On certain Konhauser and Laguerre matrix polynomials

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DOI: https://doi.org/10.47372/uajnas.2019.n1.a13

Abstract

In this paper, we generalize the well-known Laguerre and Konhauser matrix polynomials, given by Shehata (12), varma and Tas (14), respectively, and we also introduced a set of new Laguerre matrix polynomials $L_n^{(B,C)}(\alpha,x)$, we also obtained some generating relations and integral representations for these sets of polynomials.

Keywords: Laguerre and Konhauser matrix polynomials, generating relations and integral representations.

Introduction.

Recently, mathematicians have interested in some properties of orthogonal matrix polynomials, specially in Laguerre matrix polynomials and Konhauser matrix polynomials. For example, Jo'der and etal here introduced and studied Laguerre matrix polynomials [1,4,6,10,11], the Konhauser matrix polynomials [3,9,10,13,14,15].

Through this paper, we consider the complex space $C^{N\times N}$ of complex matrices of common order N.

A matrix A is said to be stable matrix in $C^{N\times N}$ if $\operatorname{Re}(\lambda)>0$ for all $\lambda\in\sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A.

The Gamma matrix function $\Gamma(A)$ have been defined by (see[5,6])

$$\Gamma(A)=\int_0^\infty e^{-t}t^{A-I}dt, \tag{1}$$

where $t^{A-I}=\exp[(A-I)\ln t]$.

$\Gamma(A)$ is invertible and this inverse coincides with $\Gamma^{-1}(A)$ and also, in [5], the following formula for Pochhammer matrix function is

$$(A)_n=\Gamma(A+nI)\Gamma^{-1}(A), \qquad n\geq 1 \tag{2}$$
$$(A)_0=I.$$

Beta matrix function can be defined by Jodar and Corte's[5]

$$B(A,C)=\int_0^1 t^{A-I}(1-t)^{C-I}dt \tag{3}$$

and

$$B(A,C)=\Gamma(A)\Gamma(C)\Gamma^{-1}(A+C). \tag{4}$$

In [2], Defez and Jo'dar have shown that, for matrices $A(k,n)$ and $B(k,n)$ in $C^{N\times N}$ when $n\geq 0,\ k\geq 0$, then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n+k). \tag{5}$$

Let A be a matrix in $C^{N\times N}$ satisfying $(-\alpha)$ is not an eigenvalue for every integer $\alpha>0$, and λ be a complex number whose real part is positive, then the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ are defined by [4]

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}(A+I)_n\left[(A+I)_k\right]^{-1}(\lambda x)^k, \quad n\geq 0 \tag{6}$$

Such matrix polynomials have the following generating matrix function:

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n = (1-t)^{-A-I} e^{\frac{-\lambda xt}{1-t}}, \tag{7}$$

$x \in C, \quad |t|<1.$

Konhauser matrix polynomials was defined in [14] ,by

$$Z_n^{(A,\lambda)}(x,k) = \frac{\Gamma(A+(kn+1)I)}{n!}\sum_{r=0}^{n}(-1)^r \binom{n}{r}\Gamma^{-1}(A+(kr+1)I)(\lambda\, x)^{kr} \tag{8}$$

Also, in [7], the generalized Laguerre matrix polynomials is given by

$$L_n^{(A,B)}(x) = \Gamma(nA+B+I)\sum_{k=0}^{n}\frac{(-1)^k\Gamma^{-1}(kA+B+I)}{k!(n-k)!}(x)^k \tag{9}$$

Here, we give a new form of generalized Konhauser matrix polynomials defined by

$$Z_n^{(A,\lambda,\alpha)}(x,k) = \Gamma(kn+A+I)\sum_{m=0}^{n}(-1)^m\frac{\Gamma^{-1}((km+1)I+A)}{m!\Gamma(\alpha n-\alpha m+1)}(\lambda x)^{km} \tag{10}$$

