University of Aden Journal of Natural and Applied Sciences., vol.28(2) (2024) 97–105

University of Aden Journal of Natural and Applied Sciences

Journal homepage: https://uajnas.adenuniv.com

Research Article

Extended Hyperbolic Function Method to Solve Two New Nonlinear Partial Differential Equations

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https://doi.org/10.47372/uajnas.2024.n2.a09

ARTICLE INFO

Received: 20 Dec 2024

Accepted: 10 Jan 2025

Keywords:

Extended hyperbolic function method (EHFM), exact solutions, c((KdV-BBM)-m(KdV-BBM)) equation and c((SWW-AKNS)-EW) equation.

Abstract

In this paper, we present two new equations, firstly a combined of Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation with modified Korteweg-de Vries-Benjamin-Bona-Mahony m(KdV-BBM) and denoted by c((KdV-BBM)-m(KdV-BBM)), secondly a combined of Shallow Water Wave-Ablowitz-Kaup-Newell-Segur (SWW-AKNS) equation with Equal-Width (EW) equation and denoted by c((SWW-AKNS)-EW). Then we apply the extended hyperbolic function method (EHFM) to solve the new equations. Exact traveling wave solutions are obtained and expresses in terms of hyperbolic functions and trigonometric functions.

1. Introduction

Nonlinear partial differential equations have been the subject of all-embracing studies in various branches of nonlinear sciences. Therefore, investigation exact solutions of NLPDEs are becoming more attractive in nonlinear sciences. As result, many methods have been successfully, such as: extended fan sub-equation method [15], the auxiliary equation method [9], the improved $(\frac{G'}{G})$-expansion method [1], the Sine-Gordon expansion method [6], the modified extended mapping method [10], the extended direct algebraic method [14], the Riccati projective equation method [3], the generalized Riccati equation method [16], the homogeneous balance method [4] and the mapping method [7]. This paper states the utilization of new analytical method called extended hyperbolic function method (EHFM) [8],[11] and [13]. This method is a promising method to handle a wide variety of such type of equations. The significant solutions are given in the form of trigonometric and hyperbolic functions.

2. Description of Extended Hyperbolic Function Method (EHFM)

For a given nonlinear partial differential equation (NLPDE), say in two variables,

$$H(v, v_t, v_x, v_{xx}, v_{tt}, v_{xt}, \dots) = 0. \qquad (1)$$

where $v = v(x,t)$ is unknown wave function, H is a polynomial in $v = v(x,t)$.

Let the wave transformation

$v(x,t) = v(\eta)$, $\eta = \lambda(x - ct)$, where λ is the wave number, c is the speed of the solitary wave.

Then Eq. (1) as per transformation reduced to a nonlinear ordinary differential equation (NLODE)

$$P(v,\ v', v'', v''', \dots) = 0. \qquad (2)$$

Now different steps of the EHFM that are displayed successively in the couple of forms like as:

Form 1: It relies on the fact that soliton solutions are usually polynomial of $\operatorname{sech}\eta$ function. We assume that the solution of Eq. (2) has the form

$$v(x,t) = v(\eta) = \sum_{i=0}^{n} b_i \Phi^i(\eta), \qquad (3)$$

where b_i are constants to be determined and $\Phi(\eta)$ satisfies a nonlinear ordinary differential equation

$$\Phi'(\eta) = \frac{d\Phi}{d\eta} = \Phi\sqrt{\tau + \mu\Phi^2}, \quad 0 \neq \tau\mu \in R. \qquad (4)$$

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The parameter n can be determined via balancing between the highest order derivative terms and the nonlinear term in Eq. (2). Substituting Eq. (3) in Eq. (2), using Eq. (4), and collecting all terms with the same power of $\Phi^i(\eta), (i = 0,1,2,\ldots,n)$. Setting the coefficients of each order of $\Phi^i(\eta)$ to zero, we get a system of algebraic equations which can be solved and obtained all the constants $b_i, (i = 0,1,2,\ldots,n), c$ and λ with the aid of Maple.
The solutions of Eq. (4), are given by
Type 1: If $\tau > 0$ and $\mu > 0$,

