The generalized q-Analogue Hermite Polynomials of two variables

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Abstract

In this paper, we introduce the q-analogue generalized Hermite polynomials of two variables. Some recurrence relations for these q-polynomials are derived.

Keywords: The generalized q- analogue Hermite polynomials, generating functions, recurrence relations.

1. Introduction.

The classical Hermite polynomials have two important properties: (i) they form a family of orthogonal polynomials, and (ii), are intimately connected with the commutation properties between the multiplication operator x and the differentiation operator D. In contrast to the discrete q-Hermite polynomials, which generalize both aspects, the continuous q-Hermite polynomials generalize only the first one.

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [3]

$$(a;q)_n = \begin{cases} 1 & , n=0 \\ \prod_{0\le j\le n-1}\left(1-aq^j\right) & , n=1,2,3,\dots \end{cases}, \tag{1.1}$$

where
$$\left(q^{-n};q\right)_k = \begin{cases} 0 & k>n \\ \dfrac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, & k\le n \end{cases}, \tag{1.2}$$

$$(0;q)_n = 1,$$

also
$$(a;q)_{n+k} = (a;q)_n\left(aq^n;q\right)_k, \tag{1.3}$$

where $\lim\limits_{q\to 1^-}\dfrac{\left(q^z;q\right)_k}{(1-q)^k} = (z)_k$.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad 0\le k\le n, \quad k,n\in N. \tag{1.4}$$

The q-derivative with index α is defined by Rajkovic [7]

$$D_\alpha = \frac{f\left(q^\alpha x\right)-f(x)}{\left(q^\alpha-1\right)x}, \qquad D_1 = D \tag{1.5}$$

which for q-derivative of the pair of functions are valid:

$$D\left(\lambda a(x)+\mu b(x)\right) = \lambda Da(x)+\mu Db(x), \tag{1.6}$$

$$D\left(a(x).b(x)\right) = a(qx)Db(x)+Da(x)b(x), \tag{1.7}$$

$$D\left(\frac{a(x)}{b(x)}\right) = \frac{Da(x)b(x)-a(x)Db(x)}{b(x)b(qx)}. \tag{1.8}$$

Exton [2] presented the following q-exponential functions:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} \frac{q^{\mu n(n-1)}}{[n]_q!} z^n,$$

where $\quad [n]_q! = \frac{(q;q)_n}{(1-q)^n}$.

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $\frac{a}{2}$, we get

$$E\left(\frac{a}{2}, \frac{x}{1-q}; q\right) = E_q(x, a),$$

where

$$E_q(x, a) = \sum_{n=0}^{\infty} \frac{q^{a\binom{n}{2}}}{(q;q)_n} x^n, \tag{1.9}$$

which satisfies the functional relation

$$E_q(x, a) - E_q(qx, a) = xE_q(q^a x, a).$$

The above q-function can be rewritten by the formula

$$D_q E_q(x, a) = \frac{1}{1-q} E_q(q^a x, a). \tag{1.10}$$

Also, the q-analogue of $(x \pm y)^n$ is given by Purohit& Raina[5]

$$(x \pm y)^n = (x \pm y)_n = x^n \left(\mp y/x; q\right)_n = x^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \left(\pm y/x\right)^k. \tag{1.11}$$

Hermite Polynomials are defined by means of generating relations [6]

$$\exp[2xt - t^2] = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}, \tag{1.12}$$

$$\exp[xt + yt^2] = \sum_{n,=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \tag{1.13}$$

Shrivastava [8] presented and studied the classical Hermite polynomials and its generalizations in the form:

$$\exp[2x(t+h) - (y+1)(t+h)^2] = \sum_{n,m=0}^{\infty} H_{n,m}(x, y) \frac{t^n h^m}{n! m!}. \tag{1.14}$$

The following double series transformations that we will occasionally use, are easy to prove

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k), \tag{1.15}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k). \tag{1.16}$$

Similarly, we can write

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n+2k), \tag{1.17}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n-k), \tag{1.18}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k), \tag{1.19}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+mk), \tag{1.20}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]} A(k,n-(m-1)k), \tag{1.21}$$

where m is a positive integer and $n > m$.

