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Research Article

On a new extension of extended (p,q) Beta function

Salem Saleh Alqasemi Barahmah[1*]

[1] Department of Mathematics Faculty of Education Aden, University of Aden

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Abstract

The main objective of this paper is to introduce a further extension of extended (p, q)-Beta function by considering product of two Mittag-Leffler functions in the kernel. We investigate various properties of this newly defined Beta function such as integral representations, summation formulas and Mellin transform. Further, some known and new relations for various forms of extended Beta functions are obtained as special cases of the main results.

1. Introduction

There are many extensions and generalizations of the (p,q) Beta function(see for example [1,3-10,13], have been considered by several authors. In this paper, we study another extension of the (p,q)-Beta function and investigate various formulas, such as integral representation, summation formula and Mellin transform.

The Gamma function Γ(z) developed by Euler [2] with the intent to extend the factorials to values between the integers is defined by the definite integral.

The Gamma function $\Gamma(z)$ developed by Euler [2] with the intent to extend the factorials to values between the integers is defined by the definite integral.

$$\Gamma(z)=\int_0^\infty e^{-t}\,t^{z-1}\,dt \quad , \quad Re(z)>0\,. \tag{1.1}$$

The Euler Beta function $B(z_1,z_2)$ (see [14]) is defined by

$$B(z_1,z_2)=\int_0^\infty t^{z_1-1}\,(1-t)^{z_2-1}\,dt.Re(z_1)>0 \quad ,Re(z_2)>0 \tag{1.2}$$

Among various extensions of gamma function, we mention here the extended gamma function [6] defined by Chaudhry and Zubair

$$\Gamma_p(z)=\int_0^\infty t^{z-1}\exp\left(-t-\frac{p}{t}\right)dt \quad , \quad (Re(p)>0). \tag{1.3}$$

In 1997, Choudhary et al. [7] introduced an extension of the Beta function defined by

$$B^p(z_1,z_2)=\int_0^1 t^{z_1-1}\,(1-t)^{z_2-1}\,exp\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.4}$$

where

$$Re(p)\geq 0 \quad , \quad (Re(z_1)>0 \quad ,Re(z_2)>0).$$

In 2011, Lee et al. [10] introduced an extension of the Beta function defined by

$$B_p(x,y;m)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,exp\left(-\frac{p}{t^m(1-t)^m}\right)dt, \tag{1.5}$$

Where

$$Re(p)\geq 0 \quad ,Re(x)>0 \quad ,Re(y)>0, \qquad m>0$$

The following extended Beta function is introduced by Choi et al. [8] respectively:

$$B(x,y,p,q)=\int_0^1 t^{x-1}\,(1-t)^{y-1}exp\left(-\frac{p}{t}-\frac{q}{(1-t)}\right)dt. \tag{1.6}$$

The following extended Beta function is given by Rahman et al. [13]:

$$B_\alpha^{(p,q)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}E_\alpha\left(-\frac{p}{t}\right)E_\alpha\left(-\frac{q}{(1-t)}\right)dt, \tag{1.7}$$

* Correspondence to: Salem.S. Alqasemi Barahmah, Department of Mathematics, University of Aden- Yemen

E-mail address: salemalqasemi@yahoo.com

$Re(x) > 0 \quad , \qquad Re(y) > 0 \quad , \quad p, q > 0$

where $E_{\alpha,\beta}(.)$ is the classica Mittag-Leffler function defined as [11]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.8}$$

Afterwards, Atash et al. [3] introduced the following extension of Beta function:

$$B_{\alpha,\beta}^{(p,q)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt, \tag{1.9}$$

$Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 0 \;,\; p, q > 0$

where $E_{\alpha,\beta}(.)$ is the generalized Mittag-Leffler function defined as [16]

$$E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \beta)}. \tag{1.10}$$

where $\qquad x \in \mathbb{C} \;, \alpha, \beta \in \mathbb{R}_0^+$

In (2019), Barahmah [4] introduces a new extended of Beta function in the following form:

$$B_{\alpha,\beta}^{(p,q,\sigma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\sigma}\left(-\frac{p}{t}\right) E_{\alpha,\beta}^{\sigma}\left(-\frac{q}{(1-t)}\right) dt, \tag{1.11}$$

$Re(p) > 0, Re(p) > 0, Re(x) > 0, Re(y) > 0,$

$\alpha, \beta, \sigma \in \mathbb{R}_0^+, \mu, v \in \mathbb{R}^+.$

where $E_{\alpha,\beta}^{\sigma}(.)$ is the generalized Mittag-Leffler function defined as [12]

$$E_{\alpha,\beta}^{\sigma}(x) = \sum_{k=0}^{\infty} \frac{(\sigma)_k}{\Gamma(\alpha k + \beta)} \frac{x^k}{k!}. \tag{1.12}$$

Khan et at. [9] introduces a new extended of Beta function in the following form:

$$B_{p,q}^{(\alpha;\mu,v)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha}\left(-\frac{p}{t^u}\right) E_{\alpha}\left(-\frac{q}{(1-t)^v}\right) dt, \tag{1.13}$$

$Re(p) \geq 0,\; Re(q) \geq 0,\; Re(x) > 0 \;, Re(y) > 0,\; \alpha, u, v > 0$

and E_{α} is a Mittag-Leffler function.