$$\Phi(\eta) = -\sqrt{\frac{\tau}{\mu}}\operatorname{csch}(\sqrt{\tau}\,\eta).$$

Type 2: If $\tau < 0$ and $\mu > 0$,

$$\Phi(\eta) = \sqrt{\frac{-\tau}{\mu}}\sec(\sqrt{-\tau}\,\eta).$$

Type 3: If $\tau > 0$ and $\mu < 0$,

$$\Phi(\eta) = \sqrt{\frac{\tau}{-\mu}}\operatorname{sech}(\sqrt{\tau}\,\eta).$$

Type 4: If $\tau < 0$ and $\mu > 0$,

$$\Phi(\eta) = \sqrt{\frac{-\tau}{\mu}}\csc(\sqrt{-\tau}\,\eta).$$

Type 5: If $\tau > 0$ and $\mu = 0$,
$\Phi(\eta) = \exp(\sqrt{\tau}\,\eta)$.
Type 6: If $\tau < 0$ and $\mu = 0$,
$\Phi(\eta) = \cos(\sqrt{\tau}\,\eta) + i\sin(\sqrt{\tau}\,\eta)$.
Type 7: If $\tau = 0$ and $\mu > 0$,

$$\Phi(\eta) = \pm\frac{1}{\sqrt{\mu}\,\eta}.$$

Type 8: If $\tau = 0$ and $\mu < 0$,

$$\Phi(\eta) = \pm\frac{1}{\sqrt{-\mu}\,\eta}.$$

Substituting the obtain coefficients and the general solutions of Eq. (4) in Eq. (3), we have the traveling wave solutions of the nonlinear partial differential equation (1).
Form 2: It relies on the fact that soliton solutions are usually polynomial of $\tanh\eta$ function. We assume Eq. (3) satisfies a nonlinear ordinary differential equation

$$\Phi'(\eta) = \frac{d\Phi}{d\eta} = \tau + \mu\,\Phi^2, \quad 0 \neq \tau\mu \in R. \tag{5}$$

The parameter n can be determined via balancing between the highest order derivative terms and the nonlinear term in Eq. (2). Substituting Eq. (3) in Eq. (2), using Eq. (5), and collecting all terms with the same power of $\Phi^i(\eta), (i = 0,1,2,\ldots,n)$. Setting the coefficients of each order of $\Phi^i(\eta)$ to zero, we get a system of algebraic equations which can be solved and obtained all the coefficients $b_i, (i = 0,1,2,\ldots,n), c$ and λ with the aid of Maple.

The solutions of Eq. (5), are given by
Type 1: If $\tau\mu > 0$,

$$\Phi(\eta) = \operatorname{sgn}(\tau)\sqrt{\frac{\tau}{\mu}}\tan(\sqrt{\tau\mu}\,\eta).$$

Type 2: If $\tau\mu > 0$,

$$\Phi(\eta) = -\operatorname{sgn}(\tau)\sqrt{\frac{\tau}{\mu}}\cot(\sqrt{\tau\mu}\,\eta).$$

Type 3: If $\tau\mu < 0$,

$$\Phi(\eta) = \operatorname{sgn}(\tau)\sqrt{\frac{\tau}{-\mu}}\tanh(\sqrt{-\tau\mu}\,\eta).$$

Type 4. If $\tau\mu < 0$,

$$\Phi(\eta) = \operatorname{sgn}(\tau)\sqrt{\frac{\tau}{-\mu}}\coth(\sqrt{-\tau\mu}\,\eta).$$

Type 5. If $\tau = 0$ and $\mu > 0$,

$$\Phi(\eta) = -\frac{1}{\mu\,\eta}.$$

Type 6. If $\tau \in R$ and $\mu = 0$,
$\Phi(\eta) = \tau\,\eta$.
Substituting the obtain coefficients and the general solutions of Eq. (5) in Eq. (3), we have the traveling wave solutions of the nonlinear partial differential equation (1).
Note: sgn is well-known sigh function.

3. Applications

3.1 Exact solution for c((KdV-BBM)-m(KdV-BBM)) equation

We consider a combined Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation with modified Korteweg-de Vries-Benjamin-Bona-Mahony m(KdV-BBM) equation as the form

$$v_t + \alpha(v + v^2)v_x + v_{xxx} - \beta v_{xxt} = 0, \quad v = v(x,t), \alpha, \beta \in R \tag{6}$$

where

$$v_t + \alpha v v_x + v_{xxx} - \beta v_{xxt} = 0, \quad v = v(x,t), \tag{7}$$

is the Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation [2]
and

$$v_t + \alpha v^2 v_x + v_{xxx} - \beta v_{xxt} = 0, \quad v = v(x,t), \tag{8}$$

is the modified Korteweg-de Vries-Benjamin-Bona-Mahony m(KdV-BBM) equation [2].
Substituting $v(x,t) = v(\eta)$, $\eta = \lambda(x - ct)$ in Eq. (6) and integrating once yields

$$-cv + \frac{\alpha}{2}v^2 + \frac{\alpha}{3}v^3 + \lambda^2(1 + \beta c)v'' = 0. \tag{9}$$

