# Some types of generalized $\beta H$ −Birecurrent Finsler space

**Fahmi Yaseen Abdo Qasem**
Dep. of Math., Faculty of Edu. - Aden, Univ. of Aden ,Khormaksar, Aden, Yemen
Email: fahmi.yaseen@yahoo.com
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## Abstract

In this paper, we defined a Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies the condition $\mathcal{B}_m\mathcal{B}_n H^i_{jkh} = a_{mn}H^i_{jkh} + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$-2y$^r$ μ$_n$$\mathcal{B}_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$, where $\mathcal{B}_m\mathcal{B}_n$ are Berwald covariant derivative of second order with respect to $x^m$ and $x^n$,respectively, $a_{mn}$ and $b_{mn}$ are non-zero covariant tensors field .

The purpose of this paper is to develop the generalized $\beta H$ −birecurrent space by studying some properties of generalized $\beta H$ −birecurrent affinely connected space, $P2$ −like generalized $\beta H$ −birecurrent space and $P^*$ −generalized $\beta H$ −birecurrent space. Some theorems and conditions have been pointed out which reduce a generalized $\beta H$ −birecurrentaffinely connected space $F_n$ $(n > 2)$ into a Finsler space of curvature scalar.

**Keywords**: Finsler space, Generalized $\beta H$ −birecurrentaffinely connected space, Finsler space ofcurvature scalar, $P2$ −like generalized $\beta H$ −birecurrent space, $P^*$ − generalized $\beta H$ −birecurrent space.

## 1.Introduction

Dikshit [1]introduced a Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies the recurrence property in the sense of Berwald . A Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies the property of generalized recurrent space in the sense of Berwald was introduced and discussed by Pandey, Saxena and Goswami[6]. Qasem and Hadi [9] introduced and studied generalized $\beta R$ −birecurrent space. Qasem and Alqashbari [8] introduced some types of generalized $H^h$ −reccurrent in spaces.

Let us consider an $n$-dimensionalFinsler space $F_n$ equipped with a metric function $F(x, y)$ satisfying the requestic condition of Finslerian metric [10] .

The vector $y_i$, its associative vector$y^i$ and the metric tensor $g_{ij}$ are given by

(1.1) a) $y_i\, y^i = F^2$,b) $g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$ and c) $g_{ij}y^j = y_i$ .

Berwald's covariant derivative of an arbitrary tensor field $T^i_j$ with respect to $x^k$ is given by

$-(\dot{\partial}_r T^i_j)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$ . $\mathcal{B}_k T^i_j := \partial_k T^i_j$

Berwald's covariant derivative of $y^i$ vanishes ,i.e.

(1.2) $\mathcal{B}_k\, y^i = 0$.

The processes of Berwald's covariant differentiation and the partial differentiation commute according to

(1.3) $(\dot{\partial}_k\mathcal{B}_h - \mathcal{B}_k\dot{\partial}_h)\, T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$.

Berwald curvature tensor $H^i_{jkh}$ satisfies the relation

(1.4)$H^i_{jkh} = \dot{\partial}_j H^i_{kh}$.

The$h(v)$ − torsion tensor $H^i_{kh}$satisfies

(1.5) a)$H^i_{kh}\, y^k = H^i_h$ and b) $H^i_{jkh}\, y^j = H^i_{kh}$,

where $H^i_h$ and $H^i_{jkh}$ are the deviation tensor of Berwald curvature tensor and Cartan's third curvature tensor , respectively

Also we have the following relations

(1.6) a) $H_{jk} = H^i_{jki}$ , b) $H_k = H^i_{ki}$ and c) $H = \frac{1}{n-1} H^i_i$,

where $H_{jk}$ , $H_k$ and $H$ are called *H-Ricci tensor* [5], *curvature vector* and *curvature scalar*, respectively. Since the contraction of the indices doesn't affect the homogeneity in $y^i$, hence $H$-Ricci tensor $H_{rk}$ , curvature vector $H_r$ and the curvature scalar $H$ are homogeneous of degree zero, one and two in $y^i$, respectively . The above tensors are also connected by

(1.7) a) $H_{jk}\, y^j = H_k$ , b) $H_{jk} = \dot{\partial}_j H_k$ and c) $H_k\, y^k = (n-1)H$ .

The necessary and sufficient condition for a Finsler space $F_n (n > 2)$ to be a Finsler space of curvature scalar is given by

(1.8) $H^i_h = F^2 R(\, \delta^i_h - l^i l_h\, )$.

