Analysis for Cartan's fourth curvature Tensor in Finsler space

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Abstract

In this paper we discussed decomposition for the curvature tensor K^i_{jkh} of three cases in generalized K^h–recurrent Finsler space, K^h–birecurrent Finsler space and K^h– trirecurrent Finsler space, some results have been obtained in such space, different identities concerning the above spaces.

Keyword: K^h– R F_n, K^h– B R F_n and K^h– T R F_n, decomposable of Cartan's fourth curvature tensorK^i_{jkh}, symmetric and skew – symmetric property

Introduction

Takano K . [10] , B . B . Sinha and S . P . Singh [9] , B . B . Sinha and G . Singh [8] and others studied the decomposition of curvature tensors in a recurrent manifold . R . Hit [6] introduced a recurrent Finsler space whose the curvature tensorH^i_{jkh} is decomposable in the form $H^i_{jkh} = X^i Y_{jkh}$ and obtained several results. H. D. Pande and H. S. Shukla [5] discussed the decomposition of the curvature tensors K^i_{jkh} and H^i_{jkh} in recurrent Finsler space and studied the properties of such decompositions . H . D . Pande and T . A . Khan [4] considered a recurrent Finsler space whose the curvature tensor$H^i_{jkh}$is decomposition in the form $H^i_{jkh} = X^i_j Y_{kh}$. M . A . A . Ali [1] discussed the symmetric and skew – symmetric property of the recurrence covariant tensor field of second order in K^h – birecurrent space. A. M. A. Al – Qashbari[2] discussed the decomposition of the curvature tensors R^i_{jkh} and K^i_{jkh} in Finsler space equipped with non – symmetric connection.

An n – dimensional space X_n equipped with a function F(x,y) satisfies the requisite conditions[7].
Let us consider a set of quantities g_{ij} defined by

(1.1) $g_{ij}(x\,,y) := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x\,,y)$.

The tensor g_{ij} is positively homogeneous of degree zero in y^i and symmetric in i and j.
The vectors y^i and y_j satisfy the relations

(1.2) $y_i\; y^i = F^2$.

The h – covariant derivative of the vectors y^ivanish identically, i. e.

(1.3) $y^i_{\;|k} = 0$.

The tensor K^i_{rkh} is called *Cartan's fourth curvature tensor* which is skew – symmetric in its last two lower indices k and h.
The curvature tensorK^i_{jkh}satisfies thefollowing identities known as *Bianchi identities*

(1.4) $K^r_{jkh} + K^r_{hjk} + K^r_{khj} = 0$.

Cartan's fourth curvature tensor K^i_{jkh} satisfies the generalized recurrence property, generalized birecurrence property and generalized trirecurrence property with respect to Cartan's coefficient parameter connectionΓ^{*i}_{kh}, respectively and denoted them briefly by G $K^h - R\,F_n$, by G $K^h - B\,R\,F_n$ and by G $K^h - T\,R\,F_n$, respectively as following ([2] ,[3]), i.e.

(1.5) $K^i_{jkh|l} = \lambda_l K^i_{jkh} + \mu_l\,(\,\delta^i_k g_{jh} - \delta^i_h g_{jk}\,)$, $K^i_{jkh} \neq 0$,

(1.6) $K^i_{jkh|l|m} = w_{lm} K^i_{jkh} + v_{lm}\,(\,\delta^i_k g_{jh} - \delta^i_h g_{jk}\,)$, $K^i_{jkh} \neq 0$

and

(1.7) $K^i_{jkh|l|m|n} = a_{lmn}K^i_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}),\ K^i_{jkh} \neq 0.$

Remark 1.1. We shall call the h – covariant derivative as *generalized h – covariant derivative* and briefly will denote by *G h – covariant derivative.*

Decomposition of Cartan's Fourth Curvature Tensor

Let us consider a Finsler space whose Cartan's fourth curvature tensor K^i_{jkh} is decomposable. Since the curvature tensor is a mixed tensor of the type (1,3) , i.e. of rank 4, it may be written as a product of a contravariant vector (or covariant vector) and a tensor of rank 3 , i.e. covariant tensor of the type (0,3) { or mixed tensor of the type (1,2)} as following [5] :

