Generating Functions for Legendre Polynomials by using Group Theoretic Method

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Abstract

In this paper we obtain generating functions for the Legendre polynomials $P_n(x)$ in its modified form by using Weisner's group theoretic method. Whereas, we deployed it to determine the new generating relations between the generalized Legendre polynomials and with easy way. The ideas in consent with proofs are originated from the book of McBride [8] and is used to determine new generating relations which involve modified Legendre polynomials.

Key words: Legendre polynomials, generating functions, differential operators & Group theoretic-method.

Introduction

The unification of generating functions has a great importance in connection with ideas and principles of special functions. Group Theoretic Method proposed by Louis Weisner in 1955, who employed this method to find generating relations for a large class of special functions. Weisner discussed the group-theoretic significance of generating functions for Hypergeometric, Hermite, and Bessel functions in his papers [13, 14] and [15] respectively. McBride [8] deployed Weisner's method to determine the new generating relations that involve Hermite, Bessel, generalized Laguerre, Gegenbauer polynomials. In this directions, some important steps has been made by researchers, namely Singhal and Srivastava [10], Chaterjea [1, 2] and Chongdar [3]. In their study, Desale and Qashash [5] have obtained a new general class of generating functions for the generalized modified Laguerre polynomials $L_n^{(\alpha)}(x)$ by group theoretic method. Also, they have introduced the bilateral generating function for the generalized modified Laguerre and Jacobi polynomials, with the help of two linear partial differential operators. Further, continuing their study [6, 7], they used the group theoretic method to obtain proper and improper partial bilateral as well as trilateral generating functions.

Now, continuing the work in connection with class of generating functions, we extend our ideas to obtain new generating relations that involve between the generalized Legendre polynomials.

Legendre Polynomials $P_n(x)$ for n = 0, 1, 2, . . . are defined as [9];

$$P_n(x) = (-1)^n {}_2F_1\left[\begin{matrix} -n, n+1; \\ & \dfrac{1+x}{2} \\ 1; \end{matrix}\right]. \tag{1.1}$$

We see that these polynomials are particular solutions of Legendre differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

One may consult Rainville [9] for more details about Legendre Polynomials.

Note that subscripts in the following relations are nonnegative integers:

$$(1-x^2)DP_n(x) = n[P_{n-1}(x) - xP_n(x)], \tag{1.2}$$

and

$$(1-x^2)DP_n(x)=(n+1)[xP_n(x)-P_{n+1}(x)]\ , \qquad (1.3)$$

where D is the differential operator $D=\frac{d}{dx}$.

Linear Differential Operators

In this section, we define some linear partial differential operators in two independent variables x and y. We will investigate their commutative properties while operating on Legendre Polynomials.

If we use (1.2) and (1.3) repeatedly, then we see that

$(1-x^2)D^2P_n(x)-2xDP_n(x)+n(n+1)P_n(x)=0$.

So, we can rewrite the above equation in the form differential operator L as;

$$L\left(x,\frac{d}{dx},n\right)P_n(x)=(1-x^2)D^2P_n(x)-2xDP_n(x)+n(n+1)P_n(x)=0 \qquad (2.1)$$

In the above equation (2.1), if we replace $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and $P_n(x) by u(x,y)$, so that

$$L\left(x,\frac{\partial}{\partial x},y\frac{\partial}{\partial y}\right)u(x,y)=0.$$

Hence, we define a linear partial differential operator L as

$$L\equiv L\left(x,\frac{\partial}{\partial x},y\frac{\partial}{\partial y}\right)=(1-x^2)\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+y^2\frac{\partial^2}{\partial y^2}\ . \qquad (2.2)$$

If we operate it on $P_n(x)$ it give us

$$L\left(x,\frac{\partial}{\partial x},y\frac{\partial}{\partial y}\right)P_n(x)=\left((1-x^2)\frac{\partial^2}{\partial x^2}-2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)P_n(x)=0. \qquad (2.3)$$

Next, we seek to define the linear differential operators A, B, and C, which will commute with L with $\psi(x)L$. Note that ψ is some function of x to be determined. Let us define these operators as follows:

$$A\equiv y\frac{\partial}{\partial y}\ , \qquad (2.4)$$

$$B\equiv(1-x^2)y^{-1}\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \qquad (2.5)$$

$$C\equiv y(1-x^2)\frac{\partial}{\partial x}-xy^2\frac{\partial}{\partial y}-xy. \qquad (2.6)$$