The $hv$-curvature tensor $P^i_{jkh}$, the $v(hv)$-torsion tensor $P^i_{kh}$, $P$ −Ricci tensor $P_{jk}$ and the curvature vector $P_k$ satisfy the following relations

(1.9) a) $P^i_{jkh}\, y^j = P^i_{kh}$ , b) $P^i_{jki} = P_{jk}$ and c) $P^i_{ki} = P_k$ .

Also, the $hv$-curvature tensor $P^i_{jkh}$ and the $v(hv)$-torsion tensor $P^i_{kh}$ satisfy the following:

(1.10) $P^i_{hjk} - P^i_{jhk} = C^i_{jk|h} + C^i_{hs} P^s_{jk} - C^i_{hk|j} - C^i_{js} P^s_{hk}$.

## 2.A Generalized $\beta H$ −Birecurrent Affinely Connected Space

An affinely connected space or Berwald space is characterized by any one of the following two equivalent conditions

(2.1) a) $G^i_{jkh} = 0$ and b) $C_{ijk|h} = 0$ .

Also, it has the following properties

(2.2) a) $\mathcal{B}_k\, g_{ij} = 0$ and b) $\mathcal{B}_k\, g^{ij} = 0$ .

Pandey, Saxena and Goswami[6] introduced and discussed a Finsler space whose Berwald curvature tensor $H^i_{jkh}$ satisfies the condition

(2.3) $\mathcal{B}_n H^i_{jkh} = \lambda_n H^i_{jkh} + \mu_n (\delta^i_k g_{jh} - \delta^i_h g_{jk})$ , $H^i_{jkh} \neq 0$,

they called it *a generalized $\beta H$- recurrent space*, where $\lambda_n$ and $\mu_n$ are non-zero covariant vectors field called the *recurrence vectors field* .

Taking the covariant derivative for the condition (2.3) with respect to $x^m$ in the sense of Berwald and using (2.2a) ,we get

(2.4) $\mathcal{B}_m \mathcal{B}_n H^i_{jkh} = a_{mn} H^i_{jkh} + b_{mn} \left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)$ , $H^i_{jkh} \neq 0$ ,

where $a_{mn} = \mathcal{B}_m\, \lambda_n + \lambda_n\, \lambda_m$ and $b_{mn} = \lambda_n\, \mu_m + \mathcal{B}_m \mu_n$ are non-zero covariant tensors field of second order .

**Definition2.1.** A Finslerspace whose Berwald curvature tensor $H^i_{jkh}$ satisfies the condition (2.4) will be called *generalized $\beta H$-birecurrentaffinely connectedspace*, we shall denote it briefly by $G\beta H - BR$ −affinely connected space.

Let us consider a $G\beta H - BR$ −affinely connected space .

Transvecting the condition (2.4) by $y^j$ , using (1.2) , (1.5b) and (1.1c),we get

(2.5) $\mathcal{B}_m \mathcal{B}_n H^i_{kh} = a_{mn} H^i_{kh} + b_{mn} \left(\delta^i_k y_h - \delta^i_h y_k\right)$.

Further , transvectingthe condition (2.5) by $y^k$ , using (1.2) , (1.5a) and (1.1a) , we get

(2.6) $\mathcal{B}_m \mathcal{B}_n H^i_h = a_{mn} H^i_h + b_{mn} \left(y^i y_h - \delta^i_h F^2\right)$.

Thus, we conclude:

**Theorem2.1.** In $G\beta H - BR$ −affinely connected space , Berwald's covariant derivative of second order for the $h(v)$ −torsion tensor $H^i_{kh}$ and the deviation tensor $H^i_h$, given by the conditions (2.5) and (2.6), respectively.

Contracting the indices $i$ and $h$ inthe conditions (2.4), (2.5) and (2.6) ,separately , using(1.6a), (1.6b) and (1.6c) , we get.

(2.7) $\mathcal{B}_m \mathcal{B}_n H_{jk} = a_{mn} H_{jk} + (1 - \mathrm{n})\, b_{mn} g_{jk}$ ,

(2.8) $\mathcal{B}_m\mathcal{B}_nH_k = a_{mn}H_k + (1-n)b_{mn}y_k$
and
(2.9) $\mathcal{B}_m\mathcal{B}_nH = a_{mn}H - b_{mn}F^2$,
respectively .