(2.1) a) $K^i_{jkh} = X^i Y_{jkh}$, b) $K^i_{jkh} = X_j Y^i_{kh}$,

c) $K^i_{jkh} = X_k Y^i_{jh}$ and d) $K^i_{jkh} = X_h Y^i_{jk}$

or

as product of two tensor each of rank 2, i.e. mixed tensors of the type (1,1) and covariant tensor of the type (0,2) as following :

(2.2) a) $K^i_{jkh} = X^i_j Y_{kh}$, b) $K^i_{jkh} = X^i_k Y_{jh}$ and c) $K^i_{jkh} = X^i_h Y_{jk}$. Cartan's fourth curvature tensor K^i_{jkh} is decomposable as (2.1a), where Y_{jkh} is non – zero and homogeneous tensor field of degree –1 in its directional argument is called *decomposition tensor field* and X^i is independent of y^j ([1] , [5]).

In this paper, we shall discuss the possible forms in three cases, two decomposition for the first case (the other are similar) and one decomposition for the second case (the other are similar).

Let us assume that X^i is covariant constant.

Taking the G h – covariant derivative for (2.1a), with respect to x^l, we get

(2.3) $K^i_{jkh|l} = X^i Y_{jkh|l}$.

Using (1.5) and (2.1a) in (2.3), we get

(2.4) $X^i Y_{jkh|l} = \lambda_l X^i Y_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Thus, we may conclude

Theorem 2.1. *In G K^h– R F_n, under the decomposition (2.1a)and ifX^i is covariant constant, then the decomposition tensor Y_{jkh} satisfies the identity (2.4).*

In view of (1.4) and using the decomposition (2.1a), we get

(2.5) $X^i Y_{jkh} + X^i Y_{hjk} + X^i Y_{khj} = 0.$

Taking the G h – covariant derivative for (2.5) with respect to x^l, we get

(2.6) $X^i Y_{jkh|l} + X^i Y_{hjk|l} + X^i Y_{khj|l} = 0.$

Using (2.4) and the symmetric property of the metric tensor in (2.6), we get

(2.7) $X^i Y_{jkh} + X^i Y_{hjk} + X^i Y_{khj} = 0$,where $\lambda_l \neq 0$.

Thus, we may conclude

Theorem 2.2. *In G K^h– R F_n, under the decomposition (2.1a) and if X^i* is covariant constant, *then we have the identity (2.7).*

Taking the G h– covariant derivative for (2.3) with respect to x^m, we get

(2.8) $K^i_{jkh|l|m} = X^i Y_{jkh|l|m}$.

Using (1.6) in (2.8), we get

(2.9) $X^i Y_{jkh|l|m} = w_{lm} X^i Y_{jkh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Thus, we conclude

Theorem 2.3. *In G K^h– R F_n, under the decomposition (2.1a) and if X^i*is covariant constant, *then the decomposition tensor Y_{jkh} satisfies the identity (2.9).*

Taking the G h– covariant derivative for (2.6) with respect to x^m, we get

(2.10) $X^i Y_{jkh|l|m} + X^i Y_{hjk|l|m} + X^i Y_{khj|l|m} = 0.$

Using (2.9) and the symmetric property of the metric tensor in (2.10), we get

(2.11) $Y_{jkh} + Y_{hjk} + Y_{khj} = 0,$ where $w_{lm} X^i \neq 0.$

Thus, we may conclude

Theorem 2.4. *In G K^h– B R F_n, under the decomposition (2.1a) and if X^i* is covariant constant,*then we have the identity (2.11).*

Taking the G h – covariant derivative for (2.8) with respect to x^n, we get

(2.12) $K^i_{jkh|l|m|n} = X^i Y_{jkh|l|m|n}$.

Using (1.8)and (2.1a) in (2.12), we get

(2.13) $X^i Y_{jkh|l|m|n} = a_{lmn} X^i Y_{jkh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Thus, we may conclude

Theorem 2.5. *In G K^h– T R F_n, under the decomposition (2.1a) and ifX^i* is covariant constant , *then the decomposition tensor Y_{jkh} satisfies the identity (2.13).*

Taking the G h – covariant derivative for (2.10), with respect to x^n, we get

(2.14) $X^i Y_{jkh|l|m|n} + X^i Y_{hjk|l|m|n} + X^i Y_{khj|l|m|n} = 0$.

