On certain $P2$ –Like and $P^*$ –Generalized $\mathcal{BK}$ –Recurrent
Finsler Space

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DOI: https://doi.org/10.47372/uajnas.2017.n1.a16

Abstract

In the present paper, we study certain types of generalized $\mathcal{BK}$-recurrent Finsler space, we shall introduce a definition for a generalized $\mathcal{BK}$-recurrent space to be $P2$ –like space and $P^*$ – space, respectively. We shall call them $P2$ – like generalized $\mathcal{BK}$ –recurrent space and $P^*$ –generalized $\mathcal{BK}$ –recurrent space, respectively. Different theorems concerning these spaces, we also plan to obtain some identities in these spaces.

Keywords: Finsler space, $P2$ – like generalized $\mathcal{BK}$ –recurrent space, $P^*$ –generalized $\mathcal{BK}$ –recurrent space.

Introduction

Verma [13] discussed recurrent property of Cartan’s fourth curvature tensor $R^i_{jkh}$. Dikshit [4] discussed birecurrent of Berwald curvature tensor $H^i_{jkh}$. Dwivedi [5] worked out the role of $P^*$ – reducible space in affinely connected space. Cartan [3] introduced it as one of particular cases and further Berwald [1], [2] showed that the space was characterized by $P^i_{jkh} = 0$, where $P^i_{jkh}$ is the hv - curvature tensor. Dwivedi [5] worked out the role of $P^*$ – reducible space in Landsberg space.

Let us consider an $n$-dimensional Finsler space $F_n$ equipped with the metric function $F(x,y)$ satisfies the request condition Rund [12].

The relation between the metric function $F$ and the corresponding metric tensor is given by

$$g_{ij}(x,y) = \frac{1}{2} \delta_{ij} \partial_i F^2 (x,y).$$

The tensor $g_{ij}(x,y)$ is symmetric and a positively homogeneous of degree zero in $y^i$.

The vector $y_i$ and its associative $y^i$ satisfy the following relation

$$g_{ij}(x,y)y^i = y_j.$$

The two sets of quantities $g_{ij}$ and its associative $g^{ij}$, which are components of a metric tensor are connected by

$$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k \end{cases} \quad \text{and} \quad \delta^i_h g_{hk} = g^i_k.$$

By differentiating (1.1) partially with respect to $y^k$, we construct a new tensor $C_{ijk}$ defined by

$$C_{ijk} = \frac{1}{2} \partial_i g_{jk}.$$

This new tensor $C_{ijk}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in all its indices called (h)hv-torsion tensor Matsumoto [10]. According to Euler’s theorem on homogeneous functions, this tensor satisfies the following:

$$C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0.$$

The tensor $C^i_{jk}$ is the associate tensor of $C_{ijk}$ defined by

$$a) \ C_{jsk} = C^i_{jk} g_{is} \quad \text{and} \quad b) \ C_{skj} g^{ji} = C^i_{sk}.$$

The tensor $C^i_{jk}$ is called (v) hv-torsion tensor, and is positively homogeneous of degree -1 in $y^i$ and symmetric in its lower indices, i.e.

$$C^i_{jk} = C^i_{kj}.$$
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This tensor satisfies the following identities

(1.5) \[ C_{ijk}^{y} = C_{ijk}^{y} = 0. \]

Berwald’s covariant derivative of the vector $y^{i}$ vanish identically, i.e.

(1.6) \[ B_{k}y^{i} = 0. \]

In general, Berwald’s covariant derivative of the metric tensor $g_{ij}$ doesn’t vanish and is given by

(1.7) \[ B_{k}g_{ij} = -2C_{ijk|h}y^{h} = -2y^{h}B_{h}C_{ijk}. \]

Remark 1.1. The symbol $|h$ is the covariant differential operator with respect to $x^{h}$ in the sense of Cartan.