The conditions (2.7), (2.8) and (2.9), show that $H-$ Ricci tensor $H_{jk}$ , the curvature vector $H_k$ and the curvature scalar $H$ can't vanish, because the vanishing of any one of them would imply $b_{mn}$= 0, a contradiction .
Thus, we conclude:

**Theorem2.2.** In $G\beta H - BR$ −affinely connected space, $H-$ Ricci tensor $H_{jk}$ ,the curvature vector $H_k$ and thecurvature scalar $H$ are non-vanishing .

Now, differentiating the condition (2.8) partially with respect to $y^j$ and using (1.1b), we get

(2.10) $\dot{\partial}_j(\mathcal{B}_m\mathcal{B}_nH_k) = (\dot{\partial}_j a_{mn})H_k + a_{mn}(\dot{\partial}_j H_k) + (1-n)(\dot{\partial}_j b_{mn})y_k + (1-n)b_{mn}g_{jk}$.

Using the commutation formula exhibited by (1.3) for $(\mathcal{B}_nH_k)$ in (2.10) , using (1.7b) and (2.1a), we get

(2.11) $\mathcal{B}_m\dot{\partial}_j(\mathcal{B}_nH_k) = (\dot{\partial}_j a_{mn})H_k + a_{mn}H_{jk} + (1-n)(\dot{\partial}_j b_{mn})y_k + (1-n)b_{mn}g_{jk}$ .

Again applying the commutation formula exhibited by (1.3) for $(H_k)$ in (2.11) and using (2.1a), we get

(2.12) $\mathcal{B}_m\mathcal{B}_nH_{jk} = (\dot{\partial}_j a_{mn})H_k + a_{mn}H_{jk} + (1-n)(\dot{\partial}_j b_{mn})y_k + (1-n)b_{mn}g_{jk}$ .

Using the condition (2.7) in (2.12), we get

(2.13) $(\dot{\partial}_j a_{mn})H_k + (1-n)(\dot{\partial}_j b_{mn})y_k = 0$ .

Transvecting (2.13) by $y^k$ , using (1.7c) and (1.1a) , we get

(2.14) $-(\dot{\partial}_j a_{mn})H + (\dot{\partial}_j b_{mn})F^2 = 0$

which can be written as

(2.15) $\dot{\partial}_j b_{mn} = \frac{(\dot{\partial}_j a_{mn})H}{F^2}$ .

If the covariant tensor field $a_{mn}$ is independent of $y^i$, (2.15) shows that the covariant tensor field $b_{mn}$ is independent of $y^i$ . Conversely , if the covariant tensor field $b_{mn}$ is independent of $y^i$ , we get $H(\dot{\partial}_j a_{mn}) = 0$.In view of theorem2.2 , the condition $H(\dot{\partial}_j a_{mn}) = 0$ implies $\dot{\partial}_j a_{mn} = 0$ ,i.e. the covariant tensor field $a_{mn}$ is also independent of $y^i$ . This leads to

**Theorem2.3.** In $G\beta H - BR$ −affinely connected space,the covariant tensor field $b_{mn}$ is independent of the directional arguments.

Suppose the covariant tensor field $a_{mn}$ is not independent of $y^i$, in view of (2.13) and (2.15), we get (2.16) $\dot{\partial}_j a_{mn}[H_k - \frac{(n-1)}{F^2}Hy_k] = 0$.

Transvecting (2.16) by $y^m$, we get

(2.17) $(\dot{\partial}_j a_{mn})y^m[H_k - \frac{(n-1)}{F^2}Hy_k]=0$

which implies

(2.18) $(\dot{\partial}_j a_n - a_{jn})[H_k - \frac{(n-1)}{F^2}Hy_k]= 0$,

where $a_{mn}y^m = a_n$ .

The equation (2.18) has at least one of the following conditions

(2.19) a) $a_{jn} = \dot{\partial}_j a_n$ , b) $H_k = \frac{(n-1)}{F^2}Hy_k$ .

Thus, we conclude

**Theorem2.4.** In $G\beta H - BR$ −affinely connected space, which the covariant tensor field $a_{mn}$ is not independent of the directional argument at least one of the conditions(2.19a) and (2.19b) hold provided (2.15) holds.