1. A new extension of the Beta function

In this section, we introduce a new extension of the extended Beta function and investigate various properties and representations

$$B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^v}\right) dt, \tag{2.1}$$

$Re(p) > 0, \quad Re(q) > 0, \quad Re(x) > 0, Re(y) > 0,$

$\alpha, \beta, \gamma, \sigma \in \mathbb{R}_0^+, \mu, v \in \mathbb{R}^+,$

where $E_{\alpha,\beta}^{\gamma,\sigma}(.)$ is the generalized Mittag-Leffler function defined as [15]

$$E_{\alpha,\beta}^{\gamma,\rho}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_{\rho k}}{\Gamma(\alpha k + \beta)} \frac{x^k}{k!}$$

Remark 2.1. The known special cases of formula (2.1) are given as follows

On setting $\mu = v = \gamma = 1$, in Eq. (2.1), it reduces to (1.11) established by Barahmah in [4].

On setting $\mu = v = \gamma = \sigma = 1$, in Eq. (2.1), it reduces to (1.9) established by Atash et al.in [3].

On setting $\mu = v = \gamma = \sigma = \beta = 1$, in Eq. (2.1), it reduces to (1.7) established by Rahman et al. [13].

On setting $\mu = v = \gamma = \sigma = \beta = \alpha = 1$, in Eq. (2.1), it reduces to (1.6) established by Choi et at. [7].

On setting $\gamma = \sigma = \beta = \alpha = 1, \mu = v = m$ and $p = q$, in Eq. (2.1), it reduces to (1.5) established by Lee et al. in [10].

On setting $\mu = v = \gamma = \sigma = \beta = \alpha = 1,$ and $p = q$, in Eq. (2.1), it reduces to (1.4) established by Chaudhry et at. in [7].

On setting $\mu = v = \gamma = \sigma = \beta = \alpha = 1$ and $p = q = 0$, in Eq. (2.1), it reduces to (1.2) established by Srivastava et at. In [14].

On setting $\gamma = \sigma = \beta = 1$, in Eq. (2.1), it reduces to (1.13) established by Khan et at. [9].

Remark 2.2. There are several new special cases obtained from formula (2.1) which are as follows:

If we take $\mu = v = 1$, in Eq. (2.1), we obtain the following result:

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{1-t}\right) dt, \tag{2.2}$$

If we take $\gamma = 1$, in Eq. (2.1), we obtain the following result

$$B_{\alpha,\beta}^{(p,q,\mu,v,\sigma)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}^{\sigma}\left(-\frac{p}{t^u}\right) E_{\alpha,\beta}^{\sigma}\left(-\frac{q}{(1-t)^v}\right) dt. \tag{2.3}$$

If we take $\gamma, \sigma = 1$, in Eq. (2.1), we obtain the following result

$$B_{\alpha,\beta}^{(p,q,\mu,v)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \times E_{\alpha,\beta}\left(-\frac{p}{t^u}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)^v}\right) dt. \tag{2.4}$$

If we take $\gamma = \sigma = \alpha = \beta = 1$, in Eq. (2.1), we obtain the following result

$$B_{\mu,v}^{(p,q)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} exp\left(-\frac{p}{t^u} - \frac{q}{(1-t)^v}\right) dt. \tag{2.5}$$

3. Some Properties of $B_{\alpha,\beta}^{(p,q,\mu,v,\delta,\varepsilon,\gamma,\sigma)}(x,y)$

In this section we get some interesting relation for summation formulas for $B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y)$ as the following theorems:

