Study in P^h − *Birecurrent Finsler space*

Fahmi Yaseen Abdo Qasem and Amani Mohammed Abdullah Hanballa
Dep. of Math., Faculty of Edu.-Aden, Univ. of Aden, Khormaksar, Aden, Yemen
Email:fahmi.yaseen@yahoo.com. Email:ahanballa@yahoo.com
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Abstract

In the present paper, a Finsler space$F_n$whose Cartan's second curvature tensor P^i_{jkh}satisfies$P^i_{jkh|\ell|m}$=$a_{\ell m}P^i_{jkh}$,$P^i_{jkh} \neq 0$,where $a_{\ell m}$ isnon-zerocovariant tensor field of second order , is introduced and such space is called as P^h–*birecurrentspace* and denoted briefly by P^h− BR− F_n.The aim of this paper is to obtain some birecurrent tensors in this space. Also, we introduced Ricci birecurrent space. We proved the projection of some curvature tensors on indicatrix are birecurrent.

Keywords: Finsler space, P^h–Birecurrent space, Ricci birecurrent tensor, projection on indicatrix.

Introduction

Ruse [15] introduced and studied a three dimensional space known*as space of recurrent curvature*. The recurrent of an n–dimensional space was extended to Finsler space by Moór ([5],[6],[7]) for the first time. Due to different connections of Finsler space, the recurrent of different curvature tensors have been discussed by Mishra and Pande [8] and Pandey [9]. Dikshit [4], discussed a Finsler space for which Cartan's third curvature tensorR^i_{jkh}, it's associative curvature tensor $R_{ijkh}$and their projections on indicatrix with respect to Berwald's and Cartan'sconnection are birecurrent. Alqufail, Qasem and Ali [1] discussed a Finsler space for which Cartan's fourth curvature tensor $K^i_{jkh}$is birecurrent. Ali [2] discussed the birecurrence property of the curvature tensor K^i_{jkh} on indicatrix with respect to Cartan's connection. Qasem [10] discussed a Finsler space for which Cartan's third curvature tensor $R^i_{jkh}$is generalized and special generalized birecurrent of the first and second kind. Qasem and Saleem ([11],[12]) discussed a Finsler space for which the$h$–curvature tensor U^i_{jkh} and Weyl's projective curvature tensorW^i_{jkh} are generalized birecurrent. Qasemand Hanballa[13] discussed a Finsler space for which Cartan's fourth curvature tensor K^i_{jkh}isgeneralized birecurrent. Verma[16]discussed the projection on indicatrix, some results have been obtained on projection of Cartan's third curvature tensor.

Let us consider an$n$ −dimensional Finsler spaceF_n equipped with a metric function $F(x\,,y)$satisfying the requestic conditions of Finslerianmetric [14].

Thevectors y^i, y_i,the metric tensor$g_{ij}$and its associative metric tensor$g^{ij}$are satisfying the following relations

(1.1) $a)\ g_{ij}(x\,,y) := \frac{1}{2}\,\dot{\partial}_i\,\dot{\partial}_j F^2(x\,,y),$ $\qquad b)\ g^{ij}(x^k,y_k) = \frac{1}{2}\bar{\partial}_i\bar{\partial}_j H^2(x^k\,,y_k),$

$c)\ \ g_{ij|k} = 0\,,\ \ d)\ \ g^{ij}_{|k} = 0\,,$ $\qquad$ e) $y^i{}_{|k} = 0$ and f) $y_{i|k} = 0,$

where $|k$ is the h −covariant derivative with respect to x^k.

The two sets of quantitiesg_{ij}andg^{ij} which are components of metric tensor and associate metric tensor which defined by the equations (1.1a) and (1.1b), respectively are related by

(1.2) $$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 & if \quad i = k\ , \\ 0 & if \quad i \neq k\ . \end{cases}$$

The $(v)hv$-torsion tensor C^i_{jk} and it'sassociativee tensor $C_{ijk}$are satisfying the following relations

(1.3) $a)\ \ C^i_{jk}y^j = 0 = C^i_{kj}\ y^j,$ $\qquad b)\ \ C_{ijk}\ g^{jk} = C_i,$

$c)\ C^i_{ji} = C_j$ and$d)\ C_{ijk} := g_{hj}C^h_{ik}.$

The tensor $R^{\,i}_{jkh}$ is called *h-curvature tensor (Cartan's third curvature tensor)* and satisfies the relation

(1.4) $R^i_{jkh}\, y^j = H^i_{kh}$.