Eq. (9) is nonlinear ordinary differential equation.
Form 1: Balancing the nonlinear term v^3 with the highest order derivative v'' gives

$3n = n + 2$ that gives $n = 1$. Thus, the solution of Eq. (9) has the form

$$v(\eta) = b_0 + b_1 \Phi(\eta), \tag{10}$$

Substituting Eq. (10) in Eq. (9) and using Eq. (4). Collecting the coefficients of power of $\Phi^i, 0 \le i \le 3$, setting each coefficient to zero, and solving the resulting system with the aid of Maple, we obtain the following sets of solutions

Set 1.

$$b_0 = b_0, b_1 = 0, c = \frac{1}{3}\alpha^2 + \frac{1}{2}\alpha, \lambda = \lambda,$$

Set 2

$$b_0 = -\frac{1}{2}, b_1 = \pm\sqrt{-\frac{\mu}{2\tau}}, c = -\frac{1}{6}\alpha,$$

$$\lambda = \pm\sqrt{-\frac{\alpha}{2\alpha\beta\tau - 12\tau}}.$$

Using Eq. (10), the solution of Eq. (4), and the above sets of solutions [1-2], we get

$v_1(x,t) = b_0 \quad \forall b_0 \in R,$

Type 1: If $\tau > 0$ and $\mu > 0$, we obtain

$v_{2,3}(x,t) =$

$$-\frac{1}{2} \pm \sqrt{-\frac{\mu}{2\tau}}\left(\sqrt{\frac{\tau}{\mu}}\operatorname{csch}\left(\sqrt{\tau}\sqrt{\frac{\alpha}{2\alpha\beta\tau - 12\tau}}\left(x + \frac{1}{6}\alpha t\right)\right)\right).$$

After simplification, we get

$v_{2,3}(x,t) =$

$$-\frac{1}{2} \pm \frac{1}{2} i\sqrt{2}\operatorname{csch}\left(\sqrt{-\frac{\alpha}{2\alpha\beta - 12}}\left(x + \frac{1}{6}\alpha t\right)\right).$$

Type 2: If $\tau < 0$ and $\mu > 0$, we obtain

$$v_{4,5}(x,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\sec\left(\sqrt{\frac{\alpha}{2\alpha\beta - 12}}\left(x + \frac{1}{6}\alpha t\right)\right).$$

Type 3: If $\tau > 0$ and $\mu < 0$, we obtain

$$v_{6,7}(x,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\operatorname{sech}\left(\sqrt{-\frac{\alpha}{2\alpha\beta - 12}}\left(x + \frac{1}{6}\alpha t\right)\right).$$

Type 4: If $\tau < 0$ and $\mu > 0$, we get

$$v_{8,9}(x,t) = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}\csc\sqrt{\frac{\alpha}{2\alpha\beta - 12}}\left(x + \frac{1}{6}\alpha t\right)$$

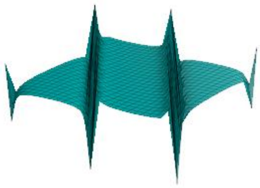

Figure 1: Graph of singular periodic solution $v_2(x,t)$ when $\alpha = 1, \beta = 2$

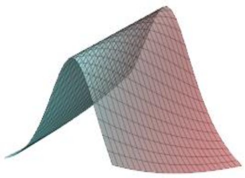

Figure 2: Graph of soliton solution $v_6(x,t)$ when $\alpha = 2, \beta = 1$

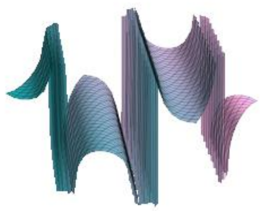

Figure 3: Graph of singular periodic solution $v_9(x,t)$ when $\alpha = \beta = 3$

Form 2. Applying the above techniques in form 1, the solution of Eq. (9) has the form

$v(\eta) = b_0 + b_1\Phi(\eta)$

Substituting Eq. (10) in Eq. (9) and using Eq. (5), collecting the coefficient of power of $\Phi^i, 0 \le i \le 3$, setting each coefficient to zero, and solving the resulting system with the aid of Maple, we obtain the following sets of solutions

