Obtaining generating relations associated with the generalized Gauss hypergeometric function

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Abstract

In this paper, some new generating relations involving the generalized hyper- geometric function and the generalized confluent hypergeometric function are established by mainly applying Taylor's Theorem. Due to their very general nature, the main results can be shown to be specialized to yield a large number of new, known, interesting and useful generating relations involving the Gauss hypergeometric function and its related functions.

Keywords: Gamma function; Beta function; Generalized Beta type function; Generating functions; Generalized confluent hypergeometric function; Generalized Gauss hypergeom- etric function; Hadamard product.

1. Introduction

Generating functions play an important role in the investigation of various useful properties of sequences which they generate. These are used to find certain properties and formulas for numbers and polynomials in a wide variety of research subjects. One can refer to the extensive work of Srivastava and Manocha [18] for a systematic introduction and several interesting(and useful) applications of the various methods of obtaining linear, bilinear, bilateral or mixed multilateral generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two or more variables, among much abundant literature. In this regard, in fact, a remarkable large number of generating functions involving a variety of special functions have been developed by many authors Agarwal et al.[1] and [4], Choi [9], Lue et al.[11], Ozergin[13], Srivastava [16], Srivastava and Manocha [18] and Srivastava [19].

Throughout this paper, let $\mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ be the sets of complex numbers, integers and positive integer, respectively, and

$$\mathbb{N}_o = \mathbb{N} \cup \{o\} \text{ and } \mathbb{Z}_o^-$$

Recently, Ozergin et al. [14] introduced and studied some fundamental properties and characteristics of the generalized Beta type function $B_p^{(\alpha,\beta)}(x,y)$ defined by Ozergin [13] and Ozergin et al. [14]

$$B_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt, \tag{1}$$

$$\left(R(p)>0, R(x)>0, R(y)>0, R(\alpha)>0, R(\beta)>0\right).$$

where $B_0^{(\alpha,\beta)}(x,y) = B(x,y)$ is the familiar Beta function which is defined by Srivastava and Choi [17] and expressed in terms of the Gamma function as follows:

$$B(x,y) = \begin{cases} \int_0^1 t^{x-1} (1-t)^{y-1} dt, & (Re(x) > 0, Re(y) > 0) \\ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} & (x, y \in \mathbb{C}|\mathbb{Z}) \end{cases} \tag{2}$$

Using the generalized Beta function (1), Ozergin et al. [14] introduced and investigated a family of potentially useful generalized Gauss hypergeometric functions defined as follows Ozergin [13] and Ozergin et al. [14]

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{3}$$

$$\left(p\geq 0;|z|<1;R(c)>0,R(b)>0,\min\left(R(\alpha),R(\beta)>0\right)\right).$$

and the generalized confluent hypergeometric function is defined by

$$\phi_p^{(\alpha,\beta)}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{4}$$

$$\left(p\geq 0;|z|<1;R(c)>0,R(b)>0,\min\left(R(\alpha),R(\beta)>0\right)\right).$$

Indeed, in their special case when $p=0$, the function $F_p^{(\alpha,\beta)}(a,b;c;z)$ and $\Phi_p^{(\alpha,\beta)}(b;c;z)$ would reduce immediately to the extensively-investigated Gauss hypergeometric function ${}_2F_1(.)$ and the confluent hypergeometric function ${}_1F_1(.)$, both of which are special cases of the well known generalized hypergeometric series ${}_pF_q(.)$ defined by Srivastava and Choi [17]

$$\begin{aligned}{}_pF_q\left(\begin{matrix}a_1,a_2,...,a_p;\\ b_1,b_2,...,b_q;\end{matrix}z\right)&=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_p)_n}{(b_1)_n(b_2)_n...(b_q)_n}\frac{z^n}{n!}\\ &={}_pF_q\left(a_1,a_2,...,a_p;b_1,b_2,...,b_q;z\right).\end{aligned} \tag{5}$$

Here $(\lambda)_n$ is the Pochhammer symbol defined by Srivastava and Monocha [18], for $\lambda\epsilon\ \mathbb{C}$

$$(\lambda)_n=\begin{cases}1 & ,if \quad n=0,\ \lambda\neq 0\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & n=1,2,\cdots\ ;\lambda\neq 0\cdot\end{cases}$$

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \tag{6}$$

We require the following concept of the Hadamard product defined by pohlen [15]

Definition1.1 let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$ be two power series whose radius of convergence are given by R_f and R_g ,respectively.

