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Research Article

A study of the concircular curvature tensor and its interactions with other tensors under the Lie derivative in G$\mathcal{B}$K- 5RF$_n$

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Abstract

This research paper delves into a comprehensive analysis of the concircular curvature tensor and its intricate relationships with other tensors under the Lie derivative. The concircular curvature tensor, a fundamental geometric invariant, plays a pivotal role in characterizing the local geometry of Riemannian manifolds. By employing the powerful tool of the Lie derivative, we explore how the concircular curvature tensor transforms under infinitesimal transformations of the underlying manifold. Our study uncovers novel connections between the concircular curvature tensor and other significant tensors, such as the Ricci tensor and Weyl tensor, providing deeper insights into the geometric structure of Riemannian spaces. The results obtained in this paper not only contribute to the advancement of differential geometry but also have potential applications in various fields, including general relativity and theoretical physics.This research expands the definition of concircular curvature tensor within the context of generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor $\boldsymbol{K^i_{jkh}}$ in sense of Berwald. By employing the **Lie**-derivative, we delve into the various connections between concircular, conformal, conharmonice curvature tensors and the Cartan's third curvature tensor $\boldsymbol{R^i_{jkh}}$ ..

1. Introduction

The concircular curvature tensor, introduced by Yano and Bochner, is a geometric invariant that has garnered significant attention due to its close connection with the conformal geometry of Riemannian manifolds. This tensor, defined as a specific linear combination of the Riemann curvature tensor and the Ricci tensor, encapsulates information about the local shape and curvature of a manifold. In this paper, we embark on a systematic investigation of the concircular curvature tensor and its interactions with other tensors under the Lie derivative.

The Lie derivative, a fundamental operation in differential geometry, provides a means to study the infinitesimal changes of geometric objects along vector fields. By applying the Lie derivative to the concircular curvature tensor, we aim to uncover new properties and relationships that can shed light on the geometric structure of Riemannian manifolds. Our study is motivated by the belief that a deeper understanding of the concircular curvature tensor and its connections with other tensors will have far-reaching implications in various areas of geometry and physics.

The Lie derivative is a fundamental concept in differential geometry that measures the rate of change of a tensor field along the flow of another vector field. In simpler terms, it tells us how a geometric object (like a vector field or a tensor) changes as we move it along a curve defined by a particular vector field.

Here's a more formal definition: Given a smooth manifold M, a vector field v on M, and a tensor field T on M, the Lie derivative of T with respect to v, denoted by L_v(T), is a new tensor field of the same type as T that measures the infinitesimal change of T along the flow generated by v .

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Ahsan and Ali [3] investigated the concircular curvature tensor, establishing relationships between it and the conformal, conharmonic, and projective curvature tensors. Ali et al. [5] further explored curvature tensors within conformally flat spaces.

Al-Qashbari et al. ([8], [9], [11]) introduced generalized recurrence in higher-order Finsler spaces and examined various recurrence curvature tensors. Abdallah [1] studied the relationship between two curvature tensors in Finsler spaces, while Al-Qashbari [7] introduced the R-projective curvature tensor in recurrent Finsler space. Abdallah and Hardan [2] explored P-third-order generalized Finsler space in the Berwald sense.

Ali et al. [6] investigated identities involving the conharmonic curvature and other tensors. Bidabad and Sepasi [13] completed the study of Finsler spaces with constant negative Ricci curvature. The curvature tensor field on D-recurrent Finsler space was studied by Atashafrouz and Najafi [12], while Voicu [18] established the Finsler spacetime condition for metrics. Al-Qashbari and Baleedi [10] introduced a Lie derivative in generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor in the Berwald sense. Ali et al. [4] studied the Lie derivative of the M-projective curvature tensor and its properties. The Lie derivative of forms and their applications were explored by various authors ([14]-[17]). We consider a generalized fifth recurrent Finsler space that fulfills the condition outlined in [9].

$$\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right) = \frac{1}{g_{rj}}\Big[(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}) + \frac{1}{2}\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\big)\big] + \Big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\Big[\frac{R}{6}\big(g_{ih}g_{jk} - g_{ik}g_{jh}\big)\Big]\Big]\Big]. \quad \text{..... (1.1)}$$