The tensor P^i_{jkh} is called hv-*curvature tensor* (*Cartan's second curvature tensor*) and is defined by

(1.5) $a)\ P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^{\,i}_{jr} P^r_{kh} - C^i_{jh|k}$

or equivalent by

$b)\ P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} C^r_{kh|s}\, y^s - C^i_{jh|k}$

or

$c)\ P^i_{jkh} := C^i_{kh|j} - g^{ir} C_{jkh|r} + C^r_{jk} P^{\,i}_{rh} - P^r_{jh} C^i_{rk}$.

The hv-curvature tensor P^i_{jkh} is positively homogeneous of degree zero in y^i and satisfies the relations

(1.6) $a)\ P^i_{jkh}\, y^j = \Gamma^{*i}_{jkh} y^j = P^i_{kh} = C^i_{kh|r} y^r$.

$b)\ P^i_{jkh}\, y^k = 0 = P^i_{jkh} y^h$

and

c) $P^i_{jk} y^j = 0$.

where $P^{\,i}_{jk}$ is called the $v(hv)$-*torsion tensor* and its associative tensor P_{rkh} is given by

(1.7) $g_{ir}\, P^{\,i}_{kh} = P_{krh}$.

The P −Ricci tensor P_{jk} is given by

(1.8) $P^i_{jki} = P_{jk}$.

The associate curvature tensor P_{ijkh} of the $hv-$ curvature tensor P^i_{jkh} is given by Rund [14]

(1.9) $a)\ P_{jikh} = g_{ir}\, P^r_{jkh}$

and $b)\ P^r_{jkh} = g^{ir} P_{jikh}$

which is skew-symmetric in the first two lower indices i and j , i.e.

$$P_{ijkh} = -P_{jikh}.$$

The tensor $(P_{ij} - P_{ji})$ is given by

(1.10) $P_{ijkh}\, g^{kh} = P_{ij} - P_{ji}$.

The curvature vector P_k is given by

(1.11) $P^{\,i}_{ki} = P_k$.

In view of (1.5c) and by using the symmetric property of the (v) hv- torsion tensor C^i_{jk} in the lower indices j and k , the hv-curvature tensor P^i_{jkh} satisfies the following:

(1.12) $P^i_{hjk} - P^i_{jhk} = C^i_{jk|h} + P^r_{jk} C^i_{rh} - h/j^*$.

The $h(v)-$ torsion tensor H^i_{kh} satisfies Pandey [9]

(1.13) $y_i\, H^{\,i}_{kh} = 0$.

In view of Euler's theorem on homogeneous functions, we have the following relation:

(1.14) $H^i_{jk}\, y^j = H^i_k = -H^i_{kj}\, y^j$.

Let the current coordinates in the tangent space at the point x_0 be x^i, then the indicatrix I_{n-1} is a hypersurface defined by

(1.15) $F(x_0\,,\, x^i\,) = 1$

or in the parametric form it is defined by

(1.16) $x^i = x^i(u^a),\ a = 1,2,\dots,n-1$.

Now, the projection of any tensor T^i_j on the indicatrix is given by

(1.17) $p\,.T^i_j := T^a_b\, h^i_a\, h^b_j$,

where

(1.18) $h^i_c := \delta^i_c - l^i l_c$.

If the projection of an arbitrary tensor T_j^i on the indicatrix I_{n-1} is the same tensor T_j^i , the tensor is called an *indicatric tensor* or an *indicatory tensor*, e.g. H_k^i,C_{jk}^i , h_j^i , S_{jkh}^i and P_{jk}^i are all indicatric tensors.

The projection of the vector y^i, the unit vector l^i and the metric tensor g_{ij} on the indicatrix are given by

(1.19) $a)\quad p.y^i = 0,$

$b)\quad p.l^i = 0$

and

$c)\quad p.\ g_{ij} = h_{ij},$

where

(1.20) $h_{ij} := g_{ij} - l_i l_j\ .$

A P^h − Birecurrent Space

Let us consider a Finsler space F_n whose Cartan's second curvature tensor P_{jkh}^i satisfies the condition

(2.1) $P_{jkh|\ell}^i = \lambda_\ell P_{jkh}^i, P_{jkh}^i \neq 0,$

$* - h/j$ means the subtraction from the former term by interchange the indices h and j.

where λ_ℓ is non-zerocovariant vector field.