The tensor $K_{ijk}^{l}$ is called Cartan’s fourth curvature tensor, and is positively homogeneous of degree zero in $y^{i}$, defined by Rund [12]

\[ K_{ijk}^{l} := \partial_{h} \Gamma^{l}_{ijk} + \left( \partial_{l} \Gamma^{i}_{jk} \right) g_{h}^{k} + \Gamma^{l}_{ik} r^{i}_{j} - h/k. \]

Also, this curvature tensor $K_{ijk}^{l}$ satisfies the following relation too

(1.8) \[ K_{ijk}^{l} = R_{ijk}^{l} - C_{ijr}^{l} H_{k}^{r}. \]

The h(ν)-torsion tensor $H_{kh}^{l}$, the curvature tensors $K_{ijk}^{l}$ and $R_{ijk}^{l}$ are connected by Rund [12]

(1.9) \[ K_{ijk}^{l}y^{j} = H_{kj}^{l} = R_{jk}^{l}. \]

The curvature tensor $R_{ijk}^{l}$ is called Cartan’s third curvature tensor, and is positively homogeneous of degree zero in $y^{i}$, defined by Rund [12]

\[ R_{ijk}^{l} = \partial_{h} \Gamma^{l}_{ijk} + \left( \partial_{l} \Gamma^{i}_{jk} \right) g_{h}^{k} + C_{jm} \left( \partial_{k} g_{h}^{m} - G_{h}^{m} G_{i}^{j} \right) + \Gamma^{m}_{nk} \Gamma^{n}_{ik} - k/h. \]

The curvature tensor $R_{ijk}^{l}$ satisfies the following:

(1.10) a) $R_{jkr}^{l} = R_{jk}^{l}$ and b) $R_{lk}^{j}g_{jk}^{l} = R_{i}^{l}$.

The curvature scalar $R$ is given by Rund [12]

(1.11) \[ R_{ijk}^{l}y_{j}^{j} = R. \]

The curvature vector tensor $R_{j}$ is given by Rund [12]

(1.12) \[ R_{j} = K_{j} + C_{jr}^{l} H_{k}^{r}. \]

The associate curvature tensor $R_{ijk}^{l}$ of the curvature tensor $R_{ijk}^{l}$ is given by Rund [12]

(1.13) \[ R_{ijk}^{l} = g_{ij} R_{ijk}^{l}. \]

Also the curvature tensor $R_{ijk}^{l}$ satisfies the following identity Rund [12]

\[ R_{ijk}^{l} + R_{ikj}^{l} + R_{jik}^{l} + y^{m} \left( R_{m}^{r} P_{jk}^{i} + R_{m}^{r} P_{jr}^{i} + R_{m}^{r} P_{jm}^{i} \right) = 0, \]

where $P_{ijk}^{l}$ is known as hv- curvature tensor (Cartan’s second curvature tensor) and is defined by Rund [12]

(1.14) \[ P_{ijk}^{l} = \partial_{h} \Gamma^{i}_{jk} + C_{jm} P_{h}^{m} - C_{jkl}^{i}. \]

or equivalent by

\[ P_{ijk}^{l} = \partial_{h} \Gamma^{i}_{jk} + C_{jm} C_{kr}^{i} y^{r} - C_{jkl}^{i}. \]

or

\[ P_{ijk}^{l} = C_{ijk}^{l} y^{i} - g^{i} C_{jk}^{r} P_{ih}^{r} + C_{jk} P_{ih}^{r} - P_{j}^{i} P_{kl}^{r} C_{i}^{r}. \]

The curvature tensor $P_{ijk}^{l}$ is positively homogeneous of degree zero in $y^{i}$ and the tensor satisfies the following:

(1.15) a) $P_{ijk}^{l}y^{j} = \Gamma^{i}_{jk} y^{j} = P_{i}^{l} y^{i}$ and b) $P_{ijk}^{l} y^{k} = P_{ijk}^{l} y^{h} = 0$.

where $P_{ijk}^{l}$ is called as (νh)-torsion tensor and the associative tensor $P_{rkh}^{l}$ is given by Rund [12]

(1.16) \[ P_{ijk}^{l} g_{lh} = P_{rkh}. \]
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The curvature vector $P_k$ is given by

\begin{equation}
P'^i_k = P_k.
\end{equation}

A Finsler space $F_n$ for which the curvature tensor $R^i_{jkh}$ satisfies the following Hussien [6]:

\begin{equation}
B^i_m R^i_{jkh} = \lambda m R^i_{jkh}, \quad R^i_{jkh} \neq 0
\end{equation}
is called $R^h$ – recurrent space, where $\lambda m$ is non-zero covariant vector field.

Transvecting the condition (1.18) by $y^i$, using (1.9) and (1.6), we get

\begin{equation}
B^i_m H^i_{kh} = \lambda m H^i_{kh}.
\end{equation}

**Definition 1.1.** A Finsler space $F_n$ for which Cartan’s fourth curvature tensor $K^i_{jkh}$ satisfies the condition Qasem and Baleedi [11]

\begin{equation}
B^i_m K^i_{jkh} = \lambda m K^i_{jkh} + \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh}), \quad K^i_{jkh} \neq 0,
\end{equation}
will be called generalized BK-recurrent space, where $\lambda m$ and $\mu_m$ are non-zero covariant vectors field and tensor will be generalized recurrent tensor. We shall denote such space and tensor briefly by $GBK - R F_n$ and $GBK - R$, respectively Qasem and Baleedi [11].

