 0$

In view of the e equations (2.10) and (2.13), we get

(2.14) $(L_v a_{lm}) N_{jkh}^i = 0, \qquad N_{jkh}^i \neq 0.$

Or $(L_v a_{lm}) = 0.$

Thus, we conclude

Theorem 2.3. *In an $NB - P\bar{F}_n$, if the a projective motion becomes an affine motion, then the recurrence tensor field a_{lm} satisfies the identity (2.14).*

Applying skew – symmetric of (1.14), we get

(2.15) $a_{[lm]} N_{jkh}^i = 0.$

Differentiating (2.15) covariantly with respect tox^n in the sense of Berwald, using (1.12) and the normal projective curvature tensor N_{jkh}^i is skew -symmetric, we get

(2.16) $\mathcal{B}_n a_{[lm]} = \lambda_n a_{[lm]}.$

Applying Lie –derivative to both sides of (1.12) and using (2.8), we get

(2.17) $L_v \mathcal{B}_l N_{jkh}^i = (L_v \lambda_l) N_{jkh}^i.$

In view of the commutation formulae (1.21) for the normal projective curvature tensor N_{jkh}^i, using (1.23) and (2.8), we get

(2.18) $L_v \mathcal{B}_l N_{jkh}^i = 0.$

In view of (2.17) and (2.18), we get

(2.19) $L_v \lambda_l = 0, \quad N_{jkh}^i \neq 0.$

Applying Lie –derivative to both sides of (2.16), using (2.14) and (2.19), we get

(2.20) $L_v \mathcal{B}_n a_{[lm]} = 0.$

Cyclic permeation with respect l, m and n in (2.20), we get

(2.21) $L_v \mathcal{B}_n a_{[lm]} + L_v \mathcal{B}_l a_{[mn]} + L_v \mathcal{B}_m a_{[nl]} = 0.$

Thus, we conclude

Theorem 2.4. *In an $NB - P\bar{F}_n$, if the a projective motion becomes an affine motion, then the recurrence tensor field a_{lm} satisfies the identity (2.21).*

3. Special cases

Let us consider an infinitesimal transformation generated by contra vector $v^i(x^j)$ characterized by (1.27a).

Taking the covariant divination for (1.27a), with respect to x^j in the sense of Berwald, we get

(3.1) $\mathcal{B}_j\mathcal{B}_k v^i = 0.$

Using equations (1.24), (1.27a), (1.4) and (3.1) in equation (1.20), we get

(3.2) $N^i_{jkh}v^h = \delta^i_j P_k + \delta^i_k P_j.$

Differentiating (3.2) covariant , with respect to $x^l$and x^min the sense of Berwald and using (1.27a), we get

(3.3) $\mathcal{B}_m\mathcal{B}_l N^i_{jkh}v^h = \delta^i_j \mathcal{B}_m\mathcal{B}_l P_k + \delta^i_k \mathcal{B}_m\mathcal{B}_l P_j.$

Using equations (1.14) and (3.2) in equation (3.3), we get

(3.4) $\delta^i_j(\mathcal{B}_m\mathcal{B}_l P_k - a_{lm}P_k) + \delta^i_k(\mathcal{B}_m\mathcal{B}_l P_j - a_{lm}P_j) = 0.$

Contracting the indies i and j in (3.4), we get

(3.5) $\mathcal{B}_m\mathcal{B}_l P_k = a_{lm}P_k.$

Thus, we conclude

Theorem 3.1. *In an NB - $P\bar{F}_n$, which admits projective motion, if the vector filed $v^i(x^j)$ spans contra affine motion, then the vector P_k is birecurrent.*

Transvecting (3.5) by y^k, using (1.11) and (1.26), we get

(3.6) $\mathcal{B}_m\mathcal{B}_l P = a_{lm}P.$

Thus, we conclude

Theorem 3.2. *In an NB - $P\bar{F}_n$, which admits projective motion, if the vector filed $v^i(x^j)$ spans contra affine motion, then the scalar function P is birecurrent.*

If we adopt the similar process for (1.27b), we get the following theorem:

Theorem 3.3.*In an NB - $P\bar{F}_n$, which admits projective motion, if the vector filed $v^i(x^j)$ spans concircular affine motion, then the vector P_k is birecurrent.*

Theorem 3.4.*In an NB - $P\bar{F}_n$, which admits projective motion, if the* vector *filed$v^i(x^j)$ spans concircular affine motion, then the scalar function P is birecurrent.*

Let us consider an infinitesimal transformation generated by contra vector $v^i(x^j)$ characterized by (1.27c).