Also, we give a new general Laguerre matrix polynomials defined by

$${}_P L_n^B(\alpha;x) = \sum_{m=0}^{n}\frac{(-1)^m (I+B)_{np}}{m!(I+B)_{mp}\Gamma(\alpha n-\alpha m+1)}(x)^{pm}. \tag{11}$$

It is clear that, if we choose $\alpha = p = 1$ in (11), we get

$${}_1 L_n^B(1;x) = L_n^B(x). \tag{12}$$

In (10), if we put $\alpha = 1$, then we get

$$Z_n^{(A,\lambda,1)}(x,k) = Z_n^{(A,\lambda)}(x,k). \tag{13}$$

Here, we also introduced the new set of matrix polynomials $L_n^{(B,C)}(\alpha;x)$ for nonnegative integer n, $\mathrm{Re}(\alpha)>0$, defined by

$$L_n^{(B,C)}(\alpha;x) = \Gamma(nB+C+I)\sum_{m=0}^{n}(-1)^m\frac{\Gamma^{-1}(mB+C+I)}{m!\Gamma(\alpha n-\alpha m+1)}(x)^m, \tag{14}$$

In (14), if we choose $\alpha = 1$, replacing B by A and C by B, we get

$$L_n^{(A,B)}(1;x) = L_n^{(A,B)}(x), \tag{15}$$

Also in (14) put $B = k$ and replacing x by λx, we get

$$L_n^{(k,C)}(\alpha;x) = Z_n^{(C,\lambda,\alpha)}(x,k), \tag{16}$$

Also, in (14) put $B = I$, we get

$$L_n^{(I,C)}(\alpha;x) = {}_1L_n^C(\alpha;x). \tag{17}$$

From equation (10), we obtain

$$Z_0^{(A,\lambda,\alpha)}(x,k) = I \tag{18}$$

And

$$Z_1^{(A,\lambda,\alpha)}(x,k) = \frac{\Gamma((k+1)I+A)\Gamma^{-1}(I+A)}{\Gamma(\alpha+1)}(\lambda x)^k \tag{19}$$

And from equation (14), we have

$$L_0^{(B,C)}(\alpha;x) = I, \tag{20}$$

$$L_1^{(B,C)}(\alpha;x) = \frac{\Gamma(B+C+I)\Gamma(C+I)}{\Gamma(\alpha+1)} - x. \tag{21}$$

Mittag-Leffler function is defined by [9]

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \tag{22}$$

And the Wright matrix function is defined by

$$W(A,B+I;-z)=\sum_{n=0}^{\infty}\frac{(-z)^n\Gamma^{-1}(nA+B+I)}{n!}\ . \tag{23}$$

Generating relations

Let B and C be matrices in $C^{N\times N}$ satisfying the above relation $(-\beta)$ is not an eigenvalue of B for every integer $\beta>0$, $(-\gamma)$ is not an eigenvalue of C for every integer $\gamma>0$, $\operatorname{Re}(\alpha)>0$.

Theorem 1. The following generating relation for the Laguerre matrix polynomials holds true

$$\sum_{n=0}^{\infty} L_n^{(B,C)}(\alpha;x)\Gamma^{-1}(nB+C+I)t^n=E_\alpha(t)w(B,C+I;-xt). \tag{24}$$

Proof. Using (14) and (5), the right hand side of (24) can be written as

$$\sum_{n=0}^{\infty}\ \Gamma(n+B+C+I)\sum_{m=0}^{n}\frac{(-1)^m x^m\Gamma^{-1}(mB+C+I)t^n}{m!\Gamma(\alpha n-\alpha m+1)}$$

$$=\sum_{n=0}^{\infty}\ \sum_{m=0}^{\infty}\frac{(-x)^m\Gamma^{-1}(mB+C+I)t^{n+m}}{m!\Gamma(\alpha n+1)}$$

$$=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(\alpha n+1)}\sum_{m=0}^{n}\frac{(-xt)^m\Gamma^{-1}(mB+C+I)}{m!}$$

Now, using (22) and (23), we get the left hand side of (24) which complete the proof.