2. The Generalized q-Analogue Hermite Polynomials of Two Variables

In this section, we introduce the generalized q-analogue Hermite polynomial of two variables and one index by the following:

$$H_n(x,y,a;q)=\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(n-2r)^2+\frac{a}{4}(r)^2}(2x)^{n-2r}y^r}{(q;q)_{n-2r}(q;q)_r} . \tag{2.1}$$

Now, we derive generating function of the generalized q-analogue Hermite polynomials in the form of the following theorem:

Theorem 2.1

The following generating function for the generalized q-analogue Hermite polynomials $H_n(x,y,a;q)$ holds true:

$$E_q\left(2q^{a/4}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}yt^2,\frac{a}{2}\right)=\sum_{n=0}^{\infty} H_n(x,y,a;q)t^n . \tag{2.2}$$

Proof. Let us denote the left hand side of (2.2) by W, then

$$W=E_q\left(2q^{a/4}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}yt^2,\frac{a}{2}\right),$$

applying relation (1.9), we get

$$W=\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}(2x)^n t^n}{(q;q)_n}\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{a}{4}r}y^r t^{2r}}{(q;q)_r}, \tag{2.3}$$

by using relation (1.15), we obtain

$$W=\sum_{n=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}\left((n-2r)^2+r^2\right)}}{(q;q)_{n-2r}(q;q)_r}(2x)^{n-2r}y^r t^n, \tag{2.4}$$

which by using definition (2.1) we get the required result (2.2).

Lemma 2.1

$$H_n(-x,y,a;q)=(-1)^n H_n(x,y,a;q). \tag{2.5}$$

Proof.

$$H_n(-x,y,a;q)=\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(n-2r)^2+\frac{a}{4}(r)^2}(-2x)^{n-2r}y^r}{(q;q)_{n-2r}(q;q)_r}$$

$$=\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(n)^2+\frac{a}{4}(r)^2}(-2x)^n y^r}{(q;q)_n(q;q)_r}$$

$$=(-1)^n\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(n-2r)^2+\frac{a}{4}(r)^2}(2x)^{n-2r}y^r}{(q;q)_{n-2r}(q;q)_r},$$

which is the required relation (2.5).

Lemma 2.2

$$\lim_{q\to 1}H_n((1-q)x,(1-q)y,0;q)=H_n(x,y). \qquad (2.6)$$

Proof.

$$\lim_{q\to 1}H_n((1-q)x,(1-q)y,0;q)=\lim_{q\to 1}\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\frac{(2(1-q)x)^{n-2r}((1-q)y)^r}{(q;q)_{n-2r}(q;q)_r}$$

$$=\sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\frac{(2x)^{n-2r}(y)^r}{[n-2r]!\,[r]!}$$

$$=H_n(x,y).$$

3. Recurrence Relations for $H_n(x,y,a;q)$

Now, we derive some recurrence relations for the polynomials $H_n(x,y,a;q)$ in the form of the following theorems:

Theorem 3.1

The q-analogue generalized Hermite polynomials of one-index and two-variable $H_n(x,y,a;q)$ satisfy the following relations:

$$\frac{\partial^s}{\partial x^s}H_n(x,y,a;q)=\frac{2^s q^{s^2 a/4}}{(1-q)^s}H_{n-s}\left(q^{s!a/2}x,y,a;q\right), \qquad (3.1)$$

and

$$\frac{\partial^s}{\partial y^s}H_n(x,y,a;q)==\frac{(-1)^{s(a+1)}q^{s^2 a/4}}{(1-q)^s}H_{n-2s}\left(x,q^{s!a/2}y,a;q\right). \qquad (3.2)$$

Proof. Differentiating (2.2) with respect to x and using (1.10), we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}H_n(x,y,a;q)t^n=\frac{2q^{a/4}t}{1-q}E_q\left(2q^{3a/4}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}yt^2,\frac{a}{2}\right)$$

$$=\frac{2q^{a/4}}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}(n(n-1)+m(m-1))+\frac{3a}{4}n+\frac{a}{4}m}(2x)^n y^m}{(q;q)_n(q;q)_m}t^{n+1+2m}, \qquad (3.3)$$

which on using relation (1.15), gives

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}H_n(x,y,a;q)t^n=\frac{2q^{a/4}}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n-1}{2}\right]}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}\left(((n-1)-2m)^2+m^2\right)+\frac{a}{2}(\{n-1)-2m)}(2x)^{(n-1)-2m}(y)^m}{(q;q)_{(n-1)-2m}(q;q)_m}t^n$$

$$=\frac{2q^{a/4}}{1-q}\sum_{n=0}^{\infty}H_{n-1}(x,y,a;q)t^n$$

By equating the coefficients of t^n, we get

$$\frac{\partial}{\partial x}H_n(x,y,a;q)=\frac{2q^{a/4}}{1-q}H_{n-1}\left(q^{a/2}x,y,a;q\right). \tag{3.4}$$