Theorem 3.1. The following integral representations holds true:

$$(i)\quad B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, E_{\alpha,\beta}^{\gamma,\sigma}(-p\,(\sec^2\theta)^{\mu}) \times E_{\alpha,\beta}^{\gamma,\sigma}(-q(cosec^2\,\theta)^{v})d\theta, \quad (3.1)$$

$$(ii)\quad B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = \int_0^{\infty} \frac{w^{x-1}}{(1+w)^{x+y}} E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(1+w)^{\mu}}{w^{\mu}}\right) E_{\alpha,\beta}^{\gamma,\sigma}(-q\,(1+w)^{v})dw, \quad (3.2)$$

$$(iii)\quad B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = (c-a)^{1-x-y}\int_a^c (1-w)^{x-1}(1-w)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(c-a)^{u}}{(1-w)^{\mu}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-q\,\frac{(c-a)^{v}}{(1-w)^{v}}\right)dw, \quad (3.3)$$

$$(iv)\quad B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = 2^{1-x-y}\int_{-1}^{1} (1+w)^{x-1}(1-w)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2^{u}}{(1+w)^{\mu}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-q\,\frac{2^{v}}{(1-w)^{v}}\right)dw\,. \quad (3.4)$$

Proof: For prove the formula (3.1), putting $t = \cos^2\theta \;\Rightarrow\; dt = -2\cos\theta\sin\theta$ in (2.1), we have

$$(i)\quad B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = 2\int_0^{\frac{\pi}{2}} (\cos^2\theta)^{x-1}\,(1-cos^2\theta)^{y-1}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(\frac{-p}{(cos^2\theta)^{u}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(\frac{-q}{(1-cos^2\theta)^{v}}\right)\cos\theta\sin\theta\, d\theta,$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta\, \sin^{2y-1}\theta\, E_{\alpha,\beta}^{\gamma,\sigma}\left(\frac{-p}{(cos^2\theta)^{u}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(\frac{-q}{(sin^2\theta)^{v}}\right)d\theta,$$

which on using simple definitions, we get the desired result.

Similarly, results (3.2), (3.3) and (3.4) can be proved by taking the transformation $= \frac{w}{1+w}$, $t = \frac{w-a}{c-a}$ and $t = \frac{1+w}{2}$ in (2.1) respectively. Thus the proof of **Theorem 3.4** is completed.

Remark 3.1.

(i) in (3.1), (3.2), (3.3) and (3.4) , if we take $\mu = v = 1$ we obtain the following new results:

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}\theta \sin^{2y-1}\theta\, E_{\alpha,\beta}^{\gamma,\sigma}(-p\,\sec^2\theta) \times E_{\alpha,\beta}^{\gamma,\sigma}(-q\,cosec^2\,\theta)d\theta, \quad (3.5)$$

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y) = \int_0^{\infty} \frac{w^{x-1}}{(1+w)^{x+y}} E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(1+w)}{w}\right) E_{\alpha,\beta}^{\gamma,\sigma}(-q\,(1+w))dw, \quad (3.6)$$

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y) = (c-a)^{1-x-y}\int_a^c (1-w)^{x-1}(1-w)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{(c-a)}{(1-w)}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-q\,\frac{(c-a)}{(1-w)}\right)dw\,, \quad (3.7)$$

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y) = 2^{1-x-y}\int_{-1}^{1} (1+w)^{x-1}(1-w)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-p\,\frac{2}{(1+w)}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-q\,\frac{2}{(1-w)}\right)dw\,. \quad (3.8)$$

(ii) If we take $\mu = v = \sigma = 1$, then we get a known result of Barahmah [4, p. 43, {(2.6) - (2.8)}].

(iii) If we take $\mu = v = \gamma = \sigma = 1$, then we get a known result of Atash et al. [3, p. 17, {(2.7) - (2.9)}].

(iv) If we take $\mu = v = \gamma = \sigma = \beta = 1$, then then we get a known result a known result of Rahman et al. [13, p. 6, {(2.5) - (2.8)}].

(v) On setting $\mu = v = \gamma = \sigma = \beta = \alpha = 1$, then then we get a known result a known result of Choi et at. [8, p. 361-362, {(2.5) – (2.8)}].

(vi) If we take $\mu = v = \gamma = \sigma = \beta = \alpha = 1$ and $p = q$, then then we get a known result a known result of Chaudhry et at. [7, p. 22, {(2.7) – (2.9) and (2.11)}].

Theorem 3.2. The following summation formula holds true:

$$B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = \sum_{k=0}^{n}\binom{n}{k} B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x+k, y+n-k).\; n \in \mathbb{N}_0 \quad (3.9)$$

Proof: To prove of (3.9) we use the mathematical induction on $(n \in \mathbb{N}_0)$ as follows:

Clearly, For $n = 0$ the equation (3.5) holds.