Differentiating the condition (2.5) partially with respect to $y^j$ , using (1.4) and (1.1b), we get

(2.20) $\dot{\partial}_j(\mathcal{B}_m\mathcal{B}_n H^i_{kh}) = (\dot{\partial}_j a_{mn})H^i_{kh} + a_{mn}H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k)$
$+b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$ .

Using the commutation formula exhibited by (1.3) for $(\mathcal{B}_n H^i_{kh})$ in (2.20) and using (2.1a), we get

(2.21) $\mathcal{B}_m(\dot{\partial}_j \mathcal{B}_n H^i_{kh}) = (\dot{\partial}_j a_{mn})H^i_{kh} + a_{mn}H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k)$
$+b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Again, applying the commutation formula exhibited by (1.3) for $(H^i_{kh})$ in (2.21) , using (1.4) and (2.1a), we get

(2.22) $\mathcal{B}_m\mathcal{B}_n H^i_{jkh} = (\dot{\partial}_j a_{mn})H^i_{kh} + a_{mn}H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using the condition (2.4) in (2.22), we get

(2.23) $(\dot{\partial}_j a_{mn})H^i_{kh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) = 0$ .

Transvecting (2.23) by $y^k$ , using (1.5a) and (1.1a) , we get

(2.24) $(\dot{\partial}_j a_{mn})H^i_h - (\dot{\partial}_j b_{mn})(\delta^i_h F^2 - y^i y_h) = 0$.

In view of (2.15) and (2.24), we get

(2.25) $(\dot{\partial}_j a_{mn})[H^i_h - H(\delta^i_h - l^i\, l_h)] = 0$ .

We have at least one of the following conditions:

(2.26) a) $\dot{\partial}_j a_{mn} = 0$ , b) $H^i_h = H(\delta^i_h - l^i\, l_h)$.

Putting $H = F^2 R$ , $R \neq 0$ ,the equation (2.26b) becomes

(2.27) $H^i_h = F^2 R(\delta^i_h - l^i\, l_h)$.

Therefore, the space is a Finsler space ofcurvature scalar.

Thus , we conclude

**Theorem2.5.**A$G\beta H - BR$ −affinely connected space, for ($n$ >2) is a Finsler space of curvature scalar provided $R$ ≠0 and the covariant tensor filed $a_{mn}$ is not independent of the directional argument.

## 3. A $P2$ −Like generalized $\beta H$ −birecurrent space

A $P2$ −Like space is characterized by Matsumoto [4]

,(3.1) $P^i_{jkh} = \varphi_j C^i_{kh} - \varphi^i C_{jkh}$

where $\varphi_j$ is non-zero covariant vector field.

**Definition 3.1.**The generalized $\beta H$ −birecurrent space which is$P2$ −Like space [satisfies the condition (3.1)], will be called a $P2$ *−Like generalized $\beta H$ −birecurrent space* and will denote it briefly by a $P2$ −Like $G\beta H - BRF_n$ .

Let us consider a $P2$ −Like $G\beta H - BR - F_n$.

Taking the covariant derivative for the condition (3.1)twice with respect to $x^n$and $x^m$, successively in the sense of Berwald , we get

(3.2) $\mathcal{B}_m\mathcal{B}_n P^i_{jkh} = (\mathcal{B}_m\mathcal{B}_n\varphi_j)C^i_{kh} + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C^i_{kh}) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C^i_{kh}) + \varphi_j(\mathcal{B}_m\mathcal{B}_n C^i_{kh}) - (\mathcal{B}_m\mathcal{B}_n\varphi^i)C_{jkh} - (\mathcal{B}_n\varphi^i)(\mathcal{B}_m C_{jkh}) - (\mathcal{B}_m\varphi^i)(\mathcal{B}_n C_{jkh}) - \varphi^i(\mathcal{B}_m\mathcal{B}_n C_{jkh})$ .