Using (2.13) in (2.14), we get

(2.15) $Y_{jkh} + Y_{hjk} + Y_{khj} = 0$, since $a_{lmn} X^i \neq 0$.

Thus, we may conclude

Theorem 2.6. *In G K^h– T R F_n, under the decomposition (2.1a) and if X^i* is covariant constant, *then we have the identity (2.15).*

Let us consider a Finsler space for which Cartan's fourth curvature tensor K^i_{jkh} is decomposable as (2.1b), where X_j is non – zero covariant vector field of order one and Y^i_{kh} is skew– symmetric decomposition tensor.

Taking G h – covariant derivative for (2.1b) with respect to x^l, we get

(2.16) $K^i_{jkh|l} = v_l X_j Y^i_{kh} + X_j Y^i_{kh|l}$,where $X_{j|l} = v_l X_j$.

Using (1.5) in (2.16) and using (2.1b), we get

(2.17) $X_j Y^i_{kh|l} = \alpha_l X_j Y^i_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, where $\alpha_l = \lambda_l - v_l$.

Thus, we may conclude

Theorem 2.7. *In G K^h– R F_n*, under the decomposition *(2.1b)* and if the covariant vector field λ_l is not equal to the covariant vector field v_l, then the decomposition $X_j$andY^i_{kh}satisfy (2.17).

Let us assume that the vector field λ_l is equal to the vector field v_l, i.e. $\lambda_l = v_l$, the equation (2.17) immediately reduces to

(2.18) $X_j Y^i_{kh|l} = \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

In view of (2.18) in (2.16), we get

(2.19) $K^i_{jkh|l} = X_{j|l} Y^i_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Adding the expression obtained by cyclic change of (2.19) with respect to indices k , h and l, we get

$K^i_{jkh|l} + K^i_{jlk|h} + K^i_{jhl|k} = v_l X_j Y^i_{kh} + v_h X_j Y^i_{lk} + v_k X_j Y^i_{hl} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_h(\delta^i_l g_{jk} - \delta^i_h g_{jk}) + \mu_k(\delta^i_h g_{jl} - \delta^i_l g_{jh})$.

In view of (1.5) and (2.1b), the above equation can be written as

(2.20) $(\lambda_l - v_l) X_j Y^i_{kh} + (\lambda_h - v_h) X_j Y^i_{lk} + (\lambda_k - v_k) X_j Y^i_{hl} = 0$.

Using (2.17) in (2.20), we get

(2.21) $X_j (Y^i_{kh|l} + Y^i_{lk|h} + Y^i_{hl|k}) - \{ \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_h(\delta^i_l g_{jk} - \delta^i_h g_{jk}) + \mu_k(\delta^i_h g_{jl} - \delta^i_l g_{jh})\} = 0$.

Thus, we may conclude

Theorem 2.8. *In G K^h– R F_n, under the decomposition (2.1b) and if the vector field λ_l is equal to the vector field v_l , then the decomposition tensor satisfies the (2.21).*

Using (1.4) and (2.1b) in (2.16), we get

(2.22) $\lambda_l X_j Y^i_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = X_{j|l} Y^i_{kh} + X_j Y^i_{kh|l}$.

Transvecting (2.22) byy^j and using (1.3), we get

(2.23) $\lambda_l X Y^i_{kh} + \mu_l(\delta^i_k y_h - \delta^i_h y_k) = X_{|l} Y^i_{kh} + X Y^i_{kh|l}$, Where X = $X_j Y^j$. If X is constant ($X_{|l} = 0$), then (2.23) can be written as

(2.24) $X\, Y^i_{kh|l} = \lambda_l X\, Y^i_{kh} + \mu_l(\delta^i_k y_h - \delta^i_h y_k)$.

Transvecting (2.24) by y^k, using (1.3) and (1.4), we get

(2.25) $X\, Y^i_{h|l} = \lambda_l X\, Y^i_h + \mu_l(y^i y_h - \delta^i_h F^2)$, where $Y^i_{kh} y^k = Y^i_h$.

Contracting the indices i and h in (2.25), we get

(2.26) $X\, Y_{|l} = \lambda_l X\, Y + \mu_l(1-\text{n})\, F^2$, where $Y^i_i = Y$.