Set 1.

$$b_0 = b_0, b_1 = 0, c = \frac{1}{3}\alpha^2 + \frac{1}{2}\alpha, \lambda = \lambda,$$

Set 2.

$$b_0 = -\frac{1}{2}, b_1 = \pm\sqrt{-\frac{\mu}{4\tau}}, c = -\frac{1}{6}\alpha, \lambda = \pm\sqrt{-\frac{\alpha}{4\alpha\beta\tau - 24\tau}}.$$

Using Eq. (10), the solution of Eq. (5), and the above sets of solutions [1-2], we get

$v_1(x,t) = b_0 \quad \forall b_0 \in R$

Type 1: If $\tau\mu > 0$, we get

$$v_{2,3}(x,t) = -\frac{1}{2} \pm \sqrt{\frac{-\mu}{4\tau}}\left(\text{sgn}\ (\tau)\sqrt{\frac{\tau}{\mu}}\tan\left(\sqrt{\mu\tau}\sqrt{-\frac{\alpha}{4\alpha\beta\tau-24\tau}}\left(x+\frac{1}{6}\alpha t\right)\right)\right).$$

After simplification, we get

$$v_{2,3}(x,t) = -\frac{1}{2} \pm \frac{1}{2} i\left(\tan\left(\sqrt{\mu}\sqrt{-\frac{\alpha}{4\alpha\beta-24}}\left(x+\frac{1}{6}\alpha t\right)\right)\right).$$

Type 2: If $\tau\mu > 0$, we obtain

$$v_{4,5}(x,t) = -\frac{1}{2} \pm \frac{1}{2} i\left(\cot\left(\sqrt{\mu}\sqrt{-\frac{\alpha}{4\alpha\beta-24}}\left(x+\frac{1}{6}\alpha t\right)\right)\right).$$

Type 3: If $\tau\mu < 0$, we get

$$v_{6,7}(x,t) = -\frac{1}{2} \pm \frac{1}{2}\left(\text{sgn}\ (\tau)\ \tanh\left(\sqrt{-\tau\mu}\sqrt{-\frac{\alpha}{4\alpha\beta\tau-24\tau}}\left(x+\frac{1}{6}\alpha t\right)\right)\right).$$

Type 4: If $\tau\mu < 0$, we get

$$v_{8,9}(x,t) = -\frac{1}{2} \pm \frac{1}{2}\left(\text{sgn}\ (\tau)\ \coth\left(\sqrt{-\tau\mu}\sqrt{-\frac{\alpha}{4\alpha\beta\tau-24\tau}}\left(x+\frac{1}{6}\alpha t\right)\right)\right).$$

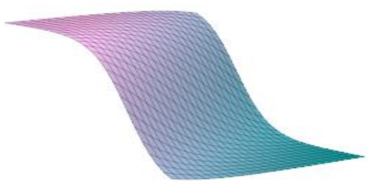

Figure 4: Graph of kink solution $v_3(x,t)$ when $\tau = \mu = 1, \alpha = \beta = 3, \lambda = 1$

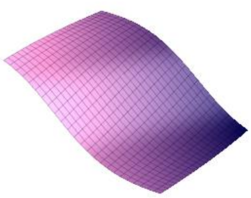

Figure 5: Graph of kink solution $v_6(x,t)$ when $\tau = -1, \mu = 1, \alpha = -2, \beta = 1, \lambda = 1$

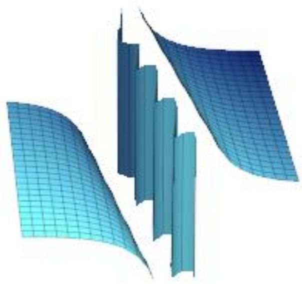

Figure 6: Graph of anti-kink solution $v_9(x,t)$ when $\tau = -1, \mu = 1, \alpha = 1, \beta = 2, \lambda = 1$

3.2 Exact solution for c((SWW-AKNS)-EW) equation

We consider a combined Shallow Water Wave-Ablowitz-Kaup-Newell-Segur (SWW-AKNS) equation with Equal-Width (EW) equation as the form

$$v_t + v_x + 2\alpha v v_x + 4 v v_t + 2 v_x \partial_x^{-1} v_t - \beta v_{xxt} = 0,$$

$$v = v(x,t), \alpha, \beta \in R,$$

where

$$v_t + v_x + 4 v v_t + 2 v_x\, \partial_x^{-1} v_t - \beta v_{xxt} = 0, v = v(x,t),$$

is the shallow water wave-Ablowitz-Kaup-Newell-Segur (SWW-AKNS) equation [12], and

$$v_t + 2\alpha v v_x - \beta v_{xxt} = 0, v = v(x,t),$$

is the Equal-Width (EW) equation [5].