As in previous paragraph, we define linear partial differential operators. It is interesting to see that how does these operators act on Legendre polynomials

Following are the actions of these operators on respective functions:

$$A\left[P_n(x)y^n\right]=nP_n(x)y^n \qquad (2.7)$$

$$B\left[P_n(x)y^n\right]=nP_{n-1}(x)y^{n-1} \qquad (2.8)$$

$$C\left[P_n(x)y^n\right]=-(n+1)P_{n+1}(x)y^{n+1} \qquad (2.9)$$

Now, we use the commutator notation [A,B] = (AB - BA). Henceforth, we consider a function $u=u(x,y)$, is a l^2 function in two independent variables x and y. Therefore, an action of a commutator operator on u is that

[A,B]u = (AB - BA)u = A(Bu) - B(Au).

Then, we have the operators A, B, and C, which satisfy the following commutator relations:

(i) [A,B] = -B,
(ii) [A,C] = C,
(iii) [B,C] = -2A - 1.

Linear differential operators defined in (2.4), (2.5) and (2.6) commute with the operator $(1-x^2)L$, where the operator L is as defined in (2.2).

Let us consider an arbitrary l^2 function u = u(x, y) in two independent variables. Hence, if we operate the operator L on u = u(x, y), we get

$$Lu = (1-x^2)\frac{\partial^2 u}{\partial x^2} - 2x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

While in computation of [B,C], we found that

$$CBu = (1-x^2)^2\frac{\partial^2 u}{\partial x^2} - 2x(1-x^2)\frac{\partial u}{\partial x} + y(1-x^2)\frac{\partial u}{\partial y} - x^2y^2\frac{\partial^2 u}{\partial y^2} - x^2y\frac{\partial u}{\partial y}.$$

Hence, we have

$$[(1-x^2)L - CB]u = y(1-x^2)\frac{\partial u}{\partial y} + x^2y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

$$= y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

$$= A^2u$$

Since u is an arbitrary function, we conclude that

$$(1-x^2)L \equiv \left(CB + A^2\right).$$

Extended form of the group of Operators

In this section, we extend the operators B and C which defined in the previous section to the exponential form. Consider an arbitrary function $f = f(x, y)$ in two independent variables. Also, we consider the arbitrary constants b and c. exp (bB) and exp (cC) are called the extended form of the transformation groups generated by B and C. For doing this business, we will follow the method suggested by Weisner [13]. One may refers to McBride [8] for similar kind of analysis.

In order to find the extended form of the group generated by operators B and C, we will change the form of differential operator $C = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} + C_0$ to $E = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y}$, and finally by the change of variable to $D = \frac{\partial}{\partial X}$. If $\phi(x, y)$ is any solution of $C[\phi(x, y)] = 0$, then

$$\phi^{-1}C\phi = C_1\frac{\partial}{\partial x} + C_2\frac{\partial}{\partial y} = E.$$

So, the operator E can be defined as

$$E = y(1-x^2)\frac{\partial}{\partial x} - xy^2\frac{\partial}{\partial y} = \phi^{-1}C\phi, \tag{3.1}$$

where $\phi = (1-x^2)^{-1}y$.

In (3.1), we have defined the operator E, which satisfies the property $C = \phi E \phi^{-1}$ and corresponding function ϕ is $\phi = (1-x^2)^{-1}y$. Now, we are in position to find $e^{cC}f(x, y)$, the extended form of transformation group generated by C. We may write $e^{cC}f(x, y)$ in the following form:

$$e^{cC}f(x, y) = e^{c(\phi E \phi^{-1})}f(x, y)$$

$$=(1-x^2)^{-1}ye^{cE}[(1-x^2)y^{-1}f(x,y)].$$

We will transform E into -D. A simple use of this substitution, and applying Taylor's theorem in the form

$$e^{c\frac{d}{dx}}F(x)=F(X+c), \qquad (3.2)$$

we find that

$$e^{cE}[(1-x^2)y^{-1}f(x,y)]=e^{-cD}\left[\left(1-\frac{X^2}{X^2+Y}\right)\left(\sqrt{X^2+Y}\right)f\left(\frac{-X}{\sqrt{X^2+Y}},\frac{1}{\sqrt{X^2+Y}}\right)\right]$$

$$=e^{-cD}\left[\frac{Y}{\sqrt{X^2+Y}}\,f\left(\frac{-X}{\sqrt{X^2+Y}},\frac{1}{\sqrt{X^2+Y}}\right)\right]$$

$$=\frac{Y}{\sqrt{(X-c)^2+Y}}\,f\left(\frac{-X+c}{\sqrt{(X-c)^2+Y}},\frac{1}{\sqrt{(X-c)^2+Y}}\right).$$