Thus, we may conclude

Theorem 2.9. *In G K^h– R F_n, under the decomposition (2.1b) and if X is constant, then we have the identities (2.24), (2.25) and (2.26).*

Taking the G h – covariant derivative for (2.19) with respect to x^m, we get

(2.27) $K^i_{jkh|l|m} = X_{j|l|m} Y^i_{kh} + X_{j|l} Y^i_{kh|m} + \mu_{l|m}\,(\delta^i_k g_{jh} - \delta^i_h g_{jk})$

or

(2.28) $K^i_{jkh|l|m} = X_{j|l|m} Y^i_{kh} + s_{lm} X_j Y^i_{kh} + t_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $X_{j|l} = m_l X_j$, $Y^i_{kh|m} = n_m Y^i_{kh}$, $m_l n_m = s_{lm}$ and $\mu_{l|m} = t_{lm}$.

Using (1.6) and (2.1b) in (2.28), we get

(2.29) $w_{lm} X_j\, Y^i_{kh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = X_{j|l|m} Y^i_{kh} + s_{lm} X_j Y^i_{kh} + t_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$

or

(2.30) $X_{j|l|m} Y^i_{kh} = (w_{lm} - s_{lm})\, X_j\, Y^i_{kh} + r_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $r_{lm} = t_{lm} - v_{lm}$ (r_{lm} is skew – symmetric tensor).

Let us assume that, if the covariant tensor field w_{lm} is not equal to covariant tensor field s_{lm} , then (2.30) may written as

(2.31) $X_{j|l|m} Y^i_{kh} = u_{lm} X_j\, Y^i_{kh} + r_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, where $u_{lm} = w_{lm} - s_{lm}$.

Using (2.31) in (2.29), we get

(2.32) $w_{lm} X_j = (u_{lm} + s_{lm})\, X_j$, $Y^i_{kh} \neq 0$.

Thus, we may conclude

Theorem 2.10.*In G K^h– B R F_n,under the decomposition (2.1b) and if the covariant vector w_{lm} is not equal to the covariant tensor field s_{lm} andthe covariant tensor fieldr_{lm} is skew – symmetric,* then the decompositions $Y^i_{kh}$and X_j satisfy (2,31) and (2.32), respectively.

Let us assume that the covariant tensor field w_{lm} is equal to the covariant tensor field s_{lm}, and if the covariant tensor field r_{lm} is symmetric, then (2.30) can be written as

(2.33) $X_{j|l|m} Y^i_{kh} = r_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using (2.33) in (2.29), we get

$w_{lm} X_j\, Y^i_{kh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = s_{lm} X_j Y^i_{kh} + (t_{lm} - v_{lm})\,(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + t_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, where $r_{lm} = t_{lm} - v_{lm}$.

or

(2.34) $w_{lm} X_j\, Y^i_{kh} = s_{lm}\, X_j Y^i_{kh} + \Psi_{lm}\,(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, where $\Psi_{lm} = 2t_{lm} - 2v_{lm}$.

Thus, we may conclude

Theorem 2.11. *In G K^h– B R F_n, under the decomposition (2.1b), if the covariant tensor fieldw_{lm} is equal to the covariant tensor field s_{lm} and the covariant tensor fieldr_{lm} is symmetric, then the decomposition tensor satisfies the identity (2.34).*

Taking the G h – covariant derivative for (2.27), with respect to x^n, we get

(2.35)$K^i_{jkh|l|m|n} = X_{j|l|m|n} Y^i_{kh} + (p_{lmn} + \varphi_{lmn} + \theta_{lmn}) X_j Y^i_{kh} + h_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $X_{j|l|m} = \rho_{lm} X_j$, $Y^i_{kh|n} = \tau_n Y^i_{kh}$, $X_{j|l|n} = d_{ln} X_j$, $Y^i_{kh|m} = e_m Y^i_{kh}$, $X_{j|l} = f_l X_j$,$Y^i_{kh|m|n} = m_{mn} Y^i_{kh}$ and$\mu_{l|m|n} = h_{lmn}$.