Taking the h–covariant derivative forthe condition (2.1) with respect to x^m, we get

(2.2) $P_{jkh|\ell|m}^i = \lambda_{\ell|m} P_{jkh}^i + \lambda_\ell P_{jkh|m}^i, P_{jkh}^i \neq 0.$

Using the condition(2.1) in (2.2), we get

$$P_{jkh|\ell|m}^i = (\lambda_{\ell|m} + \lambda_\ell \lambda_m) P_{jkh}^i$$

which can be written as

(2.3) $P_{jkh|\ell|m}^i = a_{\ell m} P_{jkh}^i, P_{jkh}^i \neq 0,$

where $a_{\ell m} = \lambda_{\ell|m} + \lambda_\ell \lambda_m$ is non–zerocovariant tensor field of second order called *therecurrence tensor field*.

Definition 2.1. A Finsler space F_n forwhich Cartan's second curvature tensor P_{jkh}^i satisfies the condition (2.3) is called P^h*–birecurrentspace* and the tensor is called *h–birecurrent tensor.* We shall denote such space and tensor briefly by P^h–BR–F_n and h–BR, respectively.

However, if we start from the condition (2.3), we can't obtain the condition (2.1) in general. Thus, we conclude

Theorem 2.1. *Every P^h–recurrent space is P^h–BR–F_n. But the converse need not be true.*

Transvecting the condition (2.3) by g_{ip}, using (1.1c) and (1.9a) yield

(2.4) $P_{pjkh|\ell|m} = a_{\ell m} P_{pjkh},$

Conversely, the transvection of the condition (2.4) by g^{ip} yield the condition (2.3). Thus, the condition (2.3) is equivalent to the condition (2.4).

Thus, we conclude

Theorem 2.2. *The P^h–BR–F_n , may characterized by the condition* (2.4).

Let us consider an P^h–BR–F_n which is characterized by the condition (2.3).

Transvecting the condition (2.3) by y^j, using (1.1e) and (1.6a), we get

(2.5) $P_{kh|\ell|m}^i = a_{\ell m} P_{kh}^i\ .$

Transvecting the condition (2.3) by g_{ip}, using (1.1c) and (1.7), we get

(2.6) $P_{pkh|\ell|m} = a_{\ell m} P_{pkh}.$

Further, transvecting the condition (2.4) by g^{kh} , using (1.1d) and (1.10), we get

(2.7) $(P_{pj} - P_{jp})_{|\ell|m} = a_{\ell m} (P_{pj} - P_{jp}).$

Thus, we conclude

Theorem 2.3. *In $P^h - BR- F_n$, the $v(hv)$–torsion tensor P^i_{kh}, it's associative torsion tensor P_{pkh} and the tensor $(P_{pj} - P_{jp})$ are $h- BR$.*

Contracting the indices i and h in the condition (2.3), (2.5), using (1.8) and (1.11), we get

(2.8) $$P_{jk|\ell|m} = a_{\ell m} P_{jk}$$

and

(2.9) $$P_{k|\ell|m} = a_{\ell m} P_k ,$$

respectively.

Thus, we conclude

Theorem 2.4. *In $P^h - BR- F_n$, $P -$ Ricci tensor P_{jk} and the curvature vector P_k are $h- BR$.*

Cartan's second curvature tensor P^i_{jkh}, the $v(hv)$ −torsion tensor P^i_{kh} and the $(v)hv$ −torsion tensor C^i_{jk} are connected by the formula (1.12).

Taking the $h-$ covariant derivative twice for (1.12) with respect to x^ℓ and x^m, successively, we get

(2.10) $$P^i_{hjk|\ell|m} - P^i_{jhk|\ell|m} = \left(C^i_{jk|h} + P^r_{jk} C^i_{rh} - h/j\right)_{|\ell|m}.$$

Using the condition (2.3) and in view of (1.12), the equation (2.10) can be written as

(2.11) $$\left(C^i_{jk|h} + P^r_{rh} C^i_{rh} - h/j\right)_{|\ell|m} = a_{\ell m} \left(C^i_{jk|h} + P^r_{jk} C^i_{rh} - h/j\right).$$

Thus, we conclude

Theorem 2.5. *In $P^h - BR- F_n$, the tensor $\left(C^i_{jk|h} + P^r_{jk} C^i_{rh} - h/j\right)$* is *$h- BR$.*