Finally, we use the inverse substitution to the right hand side of the above equation and we establish the following result:

$$e^{cC}f(x,y)=(1-x^2)^{-1}y\frac{(1-x^2)y^{-2}}{\sqrt{c^2+2cxy^{-1}+y^{-2}}}\,f\left(\frac{xy^{-1}+c}{\sqrt{c^2+2cxy^{-1}+y^{-2}}},\frac{1}{\sqrt{c^2+2cxy^{-1}+y^{-2}}}\right).$$

Further, on simplification, we have

$$e^{cC}f(x,y)=\frac{1}{\sqrt{1+2cxy+c^2y^2}}\,f\left(\frac{x+cy}{\sqrt{1+2cxy+c^2y^2}},\frac{y}{\sqrt{1+2cxy+c^2y^2}}\right). \qquad (3.3)$$

Now, we proceed to find the exp (bB) corresponding to the operator B defined in (2.5). Hence, we seek a function $\phi_1(x,y)$ such that $B[\phi_1(x,y)]=0$. Here, we find the function $\phi_1=(1-x^2)y^2$ and then we find that $\phi_1^{-1}=(1-x^2)^{-1}y^{-2}$ as in the similar way we found ϕ and ϕ^{-1} previously. So, the operator E1 can be defined as

$$E_1=(1-x^2)y^{-1}\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \qquad (3.4)$$

then the operator E_1 get transformed into $-D=-\frac{\partial}{\partial X}$. Now, we look at the action of e^{bB} on an arbitrary function $f(x,y)$;

$$e^{bB}f(x,y)=e^{(b\phi_1 E\phi_1^{-1})}f(x,y)$$

$$=(1-x^2)y^2e^{bE_1}[(1-x^2)^{-1}y^{-2}f(x,y)] \qquad (3.5)$$

Using the inverse substitution, we get

$$e^{bE_1}[(1-x^2)^{-1}y^{-2}f(x,y)]=(1-x^2)^{-1}y^{-2}f\left(\frac{xy+b}{\sqrt{y^2+2bxy+b^2}},\sqrt{y^2+2bxy+b^2}\right). \qquad (3.6)$$

Substituting (3.6) into (3.5), we get the action of e^{bB} on an arbitrary function $f(x,y)$ as follows:

$$e^{bB}f(x,y)=f\left(\frac{xy+b}{\sqrt{y^2+2bxy+b^2}},\sqrt{y^2+2bxy+b^2}\right). \qquad (3.7)$$

Now, using the similar method, we can determine the action of e^{aA} on an arbitrary function $f(x,y)$ as follows:

$$e^{aA} f(x,y) = f(x, e^{a} y) \tag{3.8}$$

in which the operator A is as defined in (2.4). Further, we proceed to determine $e^{cC}e^{bB}$, where b and c are arbitrary constants. Hence, we consider its action on an arbitrary function $f(x,y)$

$$e^{cC}e^{bB} f(x,y) = (1+2cxy+c^2y^2)^{\frac{-1}{2}} f(\varsigma,\tau), \tag{3.9}$$

where

$$\varsigma = \frac{cy^2(1+bc)+xy(1+2bc)+b}{(\sqrt{1+2cxy+c^2y^2})(\sqrt{y^2(1+bc)^2+2bxy(1+bc)+b^2})} \tag{3,10}$$

and

$$\tau = \frac{\sqrt{y^2(1+bc)^2+2bxy(1+bc)+b^2}}{\sqrt{1+2cxy+c^2y^2}} \tag{3.11}$$

Generating functions:

In this section, we have determined the new generating relations between modified Legendre polynomials. To obtain the generating function for { $P_n(x)$ }, we now transform $f(x,y)$ by means of the operator $e^{cC}e^{bB}$. We shall consider the following cases:

Case 1: b = -1, c = 0.