Taking the covariant divination for (1.27c), with respect to x^j in the sense of Berwald, we get

(3.7) $\mathcal{B}_j\rho\delta^i_k = 0.$

Using equations (1.24), (1.27c), (1.4) and (3.7) in equation (1.20), we get

(3.8) $N^i_{jkh}v^h = \delta^i_j P_k + \delta^i_k P_j - \mathcal{B}_j\rho\delta^i_k.$

Differentiating (3.8) covariant, with respect to $x^l$and x^min the sense of Berwald and using (1.27c), we get

(3.9) $\mathcal{B}_m\mathcal{B}_l N^i_{jkh}v^h = \delta^i_j \mathcal{B}_m\mathcal{B}_l P_k + \delta^i_k \mathcal{B}_m\mathcal{B}_l P_j - \delta^i_k \mathcal{B}_m\mathcal{B}_l\mathcal{B}_j\rho.$

Using equations (1.14) and (3.8) in equation (3.9), we get

(3.10) $\delta^i_j(\mathcal{B}_m\mathcal{B}_l P_k - a_{lm}P_k) + \delta^i_k(\mathcal{B}_m\mathcal{B}_l P_j - a_{lm}P_j) - \delta^i_k(\mathcal{B}_m\mathcal{B}_l\mathcal{B}_j\rho - a_{lm}\mathcal{B}_j\rho) = 0.$

Contracting the indies i and j in (3.4), we get
(3.11) $(n+1)(\mathcal{B}_m\mathcal{B}_l P_k - a_{lm}P_k) - (\mathcal{B}_m\mathcal{B}_l\mathcal{B}_k\rho - a_{lm}\mathcal{B}_k\rho) = 0.$
From the above equation, we get
(3.12) $a)\ \mathcal{B}_m\mathcal{B}_l P_k = a_{lm}P_k,$ $b)\ \mathcal{B}_m\mathcal{B}_l\mathcal{B}_k\rho = a_{lm}\mathcal{B}_k\rho.$
Thus, we conclude

Theorem 3.5. *In an NB - $P\bar{F}_n$, which admits projective motion, the vector filed $v^i(x^j)$ spans special concircular affine motion satisfy (3.12a) and (3.12b).*
Transvecting (3.11) by y^k, using (1.11) and (1.26), we get
(3.13) $\mathcal{B}_m\mathcal{B}_l P = a_{lm}P.$
Thus, we conclude

Theorem 3.6. *In an NB - $P\bar{F}_n$, which admits projective motion, if the vector filed $v^i(x^j)$ spans special concircular affine motion, then the scalar function P is birecurrent.*

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حول الحركة الإسقاطية في فضاء فنسلر – N ثنائي المعاودة

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الملخص

في هذه الورقة البحثية تم الحصول على الشروط اللازمة والكافية للحركة الأسقاطية بأن تكون حركة أفينية. وتم دراسة الحركة الإسقاطية في فضاء فنسلر ثنائي المعاودة.

وقد قدم الباحثون عدة نتائج أيضاً لفضاء فنسلر ذات التقوس أحادي المعاودة، منهم R. B. Misra, ([10], [9]) S. P. ,[2] R. P. Tripathi و Shulka , A. Kumar , [6] P. N. Pandey و N. Kishore, Singhوآخرون. وقد ناقش كل من C. K. Misra و D. D. S. Yadav [3] و S. P. Singh [11] الحركة الأفينية في فضاء ريمان ثنائي المعاودة.

الكلمات المفتاحيه: فضاء فنسلر ثنائي المعاودة، الحركة الأفينية والحركة الإسقاطية.