Some properties for Weyl's projective Curvature Tensor of Generalized W^h-Birecurrent in Finsler Space

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Abstract

In this paper, we defined a Finsler space F_n for which Weyl's projective curvature tensor W^i_{jkh} satisfies the generalized-birecurrence condition with respect to Cartan's connection parameters Γ^{*i}_{kh}, given by the condition $W^i_{jkh|l|m} = \alpha_{lm} W^i_{jkh} + \beta_{lm} \left(\delta^i_h \, g_{jk} - \delta^i_k \, g_{jh}\right)$, where $|l|m$ is h-covariant derivative of second order (Cartan's second kind covariant differential operator) with respect to x^l and x^m, successively, α_{lm} and β_{lm} are non-null covariant vectors field and such space is called as *a generalized* W^h*-birecurrent space* and denoted briefly by $G\, W^h\text{-}BRF_n$ **.** We have obtained the h-covariant derivative of the second order for Wely's projective torsion tensor W^i_{kh}, Wely's projective deviation tensor W^i_h and Weyl's projective curvature tensor W^i_{jkh} and some tensors are birecurrent in our space. We have obtained the necessary and sufficient condition for Cartan's third curvature tensor R^i_{jkh}, the associate curvature tensor R_{jpkh} to be generalized birecurrent, the necessary and sufficient condition of h-covariant derivative of second order for the h(v)-torsion tensor H^i_{kh}, the associate torsion tensor $H_{kp.h}$ and the deviation tensor H^i_h has been obtained in our space.

Key words: Finsler space, Generalized W^h- Birecurrent space, Weyl's projective curvature t ensor W^i_{jkh}, Cartan's third curvature tensor R^i_{jkh}.

1. Introduction

On account of the different connections of Finsler space, the concept of the recurrent for different curvature tensors have been discussed by Matsumot [6]. Pandey ([8], [9]), Dubey and Srivastava [4], Pandey and Misra [10], Pandey and Dwivedi [11], Verma [19], Dikshit [3], Qasem [14], Mishra and Lodhi [7], P.N. Pandey and Pal [12] and others. The generalized recurrent space was studied by De and Guha [2], Maralebhavi and Rathnamma [5], Pandey, Saxena and Goswani [13], Qasem and Al-Qashbari ([17],[18]), Qasem and Saleem[16] and others. Ahsan and Ali [1] who discussed a recurrent curvature tensor on some properties of W-curvature tensor of Weyl's projective curvature tensor W^i_{jkh} and others. Also, W-generalized birecurrent space studied by Qasem and Saleem [15] and others.

Let us consider an n-dimensional Finsler space equipped with the metric function F satisfying the requisite conditions [19].

Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk}. These are symmetric in their lower indices .

The vectors y_i and y^i satisfy the following relations [19]:

(1.1) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$ and c) $\dot{\partial}_j\, y^i = \delta^i_j$.

The h-covariant derivative of second order for an arbitrary vector field with respect to x^k and x^j, successively ,we get

(1.2) $$X^i_{|k|j} = \dot{\partial}_j \left(X^i_{|k}\right) - \left(X^i_{|r}\right)\Gamma^{*r}_{kj} + \left(X^r_{|k}\right)\Gamma^{*i}_{rj} - \dot{\partial}_r \left(X^i_{|k}\right)\Gamma^{*r}_{js}\, y^s \quad .$$

Taking skew-symmetric part with respect to the indices k and j, we get the commutation formula for h-covariant differentiation as follows [19]:

(1.3) $X^i_{|k|j} - X^i_{|j|k} = X^r K^i_{rkj} - (\dot{\partial}_r X^i) K^r_{skj} y^s$,

where

(1.4) $K^i_{rkj} := \partial_j \Gamma^{*i}_{kr} + (\dot{\partial}_l \Gamma^{*i}_{rj}) G^l_k + \Gamma^{*i}_{mj} \Gamma^{*m}_{kr} - j/k$ ** .

The tensor K^i_{rkj} as defined above is called *Cartan's fourth curvature tensor*.

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to the above process.

(1.5) a) $g_{ij|k} = 0$ and b) $y^i_{|k} = 0$.