Transvecting (2.11) by g_{iq}, using (1.1c) and (1.3d), we get

(2.12) $$\left(C_{jrk|h} + P^r_{jk} C_{hsr} - h/j\right)_{|\ell|m} = a_{\ell m} \left(C_{jrk|h} + P^r_{jk} C_{hsr} - h/j\right).$$

Transvecting (2.11) by y^h , using (1.1e), (1.3a) and (1.6c), we get

(2.13) $$\left(C^i_{jk|h} y^h\right)_{|\ell|m} = a_{\ell m}\left(C^i_{jk|h} y^h\right).$$

Further, transvecting (2.13) by g_{ir} , using (1.1c) and (1.3d), we get

(2.14) $$\left(C_{jrk|h} y^h\right)_{|\ell|m} = a_{\ell m}\left(C_{jrk|h} y^h\right).$$

Thus, we conclude

Theorem 2.6. *In $P^h - BR- F_n$, the tensors $\left(C^i_{jk|h} + P^r_{jk} C^i_{rh} - h/j\right)$, $\left(C^i_{jk|h} y^h\right)$ and $\left(C_{jrk|h} y^h\right)$ are $h- BR$.*

Contracting the indices i and k in (2.11),(2.13) and using (1.3c), we get

(2.15) $$\left(C_{j|h} + P^r_{jp} C^p_{rh} - h/j\right)_{|\ell|m} = a_{\ell m}\left(C_{j|h} + P^r_{jp} C^p_{rh} - h/j\right),$$

and

(2.16) $$\left(C_{j|h} y^h\right)_{|\ell|m} = a_{\ell m}\left(C_{j|h} y^h\right),$$

respectively.

In view of (1.5c), we have

(2.17) $$P^i_{jkh} - P^i_{jhk} = C^r_{jk} P^i_{rh} + P^r_{jk} C^i_{rh} - k/h.$$

Taking the h–covariant derivative twice for (2.17) with respect to x^ℓ and x^m , successively, we get

(2.18) $$\left(P^i_{jkh} - P^i_{jhk}\right)_{|\ell|m} = \left(C^r_{jk} P^i_{rh} + P^r_{jk} C^i_{rh} - k/h\right)_{|\ell|m}.$$

Using the condition (2.3) in (2.18) and in view of (2.17), we get

(2.19) $$\left(C^r_{jk} P^i_{rh} + P^r_{jh} C^i_{rk} - k/h\right)_{|\ell|m} = a_{\ell m}\left(C^r_{jk} P^i_{rh} + P^r_{jh} C^i_{rk} - k/h\right).$$

Thus, we conclude

Theorem 2.7. *In $P^h - BR- F_n$, the tensor $\left(C^r_{jk} P^i_{rh} + P^r_{jk} C^i_{rh} - k/h\right)$ is $h- BR$.*

Transvecting (2.19) by g_{ir}, using (1.1c), (1.7) and (1.3d), we get

(2.20) $$\left(C^r_{jk} P_{prh} + P^r_{jk} C_{prh} - k/h\right)_{|\ell|m} = a_{\ell m}\left(C^r_{jk} P_{prh} + P^r_{jk} C_{prh} - k/h\right).$$

Thus, we conclude

Theorem 2.8. *In $P^h - BR- F_n$, the tensor $\left(C^r_{jk} P_{prh} + P^r_{jk} C_{prh} - k/h\right)$ is $h- BR$.*

Cartan's second curvature tensor, the $v(hv)$–torsion tensor P^i_{rk} and the $(v)hv$–torsion tensor C^i_{jk} are connected by the formula (1.5c).

Taking the h–covariant derivative twice for (1.5c) with respect to x^ℓ and x^m, successively, we get

(2.21) $$P^i_{jkh|\ell|m} = \left(C^i_{kh|j} - g^{ir}C_{jkh|r} + C^r_{jk}P^i_{rh} - P^r_{jh}C^i_{rk}\right)_{|\ell|m}.$$

Using the condition (2.3) and (1.5c)in (2.21), we get

$$\left(C^i_{kh|j} - g^{ir}C_{jkh|r} + C^r_{jk}P^i_{rh} - P^r_{jh}C^i_{rk}\right)_{|\ell|m} = a_{\ell m}\left(C^i_{kh|j} - g^{ir}C_{jkh|r} + C^r_{jk}P^i_{rh} - P^r_{jh}C^i_{rk}\right).$$