Then, their Hadamard product is the power series defined by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n,$$

whose radius of convergence R satisfies $R_f.R_g\leq R$.

If, in particular, one of the power series defines an entire function and the radius of convergence of the other one is greater than 0, then the Hadamard product series defines an entire function, too.

Consider a function $g(p)=p^{-\lambda-n\xi}(\lambda,\xi\epsilon\mathbb{C};n\epsilon\mathbb{N})$Then, it is easy to find that the $k-th$ derivative of $\phi(p)$ can be expressed in terms of the Gamma function as follows:

$$g^{(k)}(p)=(-1)^k\ p^{-\lambda-n\xi-k}\ \frac{\Gamma(\lambda+n\xi+k)}{\Gamma(\lambda+n\xi)},\quad (k\in\mathbb{N}_0\ ). \tag{7}$$

In this paper, we consider the following generalizations of Beta function

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;\mu,\kappa)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;\frac{-p}{t^{\mu}(1-t)^{\kappa}}\right)dt, \tag{8}$$

$$\mathrm{Re}(p)>0,\min\{\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta_i)\}>0,\ i=1,2,...,r,\kappa\ge 0,\ \mu\ge 0,\mathrm{Re}(x)>-\mathrm{Re}(\kappa\alpha_i),$$
$$\mathrm{Re}(y)>-\mathrm{Re}(\mu\beta_i).$$

Using (8), we introduce a definition of generalized hypergeometric function as follows:

$$F_p^{(\alpha_1,\alpha_2,...,\alpha_r,\beta_1,\beta_2,...\beta_r;\mu,\kappa)}(a,b;c;z;m)$$
$$=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{B_p^{(\alpha_1,\alpha_2,...,\alpha_r,\beta_1,\beta_2,...\beta_r;\mu,\kappa)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^n}{n!}, \tag{9}$$
$$\left(p\ge 0;\mathrm{Re}(\kappa)>0;\mathrm{Re}(\mu)>0;m<\mathrm{Re}(b)>\mathrm{Re}(c);|z|<1\right).$$

Also, we define the generalized confluent function of (9) by

$$\phi_p^{(\alpha_1,\alpha_2,...,\alpha_r,\beta_1,\beta_2,...\beta_r;\mu,\kappa)}(b;c;z;m)$$
$$=\sum_{n=0}^{\infty}\frac{(b)_n}{(c)_n}\frac{B_p^{(\alpha_1,\alpha_2,...,\alpha_r,\beta_1,\beta_2,...\beta_r;\mu,\kappa)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^n}{n!}, \tag{10}$$
$$\left(p\ge 0;\mathrm{Re}(\kappa)>0;\mathrm{Re}(\mu)>0;m<\mathrm{Re}(b)>\mathrm{Re}(c);|z|<1\right).$$

If $\mu=\kappa$, then (8) is reduced to the extended Beta function defined by Mohsen and Kulib [12]

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;\mu)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;\frac{-p}{t^{\mu}(1-t)^{\mu}}\right)dt, \tag{11}$$

$$\mathrm{Re}(p)>0;\min\{\mathrm{Re}(x),\mathrm{Re(y)},\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta_i)\}>0;\ i=1,2,...,r.$$

Also in (11), if $r=1$ and $\mu=1$, will be reduced to Beta type function given by equation (1).