The process of h-covariant differentiation , with respect to x^k , commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to [19]

(1.6) $\dot{\partial}_j (X^i_{|k}) - (\dot{\partial}_j X^i)_{|k} = X^r (\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i) P^r_{jk}$.

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they are called *h-curvature tensor of Berwald* (*Berwald curvature tensor*) and *h(v)–torsion tensor,* respectively and are defined as follow [19]:

(1.7) a) $H^i_{jkh} := \partial_j G^i_{kh} + G^r_{kh} G^i_{rj} + G^i_{rhj} G^r_k - h/k$

and

b) $H^i_{kh} := \partial_h G^i_k + G^r_k C^i_{rh} - h/k$.

They are also related by [19]

(1.8) a) $H^i_{jkh} y^j = H^i_{kh}$, b) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$ and c) $H^i_{jk} = \dot{\partial}_j H^i_k$.

These tensors were constructed initially by means of the tensor H^i_h , called the *deviation tensor,* given by

(1.9) a) $H^i_h := 2 \partial_h G^i - \partial_r G^i_h y^r + 2 G^i_{hs} G^s - G^i_s G^s_h$,

where b) $\dot{\partial}_k G^i_h = G^i_{kh}$.

In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.8) and (1.9), we have the following:

(1.10) a) $H^i_{jk} y^j = - H^i_{kj} y^j = H^i_k$ and b) $g_{ip} H^i_{jk} = H_{jp.k}$.

The tensor W^i_{jkh} is known as *projective curvature tensor* (*generalized Wely's projective curvature tensor*), the tensor W^i_{jk} is known as *projective torsion tensor* (*Wely's torsion tensor*) and the tensor W^i_j is known as *projective deviation tensor* (*Wely's deviation tensor*) are defined by

(1.11) $$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{n+1} H_{[hk]} + \frac{2\,y^i}{n+1} \dot{\partial}_j H_{[kh]}$$
$$+ \frac{\delta^i_k}{n^2-1}\left(n\, H_{jh} + H_{hj} + y^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(n\, H_{jk} + H_{kj} + y^r \dot{\partial}_j H_{kr}\right) ,$$

(1.12) $$W^i_{jk} = H^i_{jk} + \frac{y^i}{n+1} H_{[jk]} + 2\left\{ \frac{\delta^i_{[j}}{n^2-1}\left(n\, H_{k]} - y^r H_{k]\,r}\right)\right.$$

and

(1.13) $$W^i_j = H^i_j - H\, \delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_r H^r_j - \dot{\partial}_j H\right) y^i ,$$

respectively.

The tensors W^i_{jkh} , W^i_{jk} and W^i_k are satisfying the following identities [19]

(1.14) a) $W^i_{jkh} y^j = W^i_{kh}$ and b) $W^i_{jk} y^j = W^i_k$.

The projective curvature tensor W^i_{jkh} is skew-symmetric in its indices k and h.

Cartan's third curvature tensor R^i_{jkh} and the R-Ricci tensor R_{jk} are respectively given by [19]

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* The indices $i, j, k, \ldots$ assume positive integral values from 1 to n .

** $-j/k$ means the subtraction from the former term by interchanging the indices k and j .

(1.15) a) $R^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + (\dot{\partial}_l \Gamma^{*i}_{jk})\, G^l_h + C^i_{jm}(\dot{\partial}_k\, G^m_h - G^m_{kl}\, G^l_h) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$,

b) $R^i_{jkh}\, y^j = H^i_{kh} = K^i_{jkh}\, y^j$, c) $g_{ip}\, R^i_{jkh} = R_{jpkh}$ and d) $R_{jk}\, y^j = H_k$.

2. A Generalized W^h-Birecurrent Space

A Finsler space F_n for which Weyl's projective curvature tensor W^i_{jkh} satisfies the recurrence property with respect to Cartan's coefficient connection parameters Γ^{*i}_{jk} which is characterized by the condition [16]

(2.1) $W^i_{jkh|l} = \lambda_l\, W^i_{jkh} + \mu_l\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$, $W^i_{jkh} \neq 0$,

where $|l$ is h-covariant derivative of first order (Cartan's second kind covariant differential operator) with respect to x^l, the quantities λ_l and μ_l are non-null covariant vectors field. We shall call such space as *a generalized W^h-recurrent space* and we shall denote it briefly by G W^h-RF_n .