Using (1.7)and (2.1b) in (2.35), we get

(2.36)$a_{lmn} X_j Y^i_{kh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = X_{j|l|m|n} Y^i_{kh} + q_{lmn} X_j Y^i_{kh} + h_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $p_{lmn} + \varphi_{lmn} + \theta_{lmn} = q_{lmn}$.

or
(2.37)$X_{j|l|m|n}Y^i_{kh} = (a_{lmn} - q_{lmn})X_jY^i_{kh} + \emptyset_{lnm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,
where $\emptyset_{lmn} = h_{lmn} - b_{lmn}$ ($\emptyset_{lmn}$is skew – symmetric in its last two indices) .
Let us assume that $a_{lmn} \neq q_{lmn}$, then (2.37) can be written as
(2.38)$X_{j|l|m|n}Y^i_{kh} = \beta_{lmn}X_jY^i_{kh} + \emptyset_{lnm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, where $\beta_{lmn} = a_{lmn} - q_{lmn}$.
Using (2.38) in (2.36), we get

$$a_{lmn}X_jY^i_{kh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = \beta_{lmn}X_jY^i_{kh} + \emptyset_{lnm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + q_{lmn}X_jY^i_{kh} + h_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

which can be written as
(2.43)$a_{lmn}X_j = (\beta_{lmn} + q_{lmn})X_j \quad , \quad Y^i_{kh} \neq 0.$
Thus, we may conclude

Theorem 2.12. *In GK^h–T R F_n, under the decomposition (2.1b), if the covariant tensor field $a_{lmn}$is not equal the covariant tensor field q_{lmn} and the covariant tensor field $\emptyset_{lmn}$is skew – symmetric in its last two indices, then the decomposition Y^i_{kh} and X_j*satisfy (2.38) and (2,39), respectively.

Let us assume that the covariant tensor field a_{lmn} is equal to the covariant tensor field q_{lmn} and if the covariant tensor field $\emptyset_{lmn}$ is symmetric in all lower indices m and n, then (2.37) can be written as
(2.40)$X_{j|l|m|n}Y^i_{kh} = \emptyset_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.
Using (1.7), (2.1b) and (2.40) in (2.36), we get
(2.41)$a_{lmn}X_jY^i_{kh} = q_{lmn}X_jY^i_{kh} + z_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,
Where$q_{lmn} = p_{lmn} + \varphi_{lmn} + \theta_{lmn}andz_{lmn}= 2h_{lmn} - 2b_{lmn}$.
Thus, we may conclude

Theorem 2.13. *In G K^h– T R F_n, under the decomposition (2.1b) and if the covariant tensor field a_{lmn} is equal to the covariant tensor field q_{lmn} and $\emptyset_{lmn}$* is symmetric, *then the decomposition tensor satisfies the identity (2.41).*

Let us consider a Finsler space for which Cartan's fourth curvature tensor K^i_{jkh} satisfies the conditions (1.5), (1.6) and (1.7) and consider Cartan's fourth curvature tensor K^i_{jkh} in the form (2.2a).
Taking the G h– covariant derivative for (2.2a) with respect to x^l, we get
(2.42)$K^i_{jkh|l} = X^i_{j|l}Y_{kh} + X^i_jY_{kh|l}$.
Using (1.5) and (2.2a) in (2.42), we get
(2.43)$X^i_jY_{kh|l} = (\lambda_l - v_l)X^i_jY_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $X^i_{j|l} = v_lX^i_j$.
Let us assume that $\lambda_l \neq v_l$, (2.43) can be written as
(2.44)$X^i_jY_{kh|l} = c_lX^i_jY_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $c_l = \lambda_l - v_l$.
In view of (2.42) and (2.44), we get

$$(2.45)\lambda_lK^i_{jkh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = (c_l + v_l)X^i_jY_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using (2.2a) in (2.45), we get
(2.46)$\lambda_lY_{kh} = d_lY_{kh} \quad , \quad X^i_j \neq 0$, where $d_l = c_l + v_l$.
Thus, we may conclude

Theorem 2.14. *In G K^h– R F_n,under the decomposition (2.2a) and if the covariant vector field$\lambda_l$is not equal the covariant vector fieldv_l, then the decompositions $X^i_j$and Y_{kh} satisfy (2.44) and (2.46), respectively.*