Thus, we conclude

Theorem 2.9. *In P^h-BR-F_n, the tensor $\left(C^i_{kh|j} - g^{ir}C_{jkh|r} + C^r_{jk}P^i_{rh} - P^r_{jh}C^i_{rk}\right)$ is $h-BR$.*

Transvecting (1.5c) by g_{ip} , using (1.1c), (1.3d), (1.7),(1.9) and (1.2), we get

(2.22) $$P_{jpkh} = C_{kph|j} - \delta^r_p C_{jkh|r} + C^r_{jk}P_{rph} - P^r_{jh}C_{rpk}.$$

Taking the $h-$ covariant derivative twice for (2.22) with respect to x^ℓ and x^m , successively, we get

(2.23) $$P_{jpkh|\ell|m} = \left(C_{kph|j} - \delta^r_p C_{jkh|r} + C^r_{jk}P_{rph} - P^r_{jh}C_{rpk}\right)_{|\ell|m}.$$

Using the condition (2.4)and (2.22) in (2.23), we get

$$\left(C_{kph|j} - \delta^r_p C_{jkh|r} + C^r_{jk}P_{rph} - P^r_{jh}C_{rpk}\right)_{|\ell|m} = a_{\ell m}\left(C_{kph|j} - \delta^r_p C_{jkh|r} + C^r_{jk}P_{rph} - P^r_{jh}C_{rpk}\right).$$

Thus, we conclude

Theorem 2.10. *In P^h-BR-F_n, the tensor $\left(C_{kph|j} - \delta^r_p C_{jkh|r} + C^r_{jk}P_{rph} - P^r_{jh}C_{rpk}\right)$ is $h-BR$.*

Contracting the indices i and h in (1.5c), using (1.8) and (1.11), we get

(2.24) $$P_{jk} := C_{k|j} - g^{pr}C_{jkp|r} + C^r_{jk}P_r - P^r_{jp}C^p_{rk}.$$

Taking the h–covariant derivative twice for (2.24) with respect to x^ℓ and x^m , successively, we get

(2.25) $$P_{jk|\ell|m} = \left(C_{k|j} - g^{pr}C_{jkp|r} + C^r_{jk}P_r - P^r_{jp}C^p_{rk}\right)_{|\ell|m}.$$

Using (2.8)and (2.24) in (2.25), we get

$$\left(C_{k|j} - g^{pr}C_{jkp|r} + C^r_{jk}P_r - P^r_{jp}C^p_{rk}\right)_{|\ell|m} = a_{\ell m}\left(C_{k|j} - g^{pr}C_{jkp|r} + C^r_{jk}P_r - P^r_{jp}C^p_{rk}\right).$$

Thus, we conclude

Theorem 2.11. *In P^h-BR-F_n, the tensor $(C_{k|j} - g^{pr}C_{jkp|r} + C^r_{jk}P_r - P^r_{jp}C^p_{rk})$ is $h-BR$.*

Transvecting (2.22) by g^{kh}, using (1.2), (1.3b), (1.1d) and (1.10), we get

(2.26) $$P_{jp} - P_{pj} = C_{p|j} - \delta^r_p C_{j|r} + g_{ip}g^{kh}C^r_{jk}P^i_{rh} - P^r_{jh}C^h_{rp}.$$

Taking the $h-$ covariant derivative twice for (2.26) with respect to x^ℓ and x^m , successively, we get

(2.27) $$\left(P_{jp} - P_{pj}\right)_{|\ell|m} = \left(C_{p|j} - \delta^r_p C_{j|r} + g_{ip}g^{kh}C^r_{jk}P^i_{rh} - P^r_{jh}C^h_{rp}\right)_{|\ell|m}.$$

Using (2.7) and (2.26) in (2.27), we get

$$\left(C_{p|j} - \delta^r_p C_{j|r} + g_{ip}g^{kh}C^r_{jk}P^i_{rh} - P^r_{jh}C^h_{rp}\right)_{|\ell|m} = a_{\ell m}\left(C_{p|j} - \delta^r_p C_{j|r} + g_{ip}g^{kh}C^r_{jk}P^i_{rh} - P^r_{jh}C^h_{rp}\right).$$

Thus, we conclude

Theorem 2.12. *In P^h-BR-F_n, the tensor $\left(C_{p|j} - \delta^r_p C_{j|r} + g_{ip}g^{kh}C^r_{jk}P^i_{rh} - P^r_{jh}C^h_{rp}\right)$ is $h-BR$.*