Thus, by same manner as above, we can obtain

$$\frac{\partial^2}{\partial x^2}H_n(x,y,a;q)=\frac{2^2q^{2^2a/4}}{(1-q)^2}H_{n-2}\left(q^{2!a/2}x,y,a;q\right). \tag{3.5}$$

Hence, by continuing of the above steps, we get the required relation (3.1).
Also, differentiating (2.2) with respect to y, we obtain

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}H_n(x,y,a;q)t^n=\frac{(-1)^{a+1}q^{a/4}t^2}{1-q}E_q\left(2q^{a/4}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{3a/4}yt^2,\frac{a}{2}\right)$$

$$=\frac{(-1)^{(a+1)}q^{a/4}}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}(n(n-1)+m(m-1))+\frac{a}{4}n+\frac{3a}{4}m}(2x)^n(y)^m}{(q;q)_n(q;q)_m}t^{n+2+2m}, \tag{3.6}$$

Which, on using relation (1.15), becomes

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}H_n(x,y,a;q)t^n=\frac{(-1)^{a+1}q^{a/4}}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n-2}{2}\right]}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}\left(((n-2)-2m)^2+m^2\right)}(2x)^{(n-2)-2m}\left(q^{\frac{a}{2}}y\right)^m}{(q;q)_{(n-2)-2m}(q;q)_m}t^n$$

$$=\frac{(-1)^{a+1}q^{a/4}}{1-q}\sum_{n=0}^{\infty}H_{n-2}\left(x,q^{\frac{a}{2}}y,a;q\right)t^n,$$

which by equating the coefficient of t^n, we obtain

$$\frac{\partial}{\partial y}H_n(x,y,a;q)=\frac{(-1)^{a+1}q^{a/4}}{1-q}H_{n-2}\left(x,q^{a/2}y,a;q\right). \tag{3.7}$$

Thus, by the same manner as above, we obtained

$$\frac{\partial^2}{\partial y^2}H_n(x,y,a;q)=\frac{(-1)^{2(a+1)}q^{2^2a/4}}{(1-q)^2}H_{n-4}\left(x,q^{2!a/2}y,a;q\right). \tag{3.8}$$

Hence, by continuing these steps, we get the required relation (3.2).

Theorem 3.2

The polynomials sequence $\{H_n(x,y,a;q)\}_{n=0}^{\infty}$ satisfies the following recurrence relation

$$[n+1]H_{n+1}(x,y,a;q)=\frac{2(-1)^{a+1}q^{a/4}y}{1-q}H_{n-1}\left(qx,q^{\frac{a}{2}}y,a;q\right)+\frac{2q^{a/4}x}{1-q}H_n\left(q^{\frac{a}{2}}x,y,a;q\right). \tag{3.9}$$

Proof.

Differentiating (2.2) with respect to t and using relation (1.7), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial t}H_n(x,y,a;q)t^n=\frac{2(-1)^{a+1}q^{a/4}yt}{1-q}E_q\left(2q^{a/4+1}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{3a/4}yt^2,\frac{a}{2}\right)$$

$$+\frac{2q^{a/4}x}{1-q}E_q\left(2q^{3a/4}xt,\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}yt^2,\frac{a}{2}\right),$$

Which, on using relation (1.9) gives

$$\sum_{n=0}^{\infty}[n]_q H_n(x,y,a;q)t^{n-1} = \frac{2(-1)^{a+1}q^{a/4}yt}{1-q}\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n+n}(2xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{2}\binom{m}{2}+\frac{3a}{4}m}(yt^2)^m}{(q;q)_m}$$

$$+\frac{2q^{a/4}x}{1-q}\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{3a}{4}n}(2xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{2}\binom{m}{2}+\frac{a}{4}m}(yt^2)^m}{(q;q)_m}$$

$$=\frac{2(-1)^{a+1}q^{a/4}y}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}(n^2+m^2)}(2qx)^n\left(q^{\frac{a}{2}}y\right)^m}{(q;q)_n(q;q)_m}t^{n+1+2m}$$

$$+\frac{2q^{a/4}x}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m(a+1)}\frac{q^{\frac{a}{4}(n^2+m^2)}\left(2q^{\frac{a}{2}}x\right)^n(y)^m}{(q;q)_n(q;q)_m}t^{n+2m}, \qquad (3.10)$$