Let us assume that the non – zero covariant vector field $\lambda_l$is equal to the vector field v_l, i.e.
(2.47) $\lambda_l = v_l$.
In view of (2.47), (2.43) immediately reduces to

$$(2.48)X^i_jY_{kh|l} = \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using (2.48) in (2.42), we get

$$(2.49) K^i_{jkh|l} = X^i_{j|l} Y_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

or

$(2.50) K^i_{jkh|l} = v_l X^i_j Y_{kh} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Adding the expression obtained by cyclic change of (2.50) with respect to the indices k, h and l, we get

$$(2.51) K^i_{jkh|l} + K^i_{jlk|h} + K^i_{jhl|k} = v_l X^i_j Y_{kh} + v_h X^i_j Y_{lk} + v_k X^i_j Y_{hl} + \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_h(\delta^i_l g_{jk} - \delta^i_k g_{jl}) + \mu_k(\delta^i_h g_{jl} - \delta^i_l g_{jh}).$$

In view of (1.5) and (2.2a), the equation (2.51) can be written as

$$(2.52) \quad (\lambda_l - v_l) X^i_j Y_{kh} + (\lambda_h - v_h) X^i_j Y_{lk} + (\lambda_k - v_k) X^i_j Y_{hl} = 0.$$

In view of (2.43), the above equation can be written as

$$(2.53) \quad X^i_j (Y_{kh|l} + Y_{lk|h} + Y_{hl|k}) - \{ \mu_l(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_h(\delta^i_l g_{jk} - \delta^i_k g_{jl}) + \mu_k(\delta^i_h g_{jl} - \delta^i_l g_{jh})\} = 0.$$

Thus, we mayconclude

Theorem 2.15. *In $G\ K^h - R\ F_n$, under the decomposition (2.2a) and if the vector field λ_l is equal to v_l, then the decomposition tensor satisfies the identity (2.53).*

Taking the G h – covariant derivative for (2.49), with respect to x^m, we get

$(2.54) K^i_{jkh|l|m} = X^i_{j|l|m} Y_{kh} + f_{lm} X^i_j Y_{kh} + e_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $X^i_{j|l} = v_l X^i_j$, $Y_{kh|m} = d_m Y_{kh}$, $\mu_{l|m} = e_{lm}$ and $v_l d_m = f_{lm}$.

Using (1.6) and (2.2a) in (2.54), we get

$$(2.55) w_{lm} X^i_j Y_{kh} + v_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = X^i_{j|l|m} Y_{kh} + f_{lm} X^i_j Y_{kh} + e_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$

or

$(2.56) X^i_{j|l|m} Y_{kh} = (w_{lm} - f_{lm}) X^i_j Y_{kh} + r_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $r_{lm} = e_{lm} - v_{lm}$ (r_{lm} is skew – symmetric) .

If$w_{lm} \neq f_{lm}$, then the equation (2.62) can be written as

$(2.57) X^i_{j|l|m} Y_{kh} = o_{lm} X^i_j Y_{kh} + r_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $o_{lm} = w_{lm} - f_{lm}$.

Using (2.57) in (2.55), we get

$(2.58) w_{lm} X^i_j = (o_{lm} + f_{lm}) X^i_j , \quad Y_{kh} \neq 0$.

In view of (2.56) and (2.58), we may conclude

Theorem 2.16. *In $G\ K^h - B\ R\ F_n$, under the decomposition (2.2a), if the covariant tensor field w_{lm} is not equal to the covariant tensor field f_{lm} and the covariant tensor field r_{lm} is skew–symmetric, then the decompositions Y_{kh}*and X^i_j satisfy (2.57) and (2.58), respectively.

Let us assume that the covariant tensor field w_{lm} is equal to the covariant tensor field f_{lm}, i.e.

$(2.59) \quad w_{lm} = f_{lm}$.

Using (2.59) in (2.56) and suppose that the covariant tensor field r_{ml} is symmetric , we get

$$(2.60) X^i_{j|l|m} Y_{kh} = r_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Using (2.60) in (2.56), we get

$(2.61) w_{lm} X^i_j Y_{kh} = f_{lm} X^i_j Y_{kh} + z_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $z_{lm} = 2e_{lm} - 2v_{lm}$.