For $n = 1$, we have

$$B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{v}}\right)dt,$$

$$= \int_0^1 t^{x-1}\,(1-t)^{y-1}\{\,t + (1-t)\}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{v}}\right)dt,$$

$$= \int_0^1 \{t^{x}\,(1-t)^{y-1} + \; t^{x-1}\,(1-t)^{y}\}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right) E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{v}}\right)dt,$$

$$= B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x+1,y) + B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y+1). \quad (3.10)$$

Repeating the same argument to the above two terms in (3.10), we obtain

$$B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y) = B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x+2,y) + 2B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x+1,y+1) + B_{\alpha,\beta}^{(p,q,\mu,v,\gamma,\sigma)}(x,y+2). \quad (3.11)$$

Continuing this process for all $(n \in \mathbb{N}_0)$, we finally obtain the desired relation (3.9).

Remark 3.2.

As a special cases of (3.9)

(i) If we take $\mu=\nu=1$, we obtain the following new result:

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y)=\sum_{k=0}^{n}\binom{n}{k}B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x+k,y+n-k).\ n\in\mathbb{N}_0 \quad (3.12)$$

(ii) On setting $\mu=\nu=\gamma=1$, then (3.9) reduces to a known result of Barahmah [4, p.44, (2.3)].

(iii) On setting $\mu=\nu=\gamma=\sigma=1$, then (3.9) reduces to a known result of Atash et al. [3, p. 16, (2.3)].

(iv) On setting $\mu=\nu=\gamma=\sigma=\beta=1$, then (3.9) reduces to a known result of Rahman et al. [13, p. 7, (3.1)].

(v) On setting $\mu=\nu=\gamma=\sigma=\beta=\alpha=1$, then (3.9) reduces to a known result of Choi et at. [8, p. 362 (3.1)].

(vi) On setting $\mu=\nu=\gamma=\sigma=\beta=\alpha=1$ and $p=q$ then (3.9) reduces to a known result of Chaudhry et at. [7, p. 23. (3.1)].

Theorem 3.3. The following summation relation holds

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,1-y)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x+n,1). \quad (3.13)$$

Proof: from (2.1), we have

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,1-y)=\int_0^1 t^{x-1}(1-t)^{-y} \times E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{\mu}}\right)E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{\nu}}\right)dt,$$

using the generalized binomial theorem

$$(1-t)^{-y}=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^n, \qquad |t|<1,$$

we obtain

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,1-y)$$
$$=\int_0^1\sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^{x+n-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right)dt\,E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{\nu}}\right)dt.$$

Now, interchanging the order of summation and integration in the above equation and using (2.1), we obtain the required result (3.13).

Remark 3.3.

(i) On setting $\mu=\nu=1$ in (3.13), we obtain the following new result:

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,1-y)=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x+n,1). \quad (3.14)$$

(ii) On setting $\mu=\nu=\gamma=1$, then (3.13) reduces to a known result of Barahmah [4, p. 44, (2.1)].

(iii) On setting $\mu=\nu=\gamma=\sigma=1$, then (3.13) reduces to a known result of Atash et al. [3, p. 16, (2.1)].

(iv) On setting $\mu=\nu=\gamma=\sigma=\beta=1$, then (3.13) reduces to a known result of Rahman et al. [13. P. 7, (3.5)].

(v) On setting $\mu=\nu=\gamma=\sigma=\beta=\alpha=1$, then (3.13) reduces to a known result of Choi et at. [8, p. 362 (3.2)].

(vi) On setting $\mu=\nu=\gamma=\sigma=\beta=\alpha=1$ and $p=q$ then (3.13) reduces to a known result of Chaudhry et at. [7, p. 27, (4.23)].

Theorem 3.4. The following summation relation holds

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y)=\sum_{n=0}^{\infty}B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x+n,y+1). \quad (3.15)$$

Proof: To prove (3.15), using the relation

$$(1-t)^{y-1}=(1-t)^{y}\sum_{n=0}^{\infty}t^n \qquad (|t|<1)$$

in (2.1), we obtain

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y)$$
$$=\int_0^1(1-t)^{y}\sum_{n=0}^{\infty}t^{n+x-1}\ E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right)E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{\nu}}\right)dt\,,$$

$$B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y)=\sum_{n=0}^{\infty}\int_0^1(1-t)^{y} \times t^{n+x-1}E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^{u}}\right)E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^{\nu}}\right)dt,$$

which in view of (2.1), we get the desired result (3.15).

Remark 3.4.

(i) If we take $\mu=\nu=1$ in (3.15), we obtain the following new result:

$$B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y)=\sum_{n=0}^{\infty}B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x+n,y+1). \quad (3.16)$$

(ii) On setting $\mu=\nu=\gamma=1$, then (3.15) reduces to a known result of Barahmah [4, p. 44, (2.2)].