Taking the h-covariant derivative for (2.1) with respect to x^m and using (1.5a), we get

$$W^i_{jkh|l|m} = \lambda_{l|m} W^i_{jkh} + \lambda_l W^i_{jkh|m} + \mu_{l|m}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) \quad , \text{ where } \quad g_{jk|m} = 0 \ ,$$

In view of (2.1), the above equation yields

(2.2) $W^i_{jkh|l|m} = \alpha_{lm} W^i_{jkh} + \beta_{lm}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$,

where $\alpha_{lm} = \lambda_{l|m} + \lambda_l\, \lambda_m$ and $\beta_{lm} = \lambda_l\, \lambda_m + \mu_{l|m}$.

Result 2.1. Every generalized W^h-recurrent space is generalized W^h-birecurrent space.

Transvecting the condition (2.2) by y^j, using (1.5b), (1.14a) and (1.1a), we get

(2.3) $W^i_{kh|l|m} = \alpha_{lm} W^i_{kh} + \beta_{lm}\,(\delta^i_h\, y_k - \delta^i_k\, y_h)$.

Transvecting (2.3) by y^k, using (1.5b), (1.14b) and (1.1b), we get

(2.4) $W^i_{h|l|m} = \alpha_{lm}\, W^i_h + \beta_{lm}\,(\delta^i_h\, F^2 - y_h\, y^i)$.

Thus, we conclude

Theorem 2.1. *In GW^h-BRF_n , the h-covariant derivative of the second order for Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h given by (2.3) and (2.4), respectively.*

In view of the equation (1.3), we have

(2.5) $W^i_{jkh|l|m} - W^i_{jkh|m|l} = W^r_{jkh}\, K^i_{rlm} - W^i_{rkh}\, K^r_{jlm} - W^i_{jrh}\, K^r_{klm} - W^i_{jkr}\, K^r_{hlm} -$ $(\dot{\partial}_r W^i_{jkh}) K^r_{slm} y^s$.

Using the condition (2.2) and (1.15b) in (2.5), we get

$$\alpha_{lm} W^i_{jkh} + \beta_{lm}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) - \alpha_{ml} W^i_{jkh} - \beta_{ml}\,(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$$
$$= W^r_{jkh}\, K^i_{rlm} - W^i_{rkh}\, K^r_{jlm} - W^i_{jrh}\, K^r_{klm} - W^i_{jkr}\, K^r_{hlm} - (\dot{\partial}_r W^i_{jkh})\, H^r_{lm}$$

or

(2.6) $(\alpha_{lm} - \alpha_{ml})\, W^i_{jkh} + (\beta_{lm} - \beta_{ml})(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh})$
$= W^r_{jkh}\, K^i_{rlm} - W^i_{rkh}\, K^r_{jlm} - W^i_{jrh}\, K^r_{klm} - W^i_{jkr}\, K^r_{hlm} - (\dot{\partial}_r W^i_{jkh})\, H^r_{lm}$.

If α_{lm} and β_{lm} are skew-symmetric, then (2.6) can be written as

(2.7) $a_{lm}\, W^i_{jkh} + b_{lm}(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}) = W^r_{jkh}\, K^i_{rlm} - W^i_{rkh}\, K^r_{jlm}$
$-W^i_{jrh}\, K^r_{klm} - W^i_{jkr}\, K^r_{hlm} - (\dot{\partial}_r W^i_{jkh})\, H^r_{lm}$,

where $a_{lm} = 2\, \alpha_{lm}$ and $b_{lm} = 2\, \beta_{lm}$.

In view of the condition (2.2), the equation (2.7) is reduced to

(2.8) $W^i_{jkh|l|m} = W^r_{jkh}\, K^i_{rlm} - W^i_{rkh}\, K^r_{jlm} - W^i_{jrh}\, K^r_{klm} - W^i_{jkr}\, K^r_{hlm} - (\dot{\partial}_r W^i_{jkh})\, H^r_{lm}$.

Transvecting (2.8) by y^j, using (1.5b), (1.14a), (1.15b) and (1.1c), we get

(2.9) $W^i_{kh|l|m} = W^r_{kh} K^i_{rlm} - W^i_{rkh} H^r_{lm} - W^i_{rh} K^r_{klm} - W^i_{kr} K^r_{hlm} - (\dot{\partial}_r W^i_{kh} - W^i_{rkh})\, H^r_{lm}$.