Assuming $\omega = \partial_x^{-1} v_t$ implies $\omega_x = v_t$, and using the wave transformation $\eta = \lambda(x - ct)$ in Eq. (11), we find

$$-cv' + v' + 2\alpha v v' - 4cvv' + 2v'w + c\lambda^2\beta v''' = 0,$$
$$\omega' = -cv'. \tag{14}$$

Integrating the second equation in the system (14) and neglecting the constants of integration, we find

$$\omega = -cv. \tag{15}$$

Substituting Eq. (15) into the first equation of the system (14) and integrating the resulting equation, we find

$$(1-c)v + (\alpha - 3c)v^2 + c\lambda^2\beta v'' = 0. \tag{16}$$

Equation (16) is nonlinear ordinary differential equation.

Form 1. Balancing the highest order of the nonlinear term v^2 with the highest order derivative v'' gives $2n = n + 2$ that gives $n = 2$. Thus, the solution of equation (16) has the form

$$v(\eta) = b_0 + b_1\Phi(\eta) + b_2\Phi^2(\eta). \quad (17)$$

Substituting Eq. (17) in Eq. (16) and using Eq. (4), collecting the coefficient of power $\Phi^i, 0 \le i \le 4$, setting each coefficient to zero, and solving the resulting system with the aid of Maple, we obtain the following sets of solutions

Set 1.

$$b_0 = b_0, b_1 = 0, c = \frac{\alpha b_0 + 1}{3b_0 + 1}, \lambda = \lambda,$$

Set 2.

$$b_0 = 0, b_1 = 0, b_2 = \frac{6\beta\mu\lambda^2}{4\alpha\beta\lambda^2\tau - \alpha + 3},$$

$$c = -\frac{1}{4\beta\lambda^2\tau - 1}, \lambda = \lambda,$$

Set 3.

$$b_0 = -\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3}, b_1 = 0,$$

$$b_2 = -\frac{6\beta\mu\lambda^2}{4\alpha\beta\lambda^2\tau + \alpha - 3}, c = \frac{1}{4\beta\lambda^2\tau - 1}, \lambda = \lambda.$$

Using Eq. (17), the solution of Eq. (4), and the above sets of solution [1-3], we get

$v_1(x,t) = b_0 \quad \forall b_0 \in R$

From set 2:

Type 1: If $\tau > 0$ and $\mu > 0$, we get

$$v_2(x,t) = \frac{6\beta\mu\lambda^2\tau}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\operatorname{csch}^2(\sqrt{\tau}\,\eta)\right),$$

$$w_2(x,t) = \frac{1}{4\beta\lambda^2\tau - 1}\left(\frac{6\beta\mu\lambda^2\tau}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\operatorname{csch}^2(\sqrt{\tau}\,\eta)\right)\right)$$

where $\eta = \lambda\left(x + \frac{1}{4\beta\lambda^2\tau - 1}t\right), \alpha, \beta, \mu$ and λ are arbitrary constants.

Type 2: If $\tau < 0$ and $\mu > 0$, we get

$$v_3(x,t) = \frac{-6\beta\mu\lambda^2\tau}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\sec^2(\sqrt{-\tau}\,\eta)\right),$$

$$w_3(x,t) = \frac{1}{4\beta\lambda^2\tau - 1}\left(\frac{-6\beta\mu\lambda^2\tau}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\sec^2(\sqrt{-\tau}\,\eta)\right)\right).$$

Type 3: If $\tau > 0$ and $\mu < 0$, we get

$$v_4(x,t) = \frac{6\beta\mu\lambda^2\tau}{-4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{sech}^2(\sqrt{\tau}\,\eta)\right),$$

$$w_4(x,t) = \frac{1}{4\beta\lambda^2\tau - 1}\left(\frac{6\beta\mu\lambda^2\tau}{-4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{sech}^2(\sqrt{\tau}\,\eta)\right).\right).$$

Type 4: If $\tau < 0$ and $\mu > 0$, we get

$$v_5(x,t) = \frac{-6\beta\mu\lambda^2\tau}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\csc^2(\sqrt{-\tau}\,\eta)\right),$$

$$w_5(x,t) = \frac{1}{4\beta\lambda^2\tau - 1}\left(\frac{-6\beta\mu\lambda^2}{4\alpha\beta\lambda^2\tau\mu - \alpha\mu + 3\mu}\left(\csc^2(\sqrt{-\tau}\,\eta)\right)\right).$$