2. Generating Relations

Here, we derive two generating relations involving the Gauss hypergeometric function (3) and generalized confluent hypergeometric function (4) in the form of the following theorems:

Theorem 2.1 Let $\min(\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta_i))>0,\mathrm{Re}(c)>\mathrm{Re}(b)>0,|z|<1$ and $\mathrm{Re}(p)>0,$ $(\mathrm{Re}(\mu),\mathrm{Re}(\kappa))>0.$ Then the following generating relation holds true:

$$(1+t)^{-\lambda}F_p^{(\alpha_1,...,\alpha_r;\beta_1,..,\beta_r;\mu,\kappa)}\left(a,b,c;\frac{z}{1+t};m\right)$$
$$=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{F_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;\mu,\kappa)}(a,b;c;z;m)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!},$$
$$\left(z,\lambda\in\mathbb{C};|t|<1\right).$$

(13)

Proof. For simplicity, replace $1+t\to s$ and $f(s)$ be the left hand side of (13). Then, find form (13)

$$f(s)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{B_p^{(\alpha_1,\alpha_2,\ldots,\alpha_r,\beta_1,\beta_2,\ldots\beta_r;\mu,\kappa)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{z^n}{n!}s^{-\lambda-n}, \tag{14}$$

Differentiating both sides of (14) j -times with respect to s and using (6) and (7) ,we have

$$f^{(j)}(s)=(-1)^j s^{-\lambda-j}(\lambda)_j$$

$$\times\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{B_p^{(\alpha_1,\alpha_2,\ldots,\alpha_r,\beta_1,\beta_2,\ldots\beta_r;\mu,\kappa)}(b+n,c-b+m)}{B(b+n,c-b+m)}\frac{(\lambda+j)_n}{(\lambda)_n}\frac{\left(z/s\right)^n}{n!}, \tag{15}$$

Appling the Definition 1.1 to the summation in (15), we obtain

$$f^{(j)}(s)=(-1)^j s^{-\lambda-j}(\lambda)_j\left\{F_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;\mu,\kappa)}\left(a,b;c;z/s;m\right)*{}_2F_1\left(\lambda+j,1;\lambda;z/s\right)\right\}, \tag{16}$$

Expanding $f(s+t)$ as the Taylor series, we have

$$f(s+t)=(s+t)^{-\lambda}(\lambda)_j F_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;\mu,\kappa)}\left(a,b;c;z/(s+t);m\right)$$

$$=\sum_{j=0}^{\infty}\frac{(-t)^j s^{-\lambda-j}}{j!}(\lambda)_j\left\{F_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;\mu,\kappa)}\left(a,b;c;z/s;m\right)*{}_2F_1\left(\lambda+j,1;\lambda;z/s\right)\right\}, \tag{17}$$

Finally, setting $s=1$ in (15) leads to the desired result (13).

Theorem 2.2 Let $\min(\mathrm{Re}(\alpha_i),\mathrm{Re}(\beta_i))>0,\ m<\mathrm{Re}(b)<\mathrm{Re}(c);|z|<1$ and $\mathrm{Re}(p)>0$, $\mathrm{Re}(p)>0,(\mathrm{Re}(\mu),\mathrm{Re}(\kappa))>0.$ Then, the following generating relation holds true:

$$(1+t)^{-\lambda}\phi_p^{(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r;\mu,\kappa)}\left(b,c;\frac{z}{1+t};m\right)$$

$$=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{\phi_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;\mu,\kappa)}(b;c;z;m)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \tag{18}$$

$$(z,\lambda\in\mathbb{C};|t|<1).$$

Proof. The proof of the generating relation (18) would run parallel to that of (13) in Theorem 2.1.

Remark 2.1. Result (18) can be obtained by replacing z by z/a and then taking the limit for both sides as $|a|$ tends to infinity.