by using relation (1.15), we find

$$\sum_{n=0}^{\infty}[n]_q H_n(x,y,a;q)t^{n-1} = \frac{2(-1)^{a+1}q^{a/4}y}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n-2}{2}\right]}(-1)^{a+1}\frac{q^{\frac{a}{4}\left(((n-2)-2m)^2+m^2\right)}(2qx)^{(n-2)-2m}\left(q^{\frac{a}{2}}y\right)^m}{(q;q)_{(n-2)-2m}(q;q)_m}t^{n-1}$$

$$+\frac{2q^{a/4}x}{1-q}\sum_{n=0}^{\infty}\sum_{m=0}^{\left[\frac{n-1}{2}\right]}(-1)^{a+1}\frac{q^{\frac{a}{4}\left(((n-1)-2m)^2+m^2\right)}\left(2q^{\frac{a}{2}}x\right)^{(n-1)-2m}(y)^m}{(q;q)_{(n-1)-2m}(q;q)_m}t^{n-1},$$

by equating the coefficient of t^{n-1}, we get the relation

$$[n]_q H_n(x,y,a;q) = \frac{2(-1)^{a+1}q^{a/4}y}{1-q}H_{n-2}\left(qx, q^{\frac{a}{2}}y, a;q\right)+\frac{2q^{a/4}x}{1-q}H_{n-1}\left(q^{\frac{a}{2}}x, y, a;q\right),$$

when $n \to n+1$, we get the required result (3.9).

4. The Generalized q-Analogue Associated Hermite Polynomials of Two-index and Two-Variable $H_{n,m}^k(x,y,a;q)$

We introduce the generalized q-analogue associated Hermite polynomial of two variables and two indexes by the following:

$$H_{n,m}^k(x,y,a;q)=\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}(n+m-kr-ks)^2+\frac{a}{4}(r+s)^2+\binom{m-ks}{2}+\binom{ks}{2}}(q;q)_{kr+ks}(2x)^{n+m-kr-ks}y^{r+s}}{(q;q)_{n-kr}(q;q)_{m-ks}(q;q)_{r+s}(q;q)_{kr}(q;q)_{ks}}. \quad (4.1)$$

Now, we get generating function of the generalized q-analogue associated Hermite polynomials in the form of the following theorem:

Theorem 4.1

The following generating function for the generalized q-analogue associated Hermite polynomials $H_{n,m}(x,y,a;q)$ holds true:

$$E_q\left(2q^{a/4}x(t+h),\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}y(t+h)^k,\frac{a}{2}\right)=\sum_{n,m=0}^{\infty}H_{n,m}^k(x,y,a;q)t^n h^m. \qquad (4.2)$$

Proof. Let us denote the left hand side of (4.2) by B, then

$$B = E_q\left(2q^{a/4}x(t+h),\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}y(t+h)^k,\frac{a}{2}\right),$$

appling relation (1.9), we obtain

$$B=\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}(2x)^n(t+h)^n}{(q;q)_n}\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{a}{4}r}y^r(t+h)^{kr}}{(q;q)_r}, \tag{4.3}$$

Which, using relation (1.11), we find

$$B=\sum_{n=0}^{\infty}\frac{q^{\frac{a}{4}(n(n-1))+\frac{a}{4}n}(2x)^n}{(q;q)_n}\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{\binom{m}{2}}t^{n-m}h^m$$

$$\times\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(r(r-1))+\frac{a}{4}r}y^r}{(q;q)_r}\sum_{s=0}^{kr}\begin{bmatrix}kr\\s\end{bmatrix}_q q^{\binom{s}{2}}t^{kr-s}h^s, \tag{4.4}$$

thus, by using relations (1.4) and (1.21), we get

$$B=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{q^{\frac{a}{4}n^2+\binom{m}{2}}(2x)^n}{(q;q)_{n-m}(q;q)_m}t^{n-m}h^m$$

$$\times\sum_{r=0}^{\infty}\sum_{s=0}^{r}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}r^2+\binom{ks}{2}}(q;q)_{kr}y^r}{(q;q)_r(q;q)_{kr-ks}(q;q)_{ks}}t^{kr-ks}h^{ks}$$

$$=\sum_{n,m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}(n-kr+m-ks)^2+\frac{a}{4}(r+s)^2+\binom{m-ks}{2}+\binom{ks}{2}}(q;q)_{kr+ks}(2x)^{n+m-kr-ks}y^{r+s}}{(q;q)_{n-kr}(q;q)_{m-ks}(q;q)_{r+s}(q;q)_{kr}(q;q)_{ks}}t^nh^m$$

By using definition (4.1), we obtain the required relation (4.2).