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{sqlnm}R^i_{jkh} + b_{sqlnm}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - c_{sqlnm}\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big) - d_{sqlnm}\big(\delta^i_h C_{jkl} - \delta^i_k C_{jhl}\big) - e_{sqlnm}\big(\delta^i_h C_{jkq} - \delta^i_k C_{jhq}\big) - 2b_{qlnm}y^r\mathcal{B}_r\big(\delta^i_h C_{jks} - \delta^i_k C_{jhs}\big), \quad \text{...(1.2)}$$

If and only if

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(C^i_{jt}H^t_{kh}) - a_{sqlnm}(C^i_{jt}H^t_{kh}) = 0 \quad \text{...(1.3)}$$

A conformal curvature tensor C_{ijkh}(also know as Weyl conformal curvature tensor) is defined as [3]:

$$R_{ijkh} = C_{ijkh} + \frac{1}{2}\big(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\big) + \frac{R}{6}\big(g_{ih}g_{jk} - g_{ik}g_{jh}\big). \quad (1.4)$$

A conharmonice curvature tensor L^i_{jkh} is defined as [3]:

$$L^i_{jkh} = R^i_{jkh} - \frac{1}{2}\big(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\big). \quad (1.5)$$

The following relationships hold for the metric tensor g_{ij} and the Kronecker delta δ^i_h

$$g_{ij}\, g^{ik} = \delta^k_j = \begin{cases} 1 & if \quad j = k\ , \\ 0 & if \quad j \neq k\ . \end{cases} \quad (1.6)$$

The associate curvature tensor R_{ijkh} of the Cartan's third curvature tensor R^i_{jkh} is defined by

$$R_{ijkh} = g_{rj}R^r_{ikh} \quad (1.7)$$

An affinely connected space, also known as a Berwald space, can be defined by one of several equivalent conditions (as detailed in [9]).

a) $\mathcal{B}_k g_{ij} = 0$ and b) $\mathcal{B}_k g^{ij} = 0.$ (1.8)

Using Eq. (1.8a) in Eq. (1.1) and taking the Lie- derivative of both sides of the result equation, we have

$$L_v\big(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\big) = \frac{1}{g_{rj}}\Big[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}) + \frac{1}{2}L_v\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\big)\big] + L_v\Big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\Big[\frac{R}{6}\big(g_{ih}g_{jk} - g_{ik}g_{jh}\big)\Big]\Big]\Big] - \frac{1}{g_{rj}}(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh})L_v g_{rj}\,. \quad (1.9)$$

By applying the fifth-order Berwald covariant derivative to Equation (1.5) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}) - \frac{1}{2}L_v\big[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\big(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\big)\big]$$

(1.10)

...(1.1)

2. On the Concircular Curvature Tensor in G$\mathcal{B}$K-5RF_n

In this paper, we delve into the intricate properties and applications of the concircular curvature tensor. We provide a comprehensive overview of this fundamental tensor in differential geometry, exploring its relationship with other curvature tensors and its role in characterizing various geometric structures.

Definition 2.1: A concircular curvature tensor M_{ijkh} is defined as (see [3]):

$$M_{ijkh} = R_{ijkh} - \frac{R}{12}\big(g_{jk}g_{ih} - g_{jh}g_{ik}\big) \quad (2.1)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.1) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh} - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right] \quad \ldots(2.2)$$

By applying the Lie-derivative operator to both sides of Equation (2.2), we arrive at

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}) - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right]. \quad \ldots(2.3)$$

Using Eq. (1.7) in Eq. (2.3), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (g_{rj}R^r_{ikh})) - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right]. \quad \ldots(2.4)$$

Using Eq. (1.8a) in Eq. (2.4), we get

$$L_v\left(g_{rj}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right) = L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}\right) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right]. \quad \ldots(2.5)$$

Which can be written as

$$L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right) = \frac{1}{g_{rj}}\left[L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}\right) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right]\right] - \frac{1}{g_{rj}}\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right)L_v g_{rj}. \quad \ldots(2.6)$$

Therefore, we can state the following theorem

Theorem 2.1. In $GBK - 5RF_n$ (in the sense of Berwald space),the Lie-derivative of Berwald covariant derivative of the fifth order for the Cartan's third curvature tensor R^r_{ikh} and the concircular curvature tensor M_{ijkh} linking together by the relation (2.6).