For Riemannian space V_4 , the projection curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined as [1]

(2.28) $$P^i_{jkh} = R^i_{jkh} - \tfrac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right).$$

Taking the $h-$ covariant derivative twice for (2.28) with respect to x^ℓ and x^m , successively, we get

(2.29) $$P^i_{jkh|\ell|m} = \left\{R^i_{jkh} - \tfrac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right\}_{|\ell|m}.$$

Using the condition (2.3) and (2.28) in (2.29), we get

(2.30) $$\left\{R^i_{jkh} - \tfrac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right\}_{|\ell|m} = a_{\ell m}\left\{R^i_{jkh} - \tfrac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right\}.$$

Thus, we conclude

Theorem 2.13. *For* $n = 4$, *the tensor* $\left\{R^i_{jkh} - \frac{1}{3}\left(\delta^i_h R_{jk} - \delta^i_k R_{jh}\right)\right\}$ *in* $P^h - BR - F_n$, *is* $h- BR$.

Transvecting (2.30) by y^j , using (1.1e), (1.4) and putting $\left(R_{jp}\, y^j = R_p\right)$, we get

(2.31) $$\left\{H^i_{kh} - \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right\}_{|\ell|m} = a_{\ell m}\left\{H^i_{kh} - \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right\}.$$

Thus, we conclude

Theorem 2.14. *For* $n = 4$, *the tensor* $\left\{H^i_{kh} - \frac{1}{3}\left(\delta^i_h R_k - \delta^i_k R_h\right)\right\}$ *in* $P^h - BR - F_n$, *is* $h- BR$.

Transvecting (2.31) by y^k, using (1.14), (1.1e) and putting $\left(R_k\, y^k = R\right)$, we get

$$\left\{H^i_h - \frac{1}{3}\left(\delta^i_h R - y^i R_h\right)\right\}_{|\ell|m} = a_{\ell m}\left\{H^i_h - \frac{1}{3}\left(\delta^i_h R - y^i R_h\right)\right\}.$$

Thus, we conclude

Theorem 2.15. *For* $n = 4$, *the tensor* $\left\{H^i_h - \frac{1}{3}\left(\delta^i_h R - y^i R_h\right)\right\}$ *in* $P^h - BR - F_n$ *is* $h- BR$.

Transvecting (2.31) by y_i, using (1.1f) , (1.13) and in view of $\delta^i_h y_i = y_h$, we get

$$(y_h R_k - y_k R_h)_{|\ell|m} = a_{\ell m}(y_h R_k - y_k R_h),$$

Thus, we conclude

Theorem 2.16. *For* $n = 4$, *the tensor* $(y_h R_k - y_k R_h)$ *in* $P^h - BR - F_n$ *is* $h- BR$.

3. The Projection For Cartan's Second Curvature Tensors On Indicatrix

Let us consider a Finsler space F_n for which Cartan's second curvature tensor P^i_{jkh} is $h - BR$, $i.e.$ characterized by the condition (2.3). In view of (5.17), the projection of Cartan's second curvature tensor P^i_{jkh} on the indicatrix is given by

(3.1) $$p.P^i_{jkh} = P^a_{bcd}\, h^i_a\, h^b_j\, h^c_k\, h^d_h\,.$$

Taking the h–covariant derivative twice for (3.1) with respect to x^ℓ and x^m , successively and using the fact that h^α_β is covariant constant, we get

(3.2) $$\left(p.P^i_{jkh}\right)_{|\ell|m} = P^a_{bcd|\ell|m}\, h^i_a\, h^b_j\, h^c_k\, h^d_h.$$

Using the condition (2.3) in (3.2), we get

(3.3) $$\left(p.P^i_{jkh}\right)_{|\ell|m} = a_{\ell m}\, P^a_{bcd}\, h^i_a\, h^b_j\, h^c_k\, h^d_h.$$

Using (3.1) in (3.3), we get

(3.4) $$\left(p.P^i_{jkh}\right)_{|\ell|m} = a_{\ell m}\left(p.P^i_{jkh}\right).$$

Thus, we conclude

Theorem 3.1. *In* $P^h - BR - F_n$, *the projection of Cartan's second curvature tensor* P^i_{jkh} *on indicatrix is* $h - BR$.