Theorem 2. The following generating relation for the Konhauser matrix polynomials holds true

$$\sum_{n=0}^{\infty} Z_n^{(A,\lambda,\alpha)}(x,k)\Gamma^{-1}(knI+I+A)=E_\alpha(t)W(kI+A+I;\lambda^k x^k t). \tag{25}$$

Proof. Using (10) in the right hand side of (25) and following the same procedure as in the proof of theorem (1), then we get the proof.

Integral representation.

Theorem 3. If B,C and E be positive stable matrices and $\operatorname{Re}(\alpha)>0$, then we have

$$\frac{1}{x^C}\int_0^x(x-u)^{C-E-I}u^E L_n^{(B,E)}(\alpha,u^B)du=B(nB+E+I,C-E)L_n^{(B,C)}(\alpha,x^B). \tag{26}$$

Proof. Denoting the left hand side of (26) by O and using (14), we get

$$O=\frac{1}{x^C}\Gamma(nB+E+I)\sum_{m=0}^{n}\frac{(-1)^m\Gamma^{-1}(mB+E+I)}{m!\Gamma(\alpha n-\alpha m+1)}\int_0^x(x-u)^{C-E-I}u^{mB+E}du\,.$$

Now, using [15]

$$\int_0^a x^{s-1}(a-x)^{t-1}dx=a^{s+t-1}B(s,t),$$

we get

$$O=\Gamma(nB+E+I)\sum_{m=0}^{n}\frac{(-1)^m\Gamma^{-1}(mB+E+I)x^{mB}}{m!\Gamma(\alpha n-\alpha m+1)}B(C-E,E+mB+I),$$

now using (4) and (14) we get

$$O=B(nB+E+I,C-E)L_n^{(B,C)}(\alpha,x^B),$$

which is the right hand side of (26).

Theorem 4. For the same conditions for the matrices B and C as in theorem 3, we have

$$\Gamma(knI+I+B)\Gamma^{-1}(knI+I+C)\Gamma^{-1}(B-C)x^{-B} \times\int_0^x (x-u)^{B-C-I}u^C Z_n^{(C,\lambda,\alpha)}(u;k)du = Z_n^{(B,\lambda,\alpha)}(x;k) . \quad (27)$$

Proof. Using (10) in the right hand side of (27) and then proceeding the same steps as in the proof of theorem 3, we get the required result.

Remark. If we put $\alpha=1$, in theorem 4, then we get

$$\Gamma(knI+I+B)\Gamma^{-1}(knI+I+C)\Gamma^{-1}(B-C)x^{-B} \times\int_0^x (x-u)^{B-C-I}u^C Z_n^{(C,\lambda)}(u;k)du = Z_n^{(B,\lambda)}(x;k), \quad (28)$$

which is the result given by Shehata [12] theorem 2.4.

Concluding Remark

The matrix polynomials in equations (10),(11) and (14) introduced in section 1 and also the results obtained in sections 2 and 3, seem to be new and this technique will stimulate the scope of further research in the theory of special matrix functions and polynomials.

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حول بعض كثيرات حدود كونسر ولاجير المصفوفية

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DOI: https://doi.org/10.47372/uajnas.2019.n1.a13

الملخص

في هذا البحث قمت بتعميم ما يعرف بكثيرات حدود لاجير وكونسر المصفوفية المعطاة في [12] و [14] أيضاً قدمت مجموعة جديدة من كثيرات حدود لاجير المصفوفية $L_n^{(B,C)}(\alpha, x)$، أيضاً قدمت مجموعة جديدة المولدة وكذلك التمثيل التكاملي لهذه المجموعات من كثيرات الحدود.

الكلمة المفتاحية: كثيرات حدود لاجير وكونسر المصفوفية ، العلاقات المولدة ، التمثيلات التكاملية.