From Eq. (2.6), we get

$$L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}\right) = g_{rj}L_v\left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right). \quad (2.7)$$

If and only if

$$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right] - \left(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}\right)L_v g_{rj} = 0. \quad (2.8)$$

Thus, we conclude

Corollary 2.1: In $GBK - 5RF_n$ (in the sense of Berwald space), the Lie-derivatives of Berwald covariant derivative of the fifth order for concircular curvature tensor M_{ijkh} and the Cartan's third curvature tensor R^r_{ikh} are codirectional if and only if (2.8) holds good.

Taking the Lie-derivative of both sides of Eq. (2.1), we get

$$L_v M_{ijkh} = L_v R_{ijkh} - L_v\left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]. \quad (2.9)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.9) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v M_{ijkh}\right) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v R_{ijkh}\right) - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v\left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]. \quad (2.10)$$

Using Eq. (1.7) in Eq. (2.10), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v M_{ijkh}\right) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[L_v\left(g_{rj}R^r_{ikh}\right)\right] - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v\left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]. \quad (2.11)$$

In view of Eqs. (2.4) and (2.11), we get

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(L_v M_{ijkh}\right) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}). \quad (2.12)$$

If and only if

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[L_v\left(g_{rj}R^r_{ikh}\right)\right] - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L_v\left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right] = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m (g_{rj}R^r_{ikh})) - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m \left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]\right]. \quad (2.13)$$

Hence, the following theorem holds

Theorem 2.2. In $GBK - 5RF_n$,the Lie-derivative for the concircular curvature tensor M_{ijkh} and the Berwald covariant derivative of the fifth order are commutative [provided (2.13) holds].

Transvecting Eq. (2.1) by Cartan's third curvature tensor R^i_{jkh} , we get

$$R^i_{jkh}M_{ijkh} = R^i_{jkh}R_{ijkh} - R^i_{jkh}\left[\frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right)\right]. \quad (2.14)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.14) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

…(2.7)

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}\, M_{ijkh}) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}\, R_{ijkh}) - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[R^i_{jkh}\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]. \qquad (2.15)$$

Taking the Lie-derivative of both sides of Eq. (2.15), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}\, M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^i_{jkh}\, R_{ijkh})] - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[R^i_{jkh}\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]\right]. \quad (2.16)$$

Consequently, we have the following theorem

Theorem 2.3. In $G\mathcal{B}K - 5RF_n$,the Lie-derivative of Berwald covariant derivative of the fifth order for the product of the Cartan's third curvature tensor R^i_{jkh} and the concircular curvature tensor M_{ijkh} is giving by (2.16).

Using Eqs. (1.2) and (1.6) in Eq. (2.6), we get

$$L_v(a_{sqlnm}R^r_{ikh}) = \frac{1}{g_{rj}}\left[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]\right] - \frac{1}{g_{rj}}(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh})L_v g_{rj}\,. \qquad (2.17)$$

If and only if (1.3) holds good.

Which can be written as

$$L_v(R^r_{ikh}) = \frac{1}{(a_{sqlnm}\, g_{rj})}\left[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]\right] - \frac{1}{(a_{sqlnm}\, g_{rj})}(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh})L_v g_{rj} - \frac{1}{(a_{sqlnm})}R^r_{ikh}\, L_v(a_{sqlnm})\,. \qquad (2.18)$$

If and only if (1.3) holds good.

As a result, we arrive at the following theorem

Theorem 2.4. In $G\mathcal{B}K - 5RF_n$ (in the sense of Berwald space), the Lie-derivative for the Cartan's third curvature tensor R^r_{ikh} is giving by (2.18) [provided (1.3) holds].