Let us consider $P^h - BR - F_n$ for which the associative curvature tensor P_{jikh} is $h- BR$ in the sense of Cartan, $i.e.$ characterized by the condition (2.4). In view of (1.17), the projection of the associative curvature tensor P_{jikh} on the indicatrix is given by

(3.5) $$p.P_{jikh} = P_{abcd}\, h^a_j\, h^b_i\, h^c_k\, h^d_h.$$

Taking the h–covariant derivative twice for (3.5) with respect to x^ℓ and x^m, successively and using the fact that h^α_β is covariant constant, we get

(3.6) $$\left(p.P_{jikh}\right)_{|\ell|m} = P_{abcd|\ell|m} h^a_j\, h^b_i\, h^c_k\, h^d_h.$$

Using the condition (2.4) in (3.6), we get

(3.7) $$\left(p.P_{jikh}\right)_{|\ell|m} = a_{\ell m}\, P_{abcd} h^a_j\, h^b_i\, h^c_k\, h^d_h.$$

Using (3.5) in (3.7), we get

(3.8) $$\left(p.P_{jikh}\right)_{|\ell|m} = a_{\ell m}\left(p.P_{jikh}\right).$$

Thus, we conclude

Theorem 3.2. *In* $P^h - BR - F_n$, *the projection of the associative curvature tensor* P_{jikh} *on indicatrix is* $h- BR$.

Let us consider P^h– BR– F_n for which P–Ricci tensor P_{jk} is h– BR, $i.e.$ characterized by the condition (2.8). In view of (1.17), the projection of P–Ricci tensor P_{jk} on the indicatrix is given by

(3.9) $\quad (p.P_{jk}) = P_{ab}\, h_j^a\, h_k^b$.

Taking the h–covariant derivative twice for (3.9) with respect to x^ℓ and x^m, successively and using the fact that h_β^α is covariant constant, we get

(3.10) $\quad (p.P_{jk})_{|\ell|m} = P_{ab|\ell|m}\, h_j^a\, h_k^b$.

Using the condition (2.8) in (3.10), we get

(3.11) $\quad (p.P_{jk})_{|\ell|m} = a_{\ell m}\, P_{ab}\, h_j^a\, h_k^b$.

Using (3.9) in (3.11), we get

(3.12) $\quad (p.P_{jk})_{|\ell|m} = a_{\ell m}\,(p.P_{jk})$.

Thus, we conclude

Theorem 3.3. *In P^h– BR– F_n, the projection of the P–Ricci tensor P_{jk} on indicatrix is h– BR.*

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دراسة في فضاء فنسلر – P^h ثُنائي التكرار

فهمي ياسين عبده قاسم وأماني محمد عبدالله حنبله
قسم الرياضيات، كلية التربية/ عدن، جامعة عدن، خورمكسر، عدن، اليمن
Email:ahanballa@yahoo.com Email:fahmi.yaseen@yahoo.com
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المُلخص

في هذه الورقة قدّمنا المُعادلة المُميزة للفضاء الذي يُحقق فيها الموتر التقوسي الثاني لكارتان P^i_{jkh} فيما يأتي:

$$P^i_{jkh|\ell|m}=a_{\ell m}P^i_{jkh},P^i_{jkh}\neq 0,$$

باستخدام النوع الثاني لمُشتقة كارتان (h–مُشتقة التغاير)، حيث $|\ell|m$ الموتر التفاضُلي المُتحد الاختلاف من الرُتبة الثانية بالنسبة إلى x^m و x^ℓ على التعاقب، $a_{\ell m}$ موترات حقلية مُتحدة الاختلاف غير صفرية من الرُتبة الثانية وقد سمّينا الفضاء أعلاه بفضاء فنسلرثُنائي التكرار– P^h ورمزنا له بالرمز P^h-BR-F_n.
كما أثبتنا عدد من النتائج ووجدنا عدد من المُتطابقات مثل الموتر الإلتوائيP^i_{kh}، موتر ريتشي P_{kh} التي جميعها تُحقق خاصية ثُنائي التكرار، وأيضا وجدنا إسقاطات لبعض الموترات التقوسية والإلتواءات على Indicatrix والتي يُحقق فيها خواص ذلك الفضاء.

الكلمات المُفتاحية: فضاء فنسلر، فضاء ثُنائي التكرار-P^h، موتر ريتشي ثُنائي التكرار و الإسقاطات على Indicatrix.