From set 3:

Type 1: If $\tau > 0$ and $\mu > 0$, we get

$$v_6(x,t) = -\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} - \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{csch}^2(\sqrt{\tau}\,\eta)\right),$$

$$w_6(x,t) = -\frac{1}{4\beta\lambda^2\tau - 1}\left(-\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} - \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{csch}^2(\sqrt{\tau}\,\eta)\right)\right).$$

,

where $\eta = \lambda\left(x - \frac{1}{4\beta\lambda^2\tau - 1}t\right), \alpha, \beta, \mu, \tau$ and λ are arbitrary constants.

Type 2: If $\tau < 0$ and $\mu > 0$, we obtain

$$v_7(x,t) = -\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\sec^2(\sqrt{-\tau}\,\eta)\right),$$

$$w_7(x,t) = -\frac{1}{4\beta\lambda^2\tau - 1}\left(-\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\sec^2(\sqrt{-\tau}\,\eta)\right)\right).$$

Type 3: If $\tau > 0$ and $\mu < 0$, we obtain

$$v_8(x,t) = -\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{sech}(\sqrt{\tau}\,\eta)\right)^2,$$

$$w_8(x,t) = -\frac{1}{4\beta\lambda^2\tau - 1}\left(-\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\operatorname{sech}(\sqrt{\tau}\,\eta)\right)^2\right).$$

Type 4: If $\tau < 0$ and $\mu > 0$, we obtain

$$v_9(x,t) = -\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\csc^2(\sqrt{\tau}\,\eta)\right),$$

$$w_9(x,t) = -\frac{1}{4\beta\lambda^2\tau - 1}\left(-\frac{4\beta\lambda^2\tau}{4\alpha\beta\lambda^2\tau + \alpha - 3} + \frac{6\beta\lambda^2\mu\tau}{4\alpha\beta\lambda^2\tau\mu + \alpha\mu - 3\mu}\left(\csc^2\left(\sqrt{\tau}\,\eta\right)\right)\right).$$

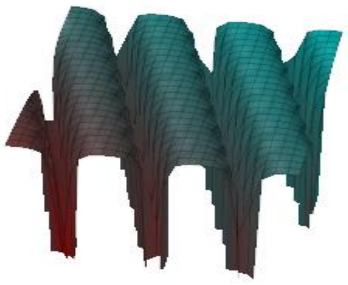

Figure 7: Graph of singular periodic solution $w_3(x,t)$ when $\tau = -1, \mu = 1, \alpha = 2, \beta = -1, \lambda = 1$

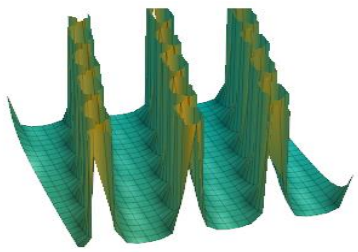

Figure 8: Graph of singular periodic solution $v_5(x,t)$ when $\tau = -1, \mu = 2, \alpha = -1, \beta = 1, \lambda = 1$

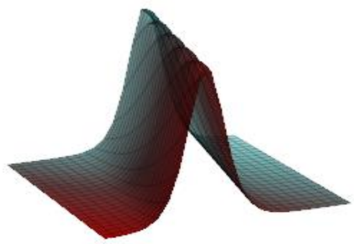

Figure 9: Graph of soliton solution $v_4(x,t)$ when $\tau = 1, \mu = -1, \alpha = -1, \beta = 1, \lambda = 1$

Form 2. Applying the above techniques in form 1, the solution of Eq. (16) has the form

$$v(\eta) = b_0 + b_1\Phi(\eta) + b_2\Phi^2(\eta). \qquad (18)$$

Substituting Eq. (18) in Eq. (16) and using Eq. (5), collecting the coefficient of power of $\Phi^i, 0 \le i \le 4$, setting each coefficient to zero, and solving the resulting system with the aid of Maple, we obtain the following sets of solutions

Set 1.

$$b_0 = b_0, a_1 = 0, c = \frac{\alpha b_0 + 1}{3b_0 + 1}, \lambda = \lambda,$$

Set 2.