Corollary 4.1

If we choose $k=2$, in (4.1), then we get the new explicit definition of the q-analogue of tow index, tow variables generalized Hermite polynomials, which is

$$H_{n,m}(x,y,a;q)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\sum_{s=0}^{\left[\frac{m}{2}\right]}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}(n+m-2r-2s)^2+\frac{a}{4}(r+s)^2+\binom{m-2s}{2}+\binom{2s}{2}}(q;q)_{2r+2s}(2x)^{n+m-2r-2s}y^{r+s}}{(q;q)_{n-2r}(q;q)_{m-2s}(q;q)_{r+s}(q;q)_{2r}(q;q)_{2s}}. \tag{4.5}$$

Also choosing $k=2$, in (4.2), then we get a new generating function for the generalized Hermite polynomials of tow index, tow variables defined by

$$E_q\left(2q^{a/4}x(t+h),\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{a/4}y(t+h)^2,\frac{a}{2}\right)=\sum_{n,m=0}^{\infty}H_{n,m}(x,y,a;q)t^nh^m. \tag{4.6}$$

Lemma 4.1.

If n,m be integer, then $H_{n,m}^k(x,y,a;q)$ satisfies

$$H_{n,m}^k(-x,y,a;q)=(-1)^{n+m}H_{n,m}^k(x,y,a;q).$$

5. Recurrence relations for $H_{n,m}^k(x,y,a;q)$

Theorem 5.1

The generalized q-analogue associated Hermite polynomials of two-index and two-variable $H_{n,m}(x,y,a;q)$ satisfy the following relations:

$$\frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) = \frac{2q^{a/4}}{1-q}\left[H_{n-1,m}^k\left(q^{a/2}x,y,a;q\right) + H_{n,m-1}^k\left(q^{a/2}x,y,a;q\right)\right], \tag{5.1}$$

and

$$\frac{\partial_q}{\partial_q y} H_{n,m}^k(x,y,a;q) = \frac{(-1)^{a+1} q^{a/4}}{1-q}\sum_{w=0}^{k} q^{\binom{k}{2}} \frac{(q;q)_k H_{n+w-k,m-w}^k\left(x, q^{a/2}y, a;q\right)}{(q;q)_{k-w}(q;q)_w}. \tag{5.2}$$

Proof. From (4.2) and using relation (1.10), we get

$$\sum_{n,m=0}^{\infty} \frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) t^n h^m = \frac{2q^{a/4}(t+h)}{1-q} E_q\left(2q^{a/4+a/2}x(t+h), \frac{a}{2}\right) E_q\left((-1)^{a+1} q^{a/4} y(t+h)^k, \frac{a}{2}\right)$$

$$= \frac{2q^{a/4}}{1-q}(t+h)\sum_{n=0}^{\infty} \frac{q^{\frac{a}{2}\binom{n}{2}+\frac{3a}{4}n}(2x)^n}{(q;q)_n}(t+h)^n \sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{a}{4}r}(y)^r}{(q;q)_r}(t+h)^{kr}$$

applying relation (1.11) gives

$$\sum_{n,m=0}^{\infty} \frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) t^n h^m = \frac{2q^{a/4}}{1-q}\sum_{w=0}^{1} \frac{(q;q)_1 t^{1-w} h^w}{(q;q)_{1-w}(q;q)_w}\sum_{n=0}^{\infty} \frac{q^{\frac{a}{4}(n(n-1))+\frac{3a}{4}n}(2x)^n}{(q;q)_n}$$

$$\times \sum_{m=0}^{n} q^{\binom{m}{2}} \frac{(q;q)_n t^{n-m} h^m}{(q;q)_{n-m}(q;q)_m}\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(r(r-1))+\frac{a}{4}r}(y)^r}{(q;q)_r}\sum_{s=0}^{[kr]} q^{\binom{s}{2}}\frac{(q;q)_{kr} t^{kr-s} h^s}{(q;q)_{kr-s}(q;q)_s}$$

$$= \frac{2q^{a/4}}{1-q}\sum_{w=0}^{1} \frac{(q;q)_1 t^{1-w} h^w}{(q;q)_{1-w}(q;q)_w}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{q^{\frac{a}{4}(n^2)+\frac{a}{2}n+\binom{m}{2}}(2x)^n}{(q;q)_{n-m}(q;q)_m} t^{n-m} h^m$$

$$\times \sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}r^2}(y)^r}{(q;q)_r}\sum_{s=0}^{r} q^{\binom{ks}{2}}\frac{(q;q)_{kr} t^{kr-ks} h^{ks}}{(q;q)_{kr-ks}(q;q)_{ks}}, \tag{5.3}$$