Suppose $C^i_{kh}$ and $C_{rkh}$ are satisfying the following:

$$(3.3)a)\quad \mathcal{B}_m\mathcal{B}_n C^i_{kh} = a_{mn}C^i_{kh} + b_{mn}(\delta^i_k y_h - \delta^i_h y_k)$$

and

.$b)$ $\mathcal{B}_m\mathcal{B}_n C_{rkh} = a_{mn}C_{rkh} + b_{mn}(g_{kr}y_h - g_{hr}y_k)$

Substituting the conditions(3.3a) and (3.3b) in (3.2) and using (3.1), we get

(3.4)$\mathcal{B}_m\mathcal{B}_n P^i_{jkh} = a_{mn}P^i_{jkh} + b_{mn}\varphi_j(\delta^i_k y_h - \delta^i_h y_k) - b_{mn}\varphi^i(g_{kj}y_h - g_{hj}y_k) +$
$(\mathcal{B}_m\mathcal{B}_n\varphi_j)C^i_{kh} + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C^i_{kh}) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C^i_{kh}) - (\mathcal{B}_m\mathcal{B}_n\varphi^i)C_{jkh} -$
. $(\mathcal{B}_n\varphi^i)(\mathcal{B}_m C_{jkh}) - (\mathcal{B}_m\varphi^i)(\mathcal{B}_n C_{jkh})$

This shows that

(3.5) $\mathcal{B}_m\mathcal{B}_n P^i_{jkh} = a_{mn}P^i_{jkh} + b_{mn}\varphi_j(\delta^i_k y_h - \delta^i_h y_k)$
if and only if
(3.6)$-b_{mn}\varphi^i\ (g_{kj}y_h - g_{hj}y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi_j)C^i_{kh} + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C^i_{kh}) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C^i_{kh}) -$
$(\mathcal{B}_m\mathcal{B}_n\varphi^i\ )C_{jkh} - (\mathcal{B}_n\varphi^i\ )(\mathcal{B}_m C_{jkh}) - (\mathcal{B}_m\varphi^i\ )(\mathcal{B}_n C_{jkh}) = 0$ .
Thus , we have

**Theorem3.1.** In $P2$ −Like $G\beta H - BR - F_n$, Berwald's covariant derivative of second order forCartan`s second curvature tensor $P^i_{jkh}$is given by the condition (3.5) if and only if (3.6) holds good [provided the conditions(3.3a)and (3.3b)hold ].
Transvecting (3.4) by $y^j$, using (1.2), (1.9a) and (1.1c), we get
(3.7)$\mathcal{B}_m\mathcal{B}_n P^i_{kh} = a_{mn}P^i_{kh} + y^j[b_{mn}\varphi_j(\delta^i_k y_h - \delta^i_h y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi_j)C^i_{kh} + (\mathcal{B}_n\varphi_j)$
$(\mathcal{B}_m C^i_{kh}) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C^i_{kh})]$.
This shows that
(3.8) $\mathcal{B}_m\mathcal{B}_n P^i_{kh} = a_{mn}P^i_{kh} + c_{mn}(\delta^i_k y_h - \delta^i_h y_k)$ ,
where $y^j\varphi_j = \varphi$ and$c_{mn} = \varphi b_{mn}$
if and only if
(3.9)$y^j[(\mathcal{B}_m\mathcal{B}_n\varphi_j)C^i_{kh} + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C^i_{kh}) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C^i_{kh})]$= 0.
Thus, we conclude

**Theorem3.2.** In $P2$ −Like $G\beta H - BR - F_n$,Berwald's covariant derivative of second order for the v(hv)- torsion tensor $P^i_{kh}$is given by the condition (3.8)if and only if (3.9) holds good[provided the conditions(3.3a)and (3.3b)hold ].
Contracting the indices $i$ and $h$ in (3.4) and using (1.9b) ,we get
(3.10)$\mathcal{B}_m\mathcal{B}_n P_{jk} = a_{mn}P_{jk} + (1-n)b_{mn}\varphi_j y_k - b_{mn}\varphi^p\ (g_{kj}y_{ph} - g_{pj}y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi_j)C_k +$
$(\mathcal{B}_n\varphi_j)(\mathcal{B}_m C_k) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C_k) - (\mathcal{B}_m\mathcal{B}_n\varphi^p\ )C_{jkp} - (\mathcal{B}_n\varphi^p\ )(\mathcal{B}_m C_{jkp}) -$
$(\mathcal{B}_m\varphi^p\ )(\mathcal{B}_n C_{jkp})$.
This shows that
(3.11)$\mathcal{B}_m\mathcal{B}_n P_{jk} = a_{mn}P_{jk} + (1-n)b_{mn}\varphi_j y_k$
if and only if
(3.12)$-b_{mn}\varphi^p\ (g_{kj}y_{ph} - g_{pj}y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi_j)C_k + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C_k) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C_k) -$
$(\mathcal{B}_m\mathcal{B}_n\varphi^p\ )C_{jkp} - (\mathcal{B}_n\varphi^p\ )(\mathcal{B}_m C_{jkp}) - (\mathcal{B}_m\varphi^p\ )(\mathcal{B}_n C_{jkp})$=0.
Contracting the indices $i$ and $h$ in (3.7) and using (1.9c) ,we get
(3.13)$\mathcal{B}_m\mathcal{B}_n P_k = a_{mn}P_k + y^j[(1-n)b_{mn}\varphi_j y_k - b_{mn}\varphi^p\ (g_{kj}y_p - g_{pj}y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi_j)C_k$
$+ (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C_k) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C_k)]$= 0 .
This shows that
(3.14)$\mathcal{B}_m\mathcal{B}_n P_k = a_{mn}P_k + (1-n)c_{mn}y_k$
if and only if
(3.15) $y^j[(\mathcal{B}_m\mathcal{B}_n\varphi_j)C_k + (\mathcal{B}_n\varphi_j)(\mathcal{B}_m C_k) + (\mathcal{B}_m\varphi_j)(\mathcal{B}_n C_k)]$ =0.
The conditions (3.11) and (3.14), show that $P$ −Ricci tensor $P_{jk}$ and the curvature vector $P_k$can't vanish, because the vanishing of any one of them would imply $b_{mn}$= 0 and $c_{mn}$ = 0, a contradiction.
Thus, we conclude