$$b_0 = -\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau + \alpha - 3}, b_1 = 0,$$
$$b_2 = -\frac{6\beta\,\mu^2\lambda^2}{4\alpha\beta\lambda^2\mu\tau + \alpha - 3}, c = \frac{1}{4\beta\lambda^2\mu\tau + 1}, \lambda = \lambda,$$

Set 3.

$$b_0 = \frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau - \alpha + 3}, b_1 = 0,$$
$$b_2 = \frac{6\;\beta\mu^2\lambda^2}{4\alpha\beta\lambda^2\mu\tau - \alpha + 3}, c = -\frac{1}{4\beta\lambda^2\mu\tau - 1}, \lambda = \lambda.$$

Using Eq. (18), the solution of Eq. (4), and the above sets of solution [1-3], we get

$$v_1(x,t) = b_0, \quad \forall b_0 \in R.$$

From set 2:

Type 1: If $\tau\mu > 0$, we get

$$v_2(x,t) = -\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau + \alpha - 3} - \frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau + \alpha\mu - 3\mu}\left(\tan^2\left(\sqrt{\mu\tau}\,\eta\right)\right),$$

$$w_2(x,t) = -\frac{1}{4\beta\lambda^2\mu\tau + 1}\left(-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau + \alpha - 3} - \frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau + \alpha\mu - 3\mu}\left(\tan^2\left(\sqrt{\mu\tau}\,\eta\right)\right)\right),$$

where $\eta = \lambda\left(x - \frac{1}{4\beta\lambda^2\mu\tau + 1}t\right), \alpha, \beta, \mu, \tau$ and λ are arbitrary constants.

Type 2: If $\tau\mu > 0$, we get

$$v_3(x,t) = -\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau + \alpha - 3} - \frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau + \alpha\mu - 3\mu}\left(\cot^2\left(\sqrt{\mu\tau}\,\eta\right)\right),$$

$$w_3(x,t) =$$

$$-\frac{1}{4\beta\lambda^2\mu\tau+1}\left(-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau+\alpha-3}-\frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau+\alpha\mu-3\mu}\left(\cot^2(\sqrt{\mu\tau}\,\eta)\right)\right).$$

Type 3: If $\tau\mu<0$, we obtain

$$v_4(x,t)=-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau+\alpha-3}+\frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau+\alpha\mu-3\mu}\left(\tanh^2(\sqrt{-\mu\tau}\,\eta)\right),$$

$$w_4(x,t)=-\frac{1}{4\beta\lambda^2\mu\tau+1}\left(-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau+\alpha-3}+\frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau+\alpha\mu-3\mu}\left(\tanh^2(\sqrt{-\mu\tau}\,\eta)\right)\right).$$

Type 4: If $\tau\mu<0$, we obtain

$$v_5(x,t)=-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau+\alpha-3}+\frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau+\alpha\mu-3\mu}\left(\coth^2(\sqrt{-\mu\tau}\,\eta)\right),$$

$$w_5(x,t)=-\frac{1}{4\beta\lambda^2\mu\tau+1}\left(-\frac{6\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau+\alpha-3}+\frac{6\beta\,\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau+\alpha\mu-3\mu}\left(\coth^2(\sqrt{-\mu\tau}\,\eta)\right)\right).$$

From set 3:

Type 1: If $\tau\mu>0$, we obtain

$$v_6(x,t)=\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}+\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\tan^2(\sqrt{\mu\tau}\,\eta)\right),$$

$$w_6(x,t)=\frac{1}{4\beta\lambda^2\mu\tau-1}\left(\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}+\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\tan^2(\sqrt{\mu\tau}\,\eta)\right)\right),$$

where $\eta=\lambda\left(x+\frac{1}{4\beta\lambda^2\mu\tau+1}t\right)$, α,β,μ,τ and λ are arbitrary constants.

Type 2: If $\tau\mu>0$, we obtain

$$v_7(x,t)=\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}+\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\cot^2(\sqrt{\mu\tau}\,\eta)\right),$$

$$w_7(x,t)=\frac{1}{4\beta\lambda^2\mu\tau-1}\left(\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}+\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\cot^2(\sqrt{\mu\tau}\,\eta)\right)\right).$$

Type 3: If $\tau\mu<0$, we obtain

$$v_8(x,t)=\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}-\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\tanh^2(\sqrt{-\mu\tau}\,\eta)\right),$$

$$w_8(x,t)=\frac{1}{4\beta\lambda^2\mu\tau-1}\left(\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}-\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\tanh^2(\sqrt{-\mu\tau}\,\eta)\right)\right).$$