Which, by using relation (1.15), gives

$$\sum_{n,m=0}^{\infty} \frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) t^n h^m = \frac{2q^{a/4}}{1-q}\sum_{w=0}^{1} \frac{(q;q)_1 t^{1-w} h^w}{(q;q)_{1-w}(q;q)_w}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{q^{\frac{a}{4}\left((n+m)^2\right)+\frac{a}{2}(n+m)+\binom{m}{2}}(2x)^{n+m}}{(q;q)_n(q;q)_m} t^n h^m$$

$$\times \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}\left((r+s)^2\right)+\binom{ks}{2}}}{(q;q)_{r+s}}\frac{(q;q)_{kr+ks}\, y^{r+s}\, t^{kr} h^{ks}}{(q;q)_{kr}(q;q)_{ks}}$$

From definition (4.1), we have

$$\sum_{n,m=0}^{\infty} \frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) t^n h^m = \frac{2q^{a/4}}{1-q}\sum_{w=0}^{1} \frac{(q;q)_1 t^{1-w} h^w}{(q;q)_{1-w}(q;q)_w}\sum_{n,m=0}^{\infty} H_{n,m}^k\left(q^{a/2}x,y,a;q\right) t^n h^m$$

$$= \frac{2q^{a/4}}{1-q}\sum_{n,m=0}^{\infty}\sum_{w=0}^{1} \frac{(q;q)_1 H_{n,m}^k\left(q^{a/2}x,y,a;q\right)}{(q;q)_{1-w}(q;q)_w} t^{n+1-w} h^{m+w},$$

by equating the coefficient of $t^n h^m$, we get

$$\frac{\partial_q}{\partial_q x} H_{n,m}^k(x,y,a;q) = \frac{2q^{a/4}}{1-q}\sum_{w=0}^{1} \frac{(q;q)_1 H_{n+w-1,m-w}^k\left(q^{a/2}x,y,a;q\right)}{(q;q)_{1-w}(q;q)_w}.$$

Expanding the summation in the above equation in series form then we get the required relation (5.1).

Thus,

$$\sum_{n,m=0}^{\infty}\frac{\partial_q}{\partial_q y}H_{n,m}^{k}(x,y,a;q)t^n h^m$$

$$=\frac{(-1)^{a+1}q^{a/4}(t+h)^k}{1-q}E_q\left(2q^{a/4}x(t+h),\frac{a}{2}\right)E_q\left((-1)^{a+1}q^{3a/4}y(t+h)^k,\frac{a}{2}\right)$$

$$=\frac{(-1)^{a+1}q^{a/4}}{1-q}(t+h)^k\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}(2x)^n}{(q;q)_n}(t+h)^n\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{3a}{4}r}(y)^r}{(q;q)_r}(t+h)^{kr}$$

by using relation (1.19), we obtain

$$\sum_{n,m=0}^{\infty}\frac{\partial_q}{\partial_q y}H_{n,m}^{k}(x,y,a;q)t^n h^m=\frac{(-1)^{a+1}q^{a/4}}{1-q}\sum_{w=0}^{k}\begin{bmatrix}k\\w\end{bmatrix}_q q^{\binom{k}{2}}t^{k-w}h^w\sum_{n=0}^{\infty}\frac{q^{\frac{a}{2}\binom{n}{2}+\frac{a}{4}n}(2x)^n}{(q;q)_n}$$

$$\times\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}_q q^{\binom{m}{2}}t^{n-m}h^m\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{2}\binom{r}{2}+\frac{3a}{4}r}(y)^r}{(q;q)_r}\sum_{s=0}^{kr}\begin{bmatrix}kr\\s\end{bmatrix}_q q^{\binom{s}{2}}t^{kr-s}h^s, \qquad (5.4)$$