Transvecting (2.9) by y^k, using (1.5b), (1.14b) and (1.15b), we get

(2.10) $$W^i_{h|l|m} = W^r_h K^i_{rlm} - W^i_{rh} H^r_{lm} - W^i_r K^r_{hlm} - \{W^i_{rkh} H^r_{lm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm}\} y^k \quad .$$

Thus, we conclude

Theorem 2.2. *In GW^h-BRF_n , the h-covariant derivative of the second order for Weyl's projective curvature tensor W^i_{jkh} , Wely's projective torsion tensor W^i_{kh} and Wely's projective deviation tensor W^i_h given by (2.8), (2.9) and (2.10), respectively, provided α_{lm}* and *β_{lm} are skew-symmetric.*

If α_{lm} and β_{lm} are symmetric, then (2.6) can be written as

(2.11) $$W^r_{jkh} K^i_{rlm} = W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm} \quad .$$

Now, taking h- covariant derivative for (2.11), with respect to x^n , we get

(2.12) $$W^r_{jkh|n} K^i_{rlm} + W^r_{jkh} K^i_{rlm|n} = \{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm}\}_{|n} \quad .$$

Again, taking h- covariant derivative for (2.12), with respect to x^p , we get

$$W^r_{jkh|n|p} K^i_{rlm} + W^r_{jkh|n} K^i_{rlm|p} + W^r_{jkh|p} K^i_{rlm|n} + W^r_{jkh} K^i_{rlm|n|p}$$
$$= \{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm}\}_{|n|p} \quad .$$

Using the condition (2.2) in the above equation, we get

(2.13) $$\alpha_{np} W^i_{jkh} K^i_{rlm} + \beta_{np} (\delta^i_h g_{jk} - \delta^i_k g_{jh}) K^i_{rlm} + W^r_{jkh|n} K^i_{rlm|p} + W^r_{jkh|p} K^i_{rlm|n}$$
$$+ W^r_{jkh} K^i_{rlm|n|p} = \{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm}\}_{|n|p} \, .$$

In view of (2.11), the equation (2.13) is reduced to

(2.14) $$\{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm} \}_{|n|p}$$
$$= \alpha_{np} \{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm}\}$$
$$+ \beta_{np} (\delta^i_h g_{jk} - \delta^i_k g_{jh}) K^i_{rlm} + W^r_{jkh|n} K^i_{rlm|p} + W^r_{jkh|p} K^i_{rlm|n} + W^r_{jkh} K^i_{rlm|n|p} \quad .$$

This shows that

$$\{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm} \}_{|n|p}$$
$$= \alpha_{np} \{ W^i_{rkh} K^r_{jlm} + W^i_{jrh} K^r_{klm} + W^i_{jkr} K^r_{hlm} + (\dot{\partial}_r W^i_{jkh}) H^r_{lm} \} \quad ,$$

if and only if

(2.15) $$\beta_{np} (\delta^i_h g_{jk} - \delta^i_k g_{jh}) K^i_{rlm} + W^r_{jkh|n} K^i_{rlm|p} + W^r_{jkh|p} K^i_{rlm|n} + W^r_{jkh} K^i_{rlm|n|p} = 0 \quad .$$

Transvecting (2.14) by y^j, using (1.5b), (1.15b), (1.14a),(1.1c) and (1.1a), we get

(2.16) $$\{ W^i_{rkh} H^r_{lm} + W^i_{rh} K^r_{klm} + W^i_{kr} K^r_{hlm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm} \}_{|n|p}$$
$$= \alpha_{np} \{ W^i_{rkh} H^r_{lm} + W^i_{rh} K^r_{klm} + W^i_{kr} K^r_{hlm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm} \}$$
$$+ \beta_{np} (\delta^i_h y_k - \delta^i_k y_h) K^i_{rlm} + W^r_{kh|n} K^i_{rlm|p} + W^r_{kh|p} K^i_{rlm|n} + W^r_{kh} K^i_{rlm|n|p} \quad .$$