Thus, we may conclude

Theorem 2.17. *In $G\ K^h - B\ R\ F_n$, under the decomposition (2.2a), if the covariant tensor field $w_{lm}$is equal to the covariant tensor field f_{lm} and the covariant tensor field r_{lm} is symmetric, then the decomposition tensor satisfies the identity (2.61).*

Taking the G h– covariant derivative for (2.54), with respect to x^n, we get

$(2.62) K^i_{jkh|l|m|n} = X^i_{j|l|m|n} Y_{kh} + (p_{lmn} + t_{lmn} + u_{lmn}) X^i_j Y_{kh} + \sigma_{lm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $e_{lm|n} = \sigma_{lmn}$, $X^i_{j|l|m} = \tau_{lm} X^i_j$, $Y_{kh|n} = d_n Y_{kh}$ and $X^i_{j|l|n} = \varepsilon_{ln} X^i_j$

Using(1.7) and (2.2a) in (2.62), we get

$$(2.63) a_{lmn} X^i_j Y_{kh} + b_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) = X^i_{j|l|m|n} Y_{kh} + q_{lmn} X^i_j Y_{kh} + \sigma_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}),$$ where $q_{lmn} = p_{lmn} + t_{lmn} + u_{lmn}$

or

(2.64)$X^i_{j|l|m|n} Y_{kh} = (a_{lmn} - q_{lmn}) X^i_j Y_{kh} + ø_{lnm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where $ø_{lmn} = \sigma_{lmn} - b_{lmn}$ ($ø_{lmn}$is skew – symmetric in its last two indices).

Let us assume that $a_{lmn} \neq q_{lmn}$, then (2.64) can be written as

(2.65)$X^i_{j|l|m|n} Y_{kh} = \alpha_{lmn} X^i_j Y_{kh} + ø_{lnm}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,where $\alpha_{lmn}= a_{lmn} - q_{lmn}$.

Using (2.65) in (2.63), we get

(2.66)$a_{lmn} X^i_j = (\alpha_{lmn} + q_{lmn}) X^i_j$, $Y_{kh} \neq 0$.

Thus, we may conclude

Theorem 2.18. *In $G\ K^h$– $T\ R\ F_n$, under the decomposition (2.2a), if the covariant tensor field a_{lmn} is not equal to the covariant tensor field q_{lmn} and the covariant tensor field $ø_{lmn}$is skew – symmetric in its last two indices, then the decompositions Y_{kh}*and X^i_j satisfy (2.65) and (2.66), respectively.

Let us assume that, the covariant tensor fielda_{lmn} is equal to the covariant tensor field q_{lmn} and the covariant tensor field $ø_{lmn}$ is symmetric in its last two indices, i.e. ($ø_{lmn} = ø_{lnm}$), the equation (2.64) immediately reduces to

(2.67)$X^i_{j|l|m|n} Y_{kh} = ø_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

Using (1.7), (2.2a) and (2.63), we get

(2.68)$a_{lmn} X^i_j Y_{kh} = q_{lmn} X^i_j Y_{kh} + \rho_{lmn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$,

where$\rho_{lmn}= 2\sigma_{lmn} - 2b_{lmn}$.

Theorem 2.19.*In $G\ K^h$– $T\ R\ F_n$, under the decomposition (2.2a), if the covariant tensor field q_{lmn} and the covariant tensor field $ø_{lmn}$ is symmetric , then the decomposition tensor satisfies the identity (2.68).*

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تحليل الموتر التقوسي الرابع لكارتان في فضاء فنسلر

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الملخص

في هذه الورقة البحثية سنناقش قابلية تحليل الموتر التقوسي الرابع لكارتان K^i_{jkh} في ثلاث حالات $(K^i_{jkh} = X^i_j Y_{kh}$ و $K^i_{jkh} = X_j Y^i_{kh}$ ،$K^i_{jkh} = X^i Y_{jkh})$ في كل من الفضاء المعمم – K^h (احادي المعاودة، ثنائي المعاودة و ثلاثي المعاودة) حيث بقية الحالات ستكون مماثله لإحدى الحالات السابقة. استخدمنا خاصية التماثل والتماثل المتخالف كما افترضنا بان المتجه X^i(covariant constant) . تم الحصول على عدد من الصيغ والمبرهنات في كل فضاء.

الكلمات المفتاحية: الفضاء المعمم احادي، ثنائي وثلاثي المعاودة، تحليل الموتر التقوسي الرابع لكارتان، خاصية التماثل والتماثل المتخالف.

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