From definition (4.1), we have

$$\sum_{n,m=0}^{\infty}\frac{\partial_q}{\partial_q y}H_{n,m}^{k}(x,y,a;q)t^n h^m=\frac{(-1)^{a+1}q^{a/4}}{1-q}\sum_{w=0}^{k}q^{\binom{k}{2}}\frac{(q;q)_k t^{k-w}h^w}{(q;q)_{k-w}(q;q)_w}$$

$$\times\sum_{n=0}^{\infty}\frac{q^{\frac{a}{4}(n(n-1))+\frac{a}{4}n}(2x)^n}{(q;q)_n}\sum_{m=0}^{n}q^{\binom{m}{2}}\frac{(q;q)_n t^{n-m}h^m}{(q;q)_{n-m}(q;q)_m}$$

$$\times\sum_{r=0}^{\infty}(-1)^{r(a+1)}\frac{q^{\frac{a}{4}(r(r-1))+\frac{a}{4}r+\frac{a}{2}r}(y)^r}{(q;q)_r}\sum_{s=0}^{kr}q^{\binom{s}{2}}\frac{(q;q)_{kr}t^{kr-s}h^s}{(q;q)_{kr-s}(q;q)_s}$$

$$=\frac{(-1)^{a+1}q^{a/4}}{1-q}\sum_{n,m=0}^{\infty}\sum_{w=0}^{k}q^{\binom{k}{2}}\frac{(q;q)_k H_{n+w-k,m-w}^{k}\left(x,q^{a/2}y,a;q\right)}{(q;q)_{k-w}(q;q)_w}t^n h^m,$$

by equating the coefficient of $t^n h^m$, we get the required relation (5.2).

Corollary 5.2

If we choose $k=2$, in (5.1), then we get the new explicit definition of $\frac{\partial_q}{\partial_q x}H_{n,m}(x,y,a;q)$ defined by

$$\frac{\partial_q}{\partial_q x}H_{n,m}(x,y,a;q)=\frac{2q^{a/4}}{1-q}\left[H_{n-1,m}\left(q^{a/2}x,y,a;q\right)+H_{n,m-1}\left(q^{a/2}x,y,a;q\right)\right], \qquad (5.5)$$

also choosing $k=2$, in (5.2), then we get the new explicit definition of $\frac{\partial_q}{\partial_q y}H_{n,m}(x,y,a;q)$, which is

$$\frac{\partial_q}{\partial_q y}H_{n,m}(x,y,a;q)=\frac{(-1)^{a+1}q^{a/4}}{1-q}\left[H_{n-2,m}\left(x,q^{a/2}y,a;q\right)+(q;q)_2 H_{n-1,m-1}\left(x,q^{a/2}y,a;q\right)+H_{n,m-2}\left(x,q^{a/2}y,a;q\right)\right]. \quad (5.6)$$

Theorem 5.2

The generalized q-analogue associated Hermite polynomials of two-index and two-variable $H_{n,m}(x,y,a;q)$ satisfy the following relations:

$$[n+1]_q H^k_{n+1,m}(x,y,a;q) = \frac{k(-1)^{a+1} q^{a/4} y}{1-q} \sum_{w=0}^{k-1} q^{\binom{w}{2}} \frac{(q;q)_{k-1} H^k_{n+w-k+1,m-w}(qx, q^{\frac{a}{2}} y, a;q)}{(q;q)_{k-1-w}(q;q)_w} + \frac{2q^{a/4} x}{1-q} H^{(k)}_{n,m}\left(q^{\frac{a}{2}} x, y, a; q\right), \tag{5.7}$$

and

$$[m+1]_q H^k_{n,m+1}(x,y,a;q) = \frac{k(-1)^{a+1} q^{a/4} y}{1-q} \sum_{w=0}^{k-1} q^{\binom{w}{2}} \frac{(q;q)_{k-1} H^k_{n+w-k+1,m-w}(qx, q^{\frac{a}{2}} y, a;q)}{(q;q)_{k-1-w}(q;q)_w} + \frac{2q^{a/4} x}{1-q} H^k_{n,m}\left(q^{\frac{a}{2}} x, y, a; q\right). \tag{5.8}$$