Adding the Cartan's third curvature tensor R^r_{ikh} of both sides of Eq. (2.1), we get

$$R^r_{ikh} + M_{ijkh} = R^r_{ikh} + R_{ijkh} - \frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\,. \quad (2.19)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.19) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + M_{ijkh}) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + R_{ijkh}) - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]. \qquad (2.20)$$

…(2.15)

Taking the Lie-derivative of both sides of Eq. (2.20), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + R_{ijkh})] - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]. \qquad (2.21)$$

…(2.16)

It follows that the following theorem is true

Theorem 2.5. In $G\mathcal{B}K - 5RF_n$,the Lie- derivative of Berwald covariant derivative of the fifth order for the addition of the Cartan's third curvature tensor R^r_{ikh} and the concircular curvature tensor M_{ijkh} is giving by (2.21).

Adding Eq. (2.6) with Eq. (2.3), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}) + L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) = \frac{1}{g_{rj}}\left[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]\right] - \frac{1}{g_{rj}}(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh})L_v g_{rj} + L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}) - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]. \qquad (2.22)$$

…(2.17)

In view of Eqs. (2.21) and (2.22), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + M_{ijkh})] = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh}) + L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}). \quad (2.23)$$

If and only if

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(R^r_{ikh} + R_{ijkh})] - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right] = \frac{1}{g_{rj}}\left[L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]\right] - \frac{1}{g_{rj}}(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^r_{ikh})L_v g_{rj} + L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}) - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}(g_{jk}g_{ih} - g_{jh}g_{ik})\right]\right]. \qquad (2.24)$$

…(2.18)

Therefore, we can state the following theorem

Theorem 2.6. In $G\mathcal{B}K - 5RF_n$ (in the sense of Berwald space), the Lie-derivative is distributive on the addition of Berwald covariant derivative of the fifth order for the Cartan's third curvature tensor R^r_{ikh} and the concircular

…(2.19)

curvature tensor M_{ijkh} [provided the equation (2.24) holds].

Using Eq. (1.4) in Eq. (2.1), we get

$$M_{ijkh} = C_{ijkh} + \frac{1}{2}\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) + \frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right). \quad (2.25)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.25) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh} = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh} + \frac{1}{2}\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right) + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]. \quad (2.26)$$

Taking the Lie-derivative of both sides of Eq. (2.26), we get

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}) + \frac{1}{2}L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right)\right] + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right]. \quad (2.27)$$

Which can be written as

$$L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{ijkh}) = L_v(\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m C_{ijkh}). \quad (2.28)$$

If and only if

$$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{ik}R_{jh} + g_{jh}R_{ik} - g_{ih}R_{jk} - g_{jk}R_{ih}\right)\right] + 2L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right] = 0. \quad (2.29)$$

From this, we may conclude the following theorem

Theorem 2.7. In $G\mathcal{B}K - 5RF_n$,the Lie-derivative of Berwald covariant derivative of the fifth order for the concircular curvature tensor M_{ijkh} and the conformal curvature tensor C_{ijkh} both are equal [provided (2.29) holds].

Adding the conharmonice curvature tensor L^i_{jkh} of both sides of Eq. (2.1), we get

$$L^i_{jkh} + M_{ijkh} = L^i_{jkh} + R_{ijkh} - \frac{R}{12}\left(g_{jk}g_{ih} - g_{jh}g_{ik}\right). \quad (2.30)$$

By applying the fifth-order Berwald covariant derivative to Equation (2.30) with respect to x^m, x^n, x^l, x^q and x^s, followed by the application of the Lie-derivative to both sides of the resulting equation, we obtain:

$$\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh} + M_{ijkh}) = \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh} + \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh} - \mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]. \quad (2.31)$$

Taking the Lie-derivative of both sides of Eq. (2.31), we get

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh} + M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh}] + L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}] - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right]. \quad (2.32)$$

…(2.25)

Which can be written as

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh} + M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m L^i_{jkh}] + L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}]. \quad (2.33)$$

If and only if

$$L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right] = 0. \quad (2.34)$$

In summary, we have proven the following theorem

Theorem 2.8. In $G\mathcal{B}K - 5RF_n$, the concircular curvature tensor M_{ijkh} is transforming to the associate curvature tensor R_{ijkh} when the Lie-derivative is distributive on the addition of Berwald covariant derivative of the fifth order for the conharmonice curvature tensor L^i_{jkh} and the concircular curvature tensor M_{ijkh} [provided (2.34) holds].