Type 4: If $\tau\mu<0$, we obtain

$$v_9(x,t)=\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}-\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\coth^2(\sqrt{-\mu\tau}\,\eta)\right),$$

$$w_9(x,t)=\frac{1}{4\beta\lambda^2\mu\tau-1}\left(\frac{2\beta\lambda^2\tau\mu}{4\alpha\beta\lambda^2\mu\tau-\alpha+3}-\frac{6\;\beta\mu^2\lambda^2\tau}{4\alpha\beta\lambda^2\mu^2\tau-\alpha\mu+3\mu}\left(\coth^2(\sqrt{-\mu\tau}\,\eta)\right)\right).$$

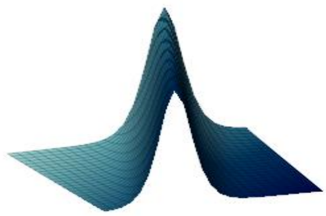

Figure 10: Graph of soliton solution $w_8(x,t)$ when $\tau=1,\mu=-1,\alpha=-1,\beta=0.5,\lambda=$

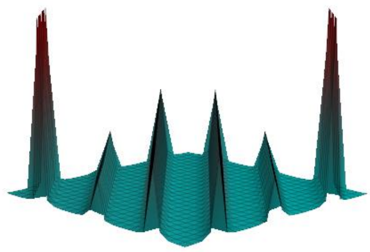

Figure 11: Graph of singular periodic solution $v_2(x,t)$ when $\tau=1,\mu=1,\alpha=-1,\beta=0.5,\lambda=1$

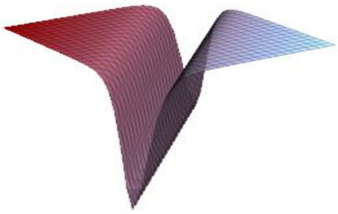

Figure 12: Graph of soliton solution $v_8(x,t)$when $\tau = 1, \mu = -1, \alpha = 1, \beta = -1, \lambda = 1$

4. Conclusion

In this paper, the extended hyperbolic function method has been achieved to find new traveling waves solutions for our proposed equations, namely a combined Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation, modified Korteweg-de Vries-Benjamin-Bona-Mahony m(KdV-BBM) equation, a combined shallow water wave-Ablowitz-Kaup-Newell-Segur (SWW-AKNS) equation and equal-width (EW) equation. Exact traveling wave solutions are constructed including soliton solutions, periodic wave solutions and kink wave solutions. Many solutions represent graphically with the aid of Scientific WorkPlace by choosing the suitable values of involved parameters. The result show that this method is powerful mathematical tool for obtain different forms of solutions for our equations. It is also a promising method to solve other nonlinear partial differential equations.

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University of Aden Journal of Natural and Applied Sciences., vol.28(2) (2024) 97–105

مجلة جامعة عدن للعلوم الطبيعية والتطبيقية

Journal homepage: https://uajnas.adenuniv.com

بحث علمي

طريقة الدالة الزائدية الممتدة لحل معادلتين تفاضليتين جزئيتين غير خطيتين جديدتين

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https://doi.org/10.47372/uajnas.2024.n2.a09

الملخص

في هذا البحث، نقدم معادلتين جديدتين، الأولى هي معادلة مركبة من معادلة كورتويغ-دي فريس-بنجامين-بونا-ماهوني (KdV-BBM)مع المعادلة المعدلة لكورتويغ-دي فريس-بنجامين-بونا-ماهوني m(KdV-BBM) والتي نرمز لها بـ c((KdV-BBM)-m(KdV-BBM)).والثانية هي معادلة مركبة من معادلة أمواج المياه الضحلة-أبلويتز-كاوب-نيويل-سيغور (SWW-AKNS) مع معادلة العرض المتساوي (EW) والتي نرمز لها بـ c((SWW-AKNS)-EW) ، ثم قمنا بتطبيق طريقة الدالة الزائدية الممتدة (EHFM) لحل المعادلات الجديدة. حيث تم الحصول على حلول دقيقة للموجات المتنقلة و التعبير عنها بالدوال الزائدية والدوال المثلثية.

مفاتيح البحث

التسليم : 20 ديسمبر 2024

القبول : 10 يناير 2025

كلمات مفتاحية:

طريقة الدالة الزائدية الممتدة، الحلول الدقيقة، معادلة c((KdV-BBM-m(KdV-BBM)) معادلة c((SWW-AKNS)-EW)