**Theorem3.3.** In $P2$ −Like $G\beta H - BR - F_n$ , $P$ −Ricci tensor $P_{jk}$and the curvature vector $P_k$ are non-vanishingif and only if (3.12) and (3.15), respectively hold[provided the conditions(3.3a)and (3.3b)hold ].
Taking the covariant derivative for the condution (1.10) twice with respect to $x^n$and $x^m$, successively, in the sense of Berwald , we get

(3.16) $\mathcal{B}_m\mathcal{B}_n P^i_{hjk} - \mathcal{B}_m\mathcal{B}_n P^i_{jhk} = \mathcal{B}_m\mathcal{B}_n\big(C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}\big)$.

Using the condition (3.5) in (3.16) and in view of (1.10), we get

(3.17) $\mathcal{B}_m\mathcal{B}_n\big(C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}\big) = a_{mn}\big(C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}\big) + b_{mn}[\varphi_h(\delta^i_j y_k - \delta^i_k y_j) - \varphi_j(\delta^i_h y_k - \delta^i_k y_h)]$.

This shows that

(3.18) $\mathcal{B}_m\mathcal{B}_n\big(C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}\big) = a_{mn}\big(C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}\big)$

if and only if

(3.19) $b_{mn}\big[\varphi_h(\delta^i_j y_k - \delta^i_k y_j) - \varphi_j(\delta^i_h y_k - \delta^i_k y_h)\big]=0$ .

**Theorem3.4.** In $P2$ −Like $G\beta H - BR - F_n$, the tensor**(**$C^i_{jk|h} + C^i_{hs}P^s_{jk} - C^i_{hk|j} - C^i_{js}P^s_{hk}$**)** behaves as birecurrent if and only if (3.6) holds[provided the conditions(3.3a)and (3.3b)hold ].

## 4. A $P^*$ − generalized $\beta H$ − birecurrent space

A $P^*$ − space is characterized by the condition ([2],[3])

(4.1) $P^i_{kh} = \varphi C^i_{kh}$ , $\varphi \neq 0$.

**Definition4.1.**The generalized $\beta H$ − birecurrent space which is a $P^*$ − space will be called a $P^*$ − *generalized $\beta H$ − birecurrent space*and will denote it briefly by $P^* - \beta H - BRF_n$.

Let us consider a $P^* - G\beta H - BRF_n$.

Now, taking the covariant derivative for the condition (4.1) twice with respect to $x^n$ and $x^m$, successivelyin the sense of Berwald, we get

(4.2) $\mathcal{B}_m\mathcal{B}_n P^i_{kh} = (\mathcal{B}_m\mathcal{B}_n\varphi)C^i_{kh} + (\mathcal{B}_m\varphi)(\mathcal{B}_n C^i_{kh}) + (\mathcal{B}_n\varphi)(\mathcal{B}_m C^i_{kh}) + \varphi(\mathcal{B}_m\mathcal{B}_n C^i_{kh})$.