This shows that

(2.17) $$\{ W^i_{rkh} H^r_{lm} + W^i_{rh} K^r_{klm} + W^i_{kr} K^r_{hlm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm} \}_{|n|p}$$
$$= \alpha_{np} \{ W^i_{rkh} H^r_{lm} + W^i_{rh} K^r_{klm} + W^i_{kr} K^r_{hlm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm} \}$$

if and only if

(2.18) $$\beta_{np} (\delta^i_h y_k - \delta^i_k y_h) K^i_{rlm} + W^r_{kh|n} K^i_{rlm|p} + W^r_{kh|p} K^i_{rlm|n} + W^r_{kh} K^i_{rlm|n|p} = 0 \quad .$$

Transvecting (2.16) by y^k, using (1.5b), (1.15b), (1.14b), (1.14a) and (1.1b), we get

(2.19) $$[W^i_{rh} H^r_{lm} + W^i_r K^r_{hlm} + \{ W^i_{rkh} H^r_{lm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm}\} y^k]_{|n|p}$$
$$= \alpha_{np} [W^i_{rh} H^r_{lm} + W^i_r K^r_{hlm} + \{ W^i_{rkh} H^r_{lm} + (\dot{\partial}_r W^i_{kh} - W^i_{rkh}) H^r_{lm}\} y^k]$$
$$+ \beta_{np} (\delta^i_h F^2 - y_h y^i) K^i_{rlm} + W^r_{h|n} K^i_{rlm|p} + W^r_{h|p} K^i_{rlm|n} + W^r_h K^i_{rlm|n|p} \quad .$$

This shows that

(2.20) $\left[W^i_{rh}\, H^r_{lm} + W^i_r\, K^r_{hlm} + \left\{ W^i_{rkh}\, H^r_{lm} + \left(\dot\partial_r W^i_{kh} - W^i_{rkh}\right) H^r_{lm}\right\} y^k \right]_{|n|p}$

$= \alpha_{np} \left[W^i_{rh}\, H^r_{lm} + W^i_r\, K^r_{hlm} + \left\{ W^i_{rkh}\, H^r_{lm} + \left(\dot\partial_r W^i_{kh} - W^i_{rkh}\right) H^r_{lm}\right\} y^k \right]$

if and only if

(2.21) $\beta_{np} \left(\delta^i_h F^2 - y_h y^i\right) K^i_{rlm} + W^r_{h|n} K^i_{rlm|p} + W^r_{h|p} K^i_{rlm|n} + W^r_h K^i_{rlm|n|p} = 0$.

Thus, we conclude

Theorem 2.3. *In* GW^h-BRF_n *, the tensors* $\{ W^i_{rkh} K^r_{jlm} + W^i_{jrh}\, K^r_{klm} + W^i_{jkr}\, K^r_{hlm} + \left(\dot\partial_r W^i_{jkh}\right) H^r_{lm} \}$, $\{ W^i_{rkh} H^r_{lm} + W^i_{rh}\, K^r_{klm} + W^i_{kr}\, K^r_{hlm} + \left(\dot\partial_r W^i_{kh} + W^i_{rkh}\right) H^r_{lm} \}$ *and* $\left[W^i_{rh}\, H^r_{lm} + W^i_r\, K^r_{hlm} + \left\{ W^i_{rkh}\, H^r_{lm} + \left(\dot\partial_r W^i_{kh} + W^i_{rkh}\right) H^r_{lm}\right\} y^k \right]$ *are behaves as birecurrent, if and only if (2.15), (2.18) and (2.21) hold, respectively (provided that* α_{lm} and β_{lm} *are symmetric).*

3. The Necessary and Sufficient Condition

In this section, we shall obtain the necessary and sufficient condition for some tensors to be generalized birecurrent in GW^h-BRF_n .

We know that Weyl's projective curvature tensor W^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} are connected by the formula ([1], [3])

(3.1) $W^i_{jkh} = R^i_{jkh} + \frac{1}{3} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)$.

Taking h-covariant derivative of (3.1), with respect to x^l, we get

(3.2) $W^i_{jkh|l} = R^i_{jkh|l} + \frac{1}{3} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)_{|l}$.

Again, taking h- covariant derivative of (3.2), with respect to x^m , we get

(3.3) $W^i_{jkh|l|m} = R^i_{jkh|l|m} + \frac{1}{3} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)_{|l|m}$.