(iii) On setting $\mu=\nu=\gamma=\sigma=1$, then (3.15) reduces to a known result of Atash et al. [3, p. 16, (2.2)].

(iv) On setting $\mu = \nu = \gamma = \sigma = \beta = 1$, then (3.15) reduces to a known result of Rahman et al. [13, p. 8, (3.7)].

(v) On setting $\mu = \nu = \gamma = \sigma = \beta = \alpha = 1$, then (3.15) reduces to a known result of Choi et at. [8, p. 363, (3.3)].

For our aim in this section, we introduce the following definition of extended Gamma function:

$$\Gamma_p^{(\alpha,\beta,\gamma,\sigma)}(x) = \int_0^\infty e^{x-1}\, E_{\alpha,\beta}^{\gamma,\sigma}\left(-t - \frac{p}{t}\right) dt, \qquad (3.17)$$

$\Re(x) > 0$, $p \geq 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\sigma) > 0$.

Theorem 3.5. The extension of extended Beta function has the following Mellin transform relation:

$$M\left\{B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y);\; p \to r, q \to s\right\} = B(x+r, y+s)\Gamma^{p,q,\gamma,\sigma}(r)\Gamma^{p,q,\gamma,\sigma}(s). \qquad (3.18)$$

Proof:

$$M\left\{B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y);\; p \to r, q \to s\right\}$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}$$

$$\times \left(\int_0^\infty p^{r-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{p}{t^u}\right) dp\right)\left(\int_0^\infty q^{s-1} E_{\alpha,\beta}^{\gamma,\sigma}\left(-\frac{q}{(1-t)^\nu}\right) dq\right) dt, \qquad (3.19)$$

Substituting $w = \frac{p}{t^u}$ and $z = \frac{q}{(1-t)^\nu}$, in (3.19), we have

$$M\left\{B_{\alpha,\beta}^{(p,q,\mu,\nu,\gamma,\sigma)}(x,y);\; p \to r, q \to s\right\} = \int_0^1 t^{x+ur-1}(1-t)^{y+\nu s-1}$$

$$\times \left(\int_0^\infty w^{r-1} E_{\alpha,\beta}^{\gamma,\sigma}(-w)\, dw\right)\left(\int_0^\infty z^{s-1} E_{\alpha,\beta}^{\gamma,\sigma}(-z)\, dz\right) dt, \qquad (3.20)$$

which on using definition (3.17) for $p = 0$, yields result (3.18).

Remark 3.5.

(i) If we take $\mu = \nu = 1$ in (3.17), we obtain the following new result:

$$M\left\{B_{\alpha,\beta}^{(p,q,\gamma,\sigma)}(x,y);\; p \to r, q \to s\right\} = B(x+r, y+s)\Gamma^{p,q,\gamma,\sigma}(r)\Gamma^{p,q,\gamma,\sigma}(s). \qquad (3.19)$$

(ii) On setting $\mu = \nu = \gamma = 1$, then (3.17) reduces to a known result of Barahmah [4].

(iii) On setting $\mu = \nu = \gamma = \sigma = 1$, then (3.17) reduces to a known result of Atash et al. [3].

(iv) On setting $\mu = \nu = \gamma = \sigma = \beta = 1$, then (3.17) reduces to a known result of Rahman et al. [11].

(v) On setting $\mu = \nu = \gamma = \sigma = \beta = \alpha = 1$, then (3.17) reduces to a known result of Choi et at. [8].

(vi) On setting $\mu = \nu = \gamma = \sigma = \beta = \alpha = 1$ and $p = q$, then (3.17) reduces to a known result of Chaudhry et at. [7].

Conclusion

This paper defines a new generalization of Beta function using product of two Mittag-Leffler functions. Most prominent properties related to the new generalized beta function, such as integral representations, summation formulas and Mellin transform are obtained. As a continuation of this work, applications of this new extension of the beta function will be presented in a forthcoming paper.

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بحث علمي

حول تمديد جديد لدالة بيتا الممتدة

سالم صالح القاسمي بارحمة
قسم الرياضيات - كلية التربية- جامعة عدن

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الملخص

الهدف الرئيسي من هذه الورقة هو تقديم امتداد أخرى لدالة بيتا الممتدة (p,q) والمتضمنة حاصل ضرب دالتي ميتاج- ليفلر. عدد من الخصائص المختلفة لدالة بيتا الممتدة استخلصت مثل التمثيلات المتكاملة وصيغ الجمع وتحويل ميلين .علاوة على ذلك، تم الحصول على بعض العلاقات المعروفة والجديدة لأشكال مختلفة من دوال بيتا الممتدة كحالات خاصة للنتائج الرئيسية.

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