If we substitute b = -1 and c = 0 in (3.9), then it will give us

$$e^{-B} f(x,y) = f\left(\frac{xy-1}{\sqrt{y^2-2xy+1}}, \sqrt{y^2-2xy+1}\right).$$

Hence, if we take $f(x,y) = P_n(x)y^n$, then it will be resulted into following relation:

$$e^{-B}[P_n(x)y^n] = (y^2-2xy+1)^{\frac{n}{2}} P_n\left(\frac{xy-1}{\sqrt{y^2-2xy+1}}\right). \tag{4.1}$$

Also, we notice from (2.8) that

$$B[P_n(x)y^n] = (1-x^2)y^{-1}\frac{\partial}{\partial x}[P_n(x)y^n] + x\frac{\partial}{\partial y}[P_n(x)y^n]$$

$$= nP_{n-1}(x)y^{n-1}.$$

On the other hand, we can expand left hand side of (4.1) in series form and then repeated application of (2.8) on the same side of (4.1), we get

$$\sum_{q=0}^{n} \frac{(-n)_q}{q!} P_{n-q}(x)y^{n-q} = (y^2-2xy+1)^{\frac{n}{2}} P_n\left(\frac{xy-1}{\sqrt{y^2-2xy+1}}\right)$$

Let us put $t = y^{-1}$ in above equation, we get

$$\sum_{q=0}^{n} \frac{(-n)_q}{q!} P_{n-q}(x)t^{q-n} = (1-2xt+t^2)^{\frac{n}{2}} t^{-n} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right)$$

Furthermore, we simplify the above equation and it will give us a new generating function between modified Legendre polynomials.

$$\sum_{q=0}^{n} \frac{(-n)_q}{q!} P_{n-q}(x)t^{q} = (1-2xt+t^2)^{\frac{n}{2}} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right). \tag{4.2}$$

Case 2: b = 0, c = 1.

If we choose b = 0 and c = 1 in (3.9) we get

$$e^{C}[P_n(x)y^n] = y^n(y^2+2xy+1)^{-(\frac{n+1}{2})} P_n\left(\frac{x+y}{\sqrt{y^2+2xy+1}}\right). \quad (4.3)$$

Since, we have

$$C\left[P_n(x)y^n\right] = -(n+1)P_{n+1}(x)y^{n+1}$$

For the purpose using it repeatedly, by expanding the left hand side of (4.3) in series form and then simplifying, we obtain the following relation

$$\sum_{k=0}^{\infty}(-1)^k \frac{(n+1)_k}{k!}P_{n+k}(x)y^k = (y^2+2xy+1)^{-(\frac{n+1}{2})} P_n\left(\frac{x+y}{\sqrt{y^2+2xy+1}}\right)$$

Let us replace $y\, by -t$ in the above simplified form so that we will be able to determine one more new generating function for the modified Legendre polynomials, which is as given below.

$$\sum_{k=0}^{\infty}\frac{(n+1)_k}{k!}P_{n+k}(x)t^k = (1-2xt+t^2)^{-(\frac{n+1}{2})} P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right). \quad (4.4)$$

As we know that the Gegenbauer polynomial $C_n^{\nu}(x)$ is the generalization of the Legendre polynomial and it defined by the generating relation $(1-2xt+t^2)^{-r} = \sum_{n=o}^{\infty} C_n^r(x)t^n$. With the aid of this relation, we can rewrite ((4.4)) in the form of

$$\sum_{k=0}^{\infty}\binom{n+k}{k}P_{n+k}(x)t^k = \sum_{r=0}^{\infty} C_r^{(\frac{n+1}{2})}(x)P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}}\right)t^r , \quad (4.5)$$

which is the new generating function that involves modified Legendre polynomial and Gegenbauer polynomial.

Case 3: b = 1, c = -1.

Now, we assign the value to arbitrary constants b and c. Let us take b = 1 and c = -1, so that (3.9) becomes

$$e^{-C}e^{B}[P_n(x)y^n] = (1-2xy+y^2)^{-(\frac{n+1}{2})} P_n\left(\frac{1-xy}{\sqrt{1-2xy+y^2}}\right). \quad (4.6)$$

Separately, we consider the left hand side of (4.6) and we write exponential operators in series form so that we have the following relation:

$$\begin{aligned} e^{-C}e^{B}[P_n(x)y^n] &= \sum_{r,s=0}^{\infty}(-1)^r\frac{C^rB^s}{r!s!}[P_n(x)y^n] \\ &= \sum_{r,s=0}^{\infty}(n+1)_r\frac{B^s}{r!s!}[P_{n+r}(x)y^{n+r}] \\ &= \sum_{s=0}^{\infty}\sum_{r=0}^{n}\frac{n!}{(n-r)!r!s!}B^s[P_n(x)y^n] \end{aligned}$$