Using the condition (2.2) in (3.3), we get

$$\alpha_{lm} W^i_{jkh} + \beta_{lm} \left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) = R^i_{jkh|l|m} + \frac{1}{3} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)_{|l|m} .$$

By using (3.1), the above equation can be written as

(3.4) $R^i_{jkh|l|m} = \alpha_{lm}\, R^i_{jkh} + \beta_{lm} \left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh} \right)$

$+ \frac{1}{3}\, \alpha_{lm} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right) - \frac{1}{3} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)_{|l|m}$.

This shows that

(3.5) $R^i_{jkh|l|m} = \alpha_{lm}\, R^i_{jkh} + \beta_{lm} \left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh} \right)$,

if and only if

(3.6) $\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)_{|l|m} = \alpha_{lm} \left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)$.

Thus, we conclude

Theorem 3.1. *In* GW^h - BRF_n *, Cartan's third curvature tensor* R^i_{jkh} *is generalized birecurrent if and only if the tensor* $\left(\delta^i_k\, R_{jh} - g_{jk}\, R^i_h \right)$ *is birecurrent.*

Transvecting (3.4) by g_{ip} , using (1.5a) and (1.15c), we get

$$R_{jpkh|l|m} = \alpha_{lm}\, R_{jpkh} + \beta_{lm} \left(g_{ph}\, g_{jk} - g_{pk}\, g_{jh} \right)$$
$$+ \frac{1}{3}\, \alpha_{lm} \left(g_{pk}\, R_{jh} - g_{jk}\, R_{ph} \right) - \frac{1}{3} \left(g_{pk}\, R_{jh} - g_{jk}\, R_{ph} \right)_{|l|m} ,$$

where $g_{ip}\, R^i_h = R_{ph}$, this shows that

(3.7) $R_{jpkh|l|m} = \alpha_{lm}\, R_{jpkh} + \beta_{lm} \left(g_{ph}\, g_{jk} - g_{pk}\, g_{jh} \right)$,

if and only if

(3.8) $\left(g_{pk}\, R_{jh} - g_{jk}\, R_{ph} \right)_{|l|m} = \alpha_{lm} \left(g_{pk}\, R_{jh} - g_{jk}\, R_{ph} \right)$.

Thus, we conclude

Theorem 3.2. *In* GW^h-BRF_n *, the associate curvature tensor* R_{jpkh} *is generalized birecurrent if and only if the tensor* $\left(g_{pk}\, R_{jh} - g_{jk}\, R_{ph} \right)$ *is birecurrent.*

Transvecting (3.4) by y^j , using (1.5b), (1.15b), (1.1a) and (1.15d), we get

(3.9) $H^i_{kh|l|m} = \alpha_{lm}\, H^i_{kh} + \beta_{lm}(\, \delta^i_h\, y_k - \delta^i_k\, y_h\,)$
$+ \frac{1}{3}\, \alpha_{lm}(\, \delta^i_k\, H_h - y_k\, R^i_h\,) - \frac{1}{3}\,(\, \delta^i_k\, H_h - y_k\, R^i_h\,)_{|l|m}$.

This shows that

(3.10) $H^i_{kh|l|m} = \alpha_{lm}\, H^i_{kh} + \beta_{lm}(\, \delta^i_h\, y_k - \delta^i_k\, y_h\,)$

if and only if

(3.11) $(\, \delta^i_k\, H_h - y_k\, R^i_h\,)_{|l|m} = \alpha_{lm}(\, \delta^i_k\, H_h - y_k\, R^i_h\,)$.

Thus, we conclude

Theorem 3.3. *In GW^h-BRF_n , the h-covariant derivative of the second order for the h(v)–torsion tensor H^i_{kh} is geven by (3.10) if and only if the tensor $(\, \delta^i_k\, H_h - y_k\, R^i_h\,)$ is birecurrent.*

Also, transvecting (3.9) by y^k , using (1.5b), (1.10a) and (1.1b), we get

$H^i_{h|l|m} = \alpha_{lm}\, H^i_h + \beta_{lm}(\, \delta^i_h\, F^2 - y_h\, y^i\,)$
$+ \frac{1}{3}\, \alpha_{lm}(\, H_h\, y^i - F^2\, R^i_h\,) - \frac{1}{3}\,(\, H_h\, y^i - F^2\, R^i_h\,)_{|l|m}$.