Proof. From (4.2) and using relation (1.7), we get

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial t} H^k_{n,m}(x,y,a;q) t^n h^m = \frac{2q^{a/4} x}{1-q} E_q\left(2q^{3a/4} x(t+h), \frac{a}{2}\right) E_q\left((-1)^{a+1} q^{a/4} y(t+h)^k, \frac{a}{2}\right) + \frac{k(-1)^{a+1} q^{a/4} y(t+h)^{k-1}}{1-q} E_q\left(2q^{a/4+1} x(t+h), \frac{a}{2}\right) E_q\left((-1)^{a+1} q^{3a/4} y(t+h)^k, \frac{a}{2}\right),$$

by using relations (1.9.1) and (1.5.1) on (5.5.14), we get

$$\sum_{n,m=0}^{\infty} \frac{\partial}{\partial t} H^k_{n,m}(x,y,a;q) t^n h^m = \frac{2q^{a/4} x}{1-q} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{q^{\frac{a}{4}(n^2) + \frac{a}{2} n} (2x)^n}{(q;q)_{n-m} (q;q)_m} t^{n-m} h^m \times \sum_{r=0}^{\infty} (-1)^{r(a+1)} \frac{q^{\frac{a}{4} r^2} (y)^r}{(q;q)_r} \sum_{s=0}^{kr} q^{\binom{s}{2}} \frac{(q;q)_{kr} t^{kr-ks} h^{ks}}{(q;q)_{kr-ks} (q;q)_{ks}},$$

$$+ \frac{k(-1)^{a+1} q^{a/4} y}{1-q} \sum_{w=0}^{k-1} q^{\binom{w}{2}} \frac{(q;q)_{k-1} t^{k-1-w} h^w}{(q;q)_{k-1-w} (q;q)_w} \times \sum_{n=0}^{\infty} \sum_{m=0}^{n} q^{\binom{m}{2}} \frac{q^{\frac{a}{4}(n^2)+n} (2x)^n}{(q;q)_{n-m} (q;q)_m} t^{n-m} h^m \times \sum_{r=0}^{\infty} (-1)^{r(a+1)} \frac{q^{\frac{a}{4} r^2 + \frac{a}{2} r} (y)^r}{(q;q)_r} \sum_{s=0}^{kr} q^{\binom{s}{2}} \frac{(q;q)_{kr} t^{kr-s} h^s}{(q;q)_{kr-s} (q;q)_s}$$

Which on using relation (1.19), gives

$$\sum_{n,m=0}^{\infty}[n+1]_q H_{n,m}^{k}(x,y,a;q)t^n h^m$$

$$=\frac{k(-1)^{a+1}q^{a/4}y}{1-q}\sum_{w=0}^{k-1}q^{\binom{w}{2}}\frac{(q;q)_{k-1}t^{k-1-w}h^w}{(q;q)_{k-1-w}(q;q)_w}\sum_{n,m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}q^{\binom{ks}{2}}(-1)^{(r+s)(a+1)}$$

$$\times\frac{q^{\frac{a}{4}\left((m+n-kr-ks)^2+(r+s)^2\right)}(2qx)^{n+m-kr-ks}}{(q;q)_{n-kr}(q;q)_{m-ks}}\frac{\left(q^{\frac{a}{2}}y\right)^{(r+s)}}{(q;q)_{r+s}}\frac{(q;q)_{kr+ks}}{(q;q)_{kr}(q;q)_{ks}}t^n h^m$$

$$+\frac{2q^{a/4}x}{1-q}\sum_{n,m=0}^{\infty}\sum_{r=0}^{\left[\frac{n}{k}\right]}\sum_{s=0}^{\left[\frac{m}{k}\right]}q^{\binom{ks}{2}}(-1)^{(r+s)(a+1)}\frac{q^{\frac{a}{4}\left((m+n-kr-ks)^2+(r+s)^2\right)}\left(2q^{\frac{a}{2}}x\right)^{n+m-kr-ks}}{(q;q)_{n-kr}(q;q)_{m-ks}}$$

$$\times\frac{(y)^{(r+s)}}{(q;q)_{r+s}}\frac{(q;q)_{kr+ks}}{(q;q)_{kr}(q;q)_{ks}}t^n h^m$$

by equating the coefficient of $t^n h^m$, we get the required relation (5.7).
Similarly, Differentiating (5.6.2) with respect to h we get relation (5.8).

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كثيرات حدود هرميت الأساسية المعممة ذات متغيرين

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الملخص

في هذا البحث قدمنا كثيرات حدود لهرميت الأساسية المعممة ذات دليل ومتغيرين و كثيرات حدود لهرميت المعدلة الأساسية المعممة ذات دليلين و متغيرين. واشتقينا بعض العلاقات التكرارية لهما.

الكلمات المفتاحية: كثيرات حدود هرميت الأساسية المعممة والدوال المولدة والعلاقات التكرارية.