Now, we take into the account the action of B on $P_n(x)y^n$. We simplify the left hand side of (4.6) as follows:

$$e^{-C}e^{B}[P_n(x)y^n] = \sum_{s=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-n)_s}{s!}[P_{n-s}(x)y^{n-s}] \quad (4.7)$$

Using (4.6) and (4.7), we get

$$(1-2xy+y^2)^{-(\frac{n+1}{2})}P_n\left(\frac{1-xy}{\sqrt{1-2xy+y^2}}\right)=\sum_{s=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-n)_s}{s!}[P_{n-s}(x)y^{n-s}]$$

Finally, if put $t=y^{-1}$ in the above equation and simplifying, we obtains the following generating function between modified Legendre polynomials, which we believe to be new generating function.

$$(t^2-2xt+1)^{-(\frac{n+1}{2})}P_n\left(\frac{t-x}{\sqrt{t^2-2xt+1}}\right)t^{2n+1}=\sum_{s=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\frac{(-n)_s}{s!}[P_{n-s}(x)t^s]$$

Conclusion

In this section, we conclude the findings of this paper. Basically, we adopt the Weisner method to determine the new generating functions for modified Legendre polynomial. In the beginning of this paper, we defined the linear operators A, B, C, and L. In section 2, we discussed the commutative properties of these operators. We have extended these operators in exponential them to determine the new generating functions such as (4.2), (4.4), (4.5) and (4.8). We also believe that the operators 1, A, B and C form a Lie group.

References

[1] Chatterjea, S. K. (1975), Unification of a class of bilateral generating relations for certain special functions, Bull. Cal. Math. Soc., 67, pp. 115-127.

[2] Chatterjea, S. K. (1977), An extension of a class of bilateral generating functions for certain special functions, Bull. Inst. Math. Acad. Sinica 5(2).

[3] Chongdar, A. K. (1987), On the uni_cation of a class of bilateral generating functions for certain special functions, Tamkang Jour. Math., 18 (3), pp. 53-59.

[4] Cohn, P. M. (1961), Lie groups, Cambridge University Press, Cambridge.

[5] Desale, B. S. and Qashash, G. A. (2011), A General Class of Generating Functions of Lagurre Polynomials, Journal of Inequality and Special Functions, 2(2), pp. 1-7.

[6] Desale, B. S. and Qashash, G. A. (2011), Improper Partial Bilateral Generating Functions for Some Orthogonal Polynomials, Int. Journal of Math. Analysis, 5, no 47, pp. 2321-2328.

[7] Desale, B. S. and Qashash, G. A. (2011), Trilateral Generating Function for Hermite, Jacobi and Bessel Polynomials, Int. Journal of Math. Analysis, 5, no 47, pp. 2329-2335.

[8] McBride, E. B. (1971), Obtaining generating function, Springer Veriag, New York, Heidelberg and Berlin.

[9] Rainville, E. D. (1960), Special Functions, Chelsea Publication Company, New York.

[10] Singhal J. P. and Srivastava, H. M. (1972), A class of bilateral generating functions for certain classical polynomials, Paci_c J. Math., 42, No.3, pp. 755-762.

[11] Srivastava, H. M. and Manocha, H. L. (1984), A Treatise on Generating Functions, Halsted Press, John Wiley & Sons, New York.

[12] Srivastava, H. M. and Karlsson, P. W. (1985), Multiple Gaussian hypergeometric series, Halsted Press, John Wiley & Sons, New York.

[13] Weisner, L. (1955), Group theoretic origin of certain generating functions, Paci_c J. Math. 5, pp.1033-1039 .

[14] Weisner, L. (1959 a), Generating function for Hermite functions, Canad. J. Math. 11, pp.141-147.

[15] Weisner, L. (1959 b), Generating function for Bessel functions, Canadian J. Math. 11, pp.148-155.

الدوال المولدة لكثيرات الحدود ليجندر باستخدام طريقة الزمر النظري

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الملخص

في هذه الورقة نحصل على وظائف توليد لكثيرات الحدود ليجندر في صيغتها المعدلة باستخدام طريقة الزمر النظري لويسنر. وإذ اننا نشرناها لتحديد علاقات التوليد الجديدة بين كثيرات الحدود للجندر المعممة وبطريقة سهلة.
الافكار والإثباتات مستوحاة من كتاب أم سي بريد (راجع 8) التي تم استخدامها لتحديد علاقات توليد جديدة محتواة على كثيرات حدود لجندر المعدلة.

الكلمات المفتاحية: كثيرات حدود ليجندر, الدوال المولدة , المؤثرات التفاضلية, طريقة الزمر النظري.