Using Eq. (1.10) in Eq. (2.32), we get

…(2.27)

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh} + M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}] + L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}] - \frac{1}{2}L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\right)\right] - L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right]. \quad (2.35)$$

Which can be written as

$$L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(L^i_{jkh} + M_{ijkh})] = L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh}] + L_v[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{ijkh}]. \quad (2.36)$$

If and only if

$$\frac{1}{2}L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k\right)\right] + L_v\left[\mathcal{B}_s\mathcal{B}_q\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left[\frac{R}{12}\left(g_{ih}g_{jk} - g_{ik}g_{jh}\right)\right]\right] = 0. \quad (2.37)$$

To summarize, the following theorem holds

Theorem 2.9. In $G\mathcal{B}K - 5RF_n$, the conharmonice curvature tensor L^i_{jkh} and the concircular curvature tensor M_{ijkh} are transforming to the Cartan's third curvature tensor R^i_{jkh} and the associate curvature tensor R_{ijkh} respectively when the Lie-derivative is distributive on the addition of Berwald covariant derivative of the fifth order for the conharmonice curvature tensor L^i_{jkh} and the concircular curvature tensor M_{ijkh} [provided (2.37) holds].

…(2.31)

3. Conclusions

In this research, we have conducted a comprehensive study of the concircular curvature tensor and its interactions with other tensors under the Lie derivative. Our findings have revealed novel connections between the concircular curvature tensor and the Ricci and Weyl tensors, providing deeper insights into the local geometry of Riemannian manifolds. This research contributes significantly to the field of differential geometry by offering a more precise description of concircular geometry and providing new tools for analyzing the geometric properties of Riemannian manifolds. Moreover, our results have potential applications in general relativity, where they can be used to study the geometric properties of spacetime. In this paper an attempt has been made to investigate the relationship between concircular curvature tensor and other curvature tensors especially conformal, conharmonice curvature tensor and Cartan's third curvature tensor R^i_{jkh} by **Lie**-derivative in generalized fifth recurrent Finsler space for Cartan's fourth curvature tensor K^i_{jkh} in sense of Berwald.

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بحث علمي

دراسة موتر الانحناء الدائري وتفاعلاته مع موترات أخرى تحت مشتقة لي في $G\mathcal{B}K - 5RF_n$

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الملخص

تلخص هذه الورقة البحثية تحليلاً شاملاً لموتر الانحناء الدائري وعلاقاته المعقدة مع موترات أخرى تحت مشتقة لي. يعتبر موتر الانحناء الدائري، وهو ثابت هندسي أساسي، عنصراً محورياً في تحديد الهندسة المحلية للمفكوكات الريمانية. من خلال استخدام الأداة القوية لمشتقة لي، نستكشف كيف يتحول موتر الانحناء الدائري تحت التحويلات اللامتناهية الصغيرة للمفكوك الأساسي. تكشف دراستنا عن روابط جديدة بين موتر الانحناء الدائري وموترات مهمة أخرى، مثل موتر ريتشي وموتر وايل، مما يوفر رؤى أعمق في البنية الهندسية للمفكوكات الريمانية. لا تساهم النتائج التي تم الحصول عليها في هذه الورقة فقط في تقدم الهندسة التفاضلية، بل لها أيضًا تطبيقات محتملة في مجالات مختلفة، بما في ذلك النسبية العامة والفيزياء النظرية. يوسع هذا البحث تعريف موتر الانحناء الدائري في سياق فضاء فينسلر العام المتكرر من الدرجة الخامسة لموتر انحناء كارتان الرابع K^i_{jkh} بحسب بيروالد. من خلال استخدام المشتقة، نتعمق في الروابط المختلفة بين موترات الانحناء الدائري، والتوافقي، والتوافقي المضاد، وموتر انحناء كارتان الثالث R^i_{jkh} .

مفاتيح البحث

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كلمات مفتاحية:
فضاء فينسلر العام للاشتقاق من الدرجة الخامسة, مشتقة لي L_v ، موتر الانحناء المطابق C_{ijkh} ، موتر الانحناء التوافقي المضاد L^i_{jkh} ، موتر الانحناء الدائري M^i_{jkh} .