3. Special Cases

I. Setting $m=0, r=1$ and $\mu=\kappa$ in Theorem 2.1, we have

$$(1+t)^{-\lambda}F_p^{(\alpha,\beta;\kappa)}\left(a,b,c;\frac{z}{1+t}\right)$$

$$=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{F_p^{(\alpha,\beta;\kappa)}(a,b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \tag{19}$$

$$(z,\lambda\in\mathbb{C};|t|<1),\min(\mathrm{Re}(\alpha),\mathrm{Re}(\beta))>0,\ m<\mathrm{Re}(b)<\mathrm{Re}(c);|z|<1.$$

Putting $\kappa=1$, in (19), we get

$$(1+t)^{-\lambda} F_p^{(\alpha,\beta)}\left(a,b,c;\frac{z}{1+t}\right)$$
$$=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{F_p^{(\alpha,\beta)}(a,b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (20)$$

$$(z,\lambda\in\mathbb{C};|t|<1), \min(\operatorname{Re}(\alpha),\operatorname{Re}(\beta))>0,\ m<\operatorname{Re}(b)<\operatorname{Re}(c);|z|<1.$$

Which is the revised result given by Chand et. al [5].

Now, for $\alpha=\beta$ in (20), we get

$$(1+t)^{-\lambda} F_p\left(a,b,c;\frac{z}{1+t}\right)=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{F_p(a,b;c;z)*{}_2F_1(\lambda+j;1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (21)$$

$$\min(\operatorname{Re}(\alpha),\operatorname{Re}(\beta))>0, |z|<1, (z,\lambda\in \mathbb{C}\ |t|<1) \text{ and } \operatorname{Re}(p)>0.$$

Which is the revised result given by Chand et. al [5].

II. In Theorem (2.2), put $m=0$, we get the following result:

$$(1+t)^{-\lambda}\phi_p^{(\alpha_1,\dots,\alpha_r;\beta_1,\dots,\beta_r;\mu;\kappa)}\left(b,c;\frac{z}{1+t}\right)$$
$$=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{\phi_p^{(\alpha_1,\alpha_2,\dots\alpha_r;\beta_1,\beta_2,\dots,\beta_r;\mu;\kappa)}(b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (22)$$

$$\min(\operatorname{Re}(\alpha),\operatorname{Re}(\beta))>0, |z|<1, \text{ and } \operatorname{Re}(p)>0, \operatorname{Re}(\mu)>0,\ \mu=\kappa,\ (z,\lambda\in\mathbb{C};|t|<1).$$

For $r=1$ and $\mu=\kappa$, in (22), we get the following generating relation associated with the generalized confluent hypergeometric function $\phi_p^{(\alpha,\beta;\kappa)}(b,c;z)$:

$$(1+t)^{-\lambda}\phi_p^{(\alpha,\beta;\kappa)}\left(b,c;\frac{z}{1+t}\right)=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{\phi_p^{(\alpha,\beta;\kappa)}(b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (23)$$

Putting $\kappa=1$, in (22), we get

$$(1+t)^{-\lambda}\phi_p^{(\alpha,\beta)}\left(b,c;\frac{z}{1+t}\right)=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{\phi_p^{(\alpha,\beta)}(b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (24)$$

Which is the revised result given by Chand et. al [5].

Now, for $\alpha=\beta$ in (24), we get

$$(1+t)^{-\lambda}\phi_p\left(b,c;\frac{z}{1+t}\right)=\sum_{j=0}^{\infty}(-1)^j(\lambda)_j\left\{\phi_p(b;c;z)*{}_2F_1(\lambda+j,1;\lambda;z)\right\}\frac{t^j}{j!}, \qquad (25)$$

$$(z,\lambda\in\mathbb{C};|t|<1).$$

Which is the revised result given by Chand et. al [5].

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الحصول على علاقات مولدة مرتبطة بدالة جاوس المُعّممة

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الملخص

في هذه البحث تم إيجاد علاقات مولدة جديدة لتعميم دالة جاوس الفوق هندسية ودالة الفوق هندسية المندمجة من خلال تطبيق نظرية تايلور بشكل رئيسي. ونتيجة لطابعها العام فإنًه يمكن استنباط الكثير من العلاقات المولدة الجديدة والمعروفة والمهمة لدالة جاوس الفوق هندسية والدوال الأخرى ذات الصلة.

الكلمات المفتاحية: دالة جاما, دالة بيتا, دالة بيتا المُوسّعة, الدوال المُولّدة, الدالة الفوق هندسية المُعمّمة المندمجة, دالة جاوس الفوق هندسية المُعمّمة, ضرب هادما رد.