Using the condition (3.3a) and (3.3b)in (4.2), we get

(4.3) $\mathcal{B}_m\mathcal{B}_n P^i_{kh} = a_{mn}P^i_{kh} + d_{mn}(\delta^i_k y_h - \delta^i_h y_k) + (\mathcal{B}_m\mathcal{B}_n\varphi)C^i_{kh} + (\mathcal{B}_m\varphi)(\mathcal{B}_n C^i_{kh}) + (\mathcal{B}_n\varphi)(\mathcal{B}_m C^i_{kh})$,

where $b_{mn} = d_{mn}$ .

This shows that

(4.4) $\mathcal{B}_m\mathcal{B}_n P^i_{kh} = a_{mn}P^i_{kh} + d_{mn}(\delta^i_k y_h - \delta^i_h y_k)$

if and only if

(4.5) $(\mathcal{B}_m\mathcal{B}_n\varphi)C^i_{kh} + (\mathcal{B}_m\varphi)(\mathcal{B}_n C^i_{kh}) + (\mathcal{B}_n\varphi)(\mathcal{B}_m C^i_{kh})$= 0 .

Thus, we conclude

**Theorem4.1.** In $P^* - G\beta H - BRF_n$, Berwald's covariant derivative of second order for the $v(hv)$ −torsion tensor $P^i_{kh}$is given by the condition (4.4) if and only if (4.5) holds[provided the conditions(3.3a) and (3.3b)hold ].

Contracting the indices $i$ and $h$ in (4.3) and using (1.9c) , we get

(4.6) $\mathcal{B}_m\mathcal{B}_n P_k = a_{mn}P_k + (1-n)d_{mn}y_k + (\mathcal{B}_m\mathcal{B}_n\varphi)C_k + (\mathcal{B}_m\varphi)(\mathcal{B}_n C_k) + (\mathcal{B}_n\varphi)(\mathcal{B}_m C_k)$

This shows that

(4.7) $\mathcal{B}_m\mathcal{B}_n P_k = a_{mn}P_k + (1-n)d_{mn}y_k$

if and only if

(4.8) $(\mathcal{B}_m\mathcal{B}_n\varphi)C_k + (\mathcal{B}_m\varphi)(\mathcal{B}_n C_k) + (\mathcal{B}_n\varphi)(\mathcal{B}_m C_k) = 0$.

Thus, we conclude

**Theorem4.2.**$P^* - G\beta H - BRF_n$ , the curvature vector $P_k$ is non-vanishing if and only if (4.8) holds[provided the conditions(3.3a) and (3.3b)hold ].

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# بعض الأنواع لتعميم فضاء فنسلر βH – ثُنائي المُعاودة

**فهمي ياسين عبده قاسم**

**Email: Fahmi.yaseen@yahoo.com**

قسم الرياضيات، كلية التربية، عدن، جامعة عدن، خور مكسر، عدن، اليمن

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## المُلخص

في هذه الورقة عرفنا فضاء فنسلر الذي يُحقق موتر بروالاد التقوسي $H^i_{jtl}$ الحالة

$$\mathcal{B}_m\mathcal{B}_n H^i_{jkh} = a_{mn}H^i_{jkh} + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r \mu_n \mathcal{B}_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}),$$

حيث $\mathcal{B}_m\mathcal{B}_n$ هي المُشتقة الثانية لبروالاد المُتحدة الاختلاف بالنسبة الى $x^m$ ، $x^n$ على الترتيب , $a_{mn}$ ، $b_{mn}$ هي حقول موترات مُتحدة الاختلاف غير صفرية.

الغرض من هذه الورقة هو تطوير تعميم فضاء βH- ثُنائي المُعاودة بدراسة بعض الخواص لتعميم فضاء βH- affinely connected ثُنائي المُعاودة، تعميم فضاء βH – like – P2 ثُنائي المُعادة وتعميم فضاء $P^*$-βH ثُنائي المُعاودة، تم الحصول على بعض المُبرهنات والحالات التي تختزل تعميم فضاء βH- affinely connected ثُنائي المُعاودة $F_n$ (n>2) إلى فضاء فنسلر ذات الثابت التقوسي.

**الكلمات المفتاحية:** فضاء فنسلر، تعميم فضاء βH- affinely connection ثُنائي المُعاودة، فضاء فنسلر ذات الثابت التقوسي، تعميم فضاء تعميم فضاء βH – like – P2 ثُنائي المُعادة وتعميم فضاء $P^*$-βH ثُنائي المُعاودة.