This shows that

(3.12) $H^i_{h|l|m} = \alpha_{lm}\, H^i_h + \beta_{lm}(\, \delta^i_h\, F^2 - y_h\, y^i\,)$

if and only if

(3.13) $(\, H_h\, y^i - F^2\, R^i_h\,)_{|l|m} = \alpha_{lm}(\, H_h\, y^i - F^2\, R^i_h\,)$.

Thus, we conclude

Theorem 3.4. *In GW^h-BRF_n , the h-covariant derivative of the second order for the deviation tensor H^i_h is geven by (3.12) if and only if the tensor $(H_h\, y^i - F^2\, R^i_h\,)$ is birecurrent.*

Further, transvecting (3.9) by g_{ip} , using (1.5a) and (1.10b), we get

$H_{pk.h|l|m} = \alpha_{lm}\, H_{pk.h} + \beta_{lm}\,(\, g_{ph}\, y_k - g_{pk}\, y_h\,)$
$+ \frac{1}{3}\, \alpha_{lm}(\, g_{pk}\, H_h - y_k\, R_{ph}) - \frac{1}{3}\,(\, g_{pk}\, H_h - y_k\, R_{ph})_{|l|m}$,

where $g_{ip}\, R^i_h = R_{ph}$, this shows that

(3.14) $H_{pk.h|l|m} = \alpha_{lm}\, H_{pk.h} + \beta_{lm}\,(\, g_{ph}\, y_k - g_{pk}\, y_h\,)$

if and only if

(3.15) $(\, g_{pk}\, H_h - y_k\, R_{ph})_{|l|m} = \alpha_{lm}(\, g_{pk}\, H_h - y_k\, R_{ph})$.

Thus, we conclude

Theorem 3.5. *In GW^h-BRF_n , the h-covariant derivative of the second order for the associate curvature tensor $H_{kp.h}$ is geven by (3.14) if and only if the tensor $(g_{pk}\, H_h - y_k\, R_{ph})$ is birecurrent.*

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بعض الخصائص للموتر الإسقاطي لويلي في تعميم فضاء فنسلر W^h- ثُنائي المُعاودة

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الملخص

في هذه الورقة، عرفنا فضاء فنسلر F_n الذي يحقق الموتر التقوسي لويلي W^i_{jkh} في مفهوم كارتان بالحالة الآتية :

$$W^i_{jkh|l|m} = \alpha_{lm} W^i_{jkh} + \beta_{lm}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) \qquad , \qquad W^i_{jkh} \neq 0 \quad ,$$

حيث $|l|m$ هي مشتقة h– متحدة الاختلاف من الرتبة الثانية (مشتقة كارتان من النوع الثاني) بالنسبة إلى المسقطين الوضعيين x^m و x^l على التعاقب إذ أن α_{lm} و β_{lm} هي حقول غير صفرية لمتجهات متحدة الاختلاف وأطلقنا عليه تعميم فضاء فنسلر W^h - ثنائي المعاودة ورمزنا إليه بالرمز التالي $GW^h\text{-}BRF_n$. أوجدنا المشتقة h– المتحدة الاختلاف من المرتبة الثانية للموتر الألتوائي لويلي W^i_{kh} و للموتر الأنحرافي لويلي W^i_h و كذلك للموتر الاسقاطي لويلي W^i_{jkh} في هذا الفضاء . كذلك أوجدنا الشرط اللازم والكافي للموتر القوسي الثالث لكارتان R^i_{jkh} ومرافقه لتكون ثناية المعاودة المعممة ، ثم أوجدنا الشرط اللازم والكافي لإيجاد المشتقة h– المتحدة الاختلاف من المرتبة الثانية للموتر الإلتوائي H^i_{kh} ، الموتر المرافق الإلتوائي $H_{kp.h}$ وكذلك للموتر الأنحرافي H^i_h في هذا الفضاء.

الكلمات مفتاحية: فضاء فنسلر ، تعميم فضاء فنسلر W^h - ثُنائي المُعاودة ، الموتر الأسقاطي لويلي W^i_{jkh} ، الموتر التقوسي الثالث لكارتان R^i_{jkh} .