Transvecting the condition (1.20) by $y^i$, using (1.9), (1.6) and (1.2), we get

\begin{equation}
B^i_m H^i_{kh} = \lambda m H^i_{kh} + \mu_m (\delta^i h y_k - \delta^i k y_h).
\end{equation}

Taking the covariant derivative for (1.8) with respect to $x^m$ in the sense of Berwald, using the condition (1.20) and (1.21), we get

\begin{equation}
B^i_m R^i_{jkh} = \lambda m K^i_{jkh} + \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh}) + (B^i_m C_{jr}) H^i_{kh} + C_{jr} [\lambda m H^i_{kh} + \mu m (\delta^i h y_k - \delta^i k y_h)].
\end{equation}

Using (1.8) in (1.22), we get

\begin{equation}
B^i_m R^i_{jkh} = \lambda m R^i_{jkh} + \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh} + C^i_{jr} y_k - C^i_{jk} y_h) + (B^i_m C_{jr}) H^i_{kh}.
\end{equation}

2. A $P^2$ –Like and $GBK - RF_n$

**Definition 2.1.** A $P^2$ –like space is characterized by Matsomoto[10]

\begin{equation}
P^i_{jkh} = \varphi_j C_{kh} - \varphi^i C_{jkh},
\end{equation}
where $\varphi_j$ and $\varphi^i$ are non-zero covariant and contravariant vector fields, respectively.

**Definition 2.2.** The $GBK - RF_n$ which is $P^2$ –like space [satisfies the condition (2.1)] will be called $P^2$ –Like and $GBK$ –$RF_n$ and is denoted briefly by $P^2$ –Like and $GBK - RF_n$.

Transvecting (1.8) by $\varphi_r$ and using (2.1), we get

\begin{equation}
\varphi_r R^i_{jkh} = \varphi_r K^i_{jkh} + (P^i_{rjm} + \varphi^i C_{rjm}) H^m_{kh}.
\end{equation}

Transvecting (2.2) by $y^r$, using (1.15a) and (1.4), we get

\begin{equation}
\varphi R^i_{jkh} = \varphi K^i_{jkh} + P^i_{jm} H^m_{kh},
\end{equation}
where $\varphi = \varphi_r y^r$.

Using (1.8) in (2.3), we get

\begin{equation}
P^i_{jm} = \varphi^i C_{jm},
\end{equation}
since $H^m_{kh} \neq 0$. Thus, we conclude that

**Theorem 2.1.** In $P^2$ –Like and $GBK - RF_n$, the torsion tensor $P^i_{jm}$ is proportional to the torsion tensor $C^i_{jm}$.

Taking the covariant derivative for (2.2) with respect to $x^m$ in the sense of Berwald and using (1.20), we get

\begin{equation}
B^i_m (\varphi_r R^i_{jkh}) = \varphi_r \lambda m K^i_{jkh} + \varphi_r \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh}) + K^i_{jkh} B^i_m \varphi_r
+ B^i_m [(P^i_{rjm} + \varphi^i C_{rjm}) H^m_{kh}],
\end{equation}
Using (2.2) in (2.5), we get

\begin{equation}
B^i_m (\varphi_r R^i_{jkh}) = \lambda m (\varphi_r R^i_{jkh}) + \varphi_r \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh}) + K^i_{jkh} B^i_m \varphi_r
+ B^i_m [(P^i_{rjm} + \varphi^i C_{rjm}) H^m_{kh}] - \lambda m (P^i_{rjm} + \varphi^i C_{rjm}) H^m_{kh},
\end{equation}
which can be written as

\begin{equation}
\varphi_r B^i_m R^i_{jkh} + R^i_{jkh} B^i_m \varphi_r = \lambda m (\varphi_r R^i_{jkh}) + \varphi_r \mu_m (\delta^i h g_{jk} - \delta^i k g_{jh})
\end{equation}
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This shows that
\[ (2.8) \quad B_m R^i_{jkh} = \lambda_m R^i_{jkh} + \mu_m (\delta^i_{h} g_{jk} - \delta^i_{k} g_{jh}), \]
if and only if
\[ (2.9) \quad B_m \left( (P^i_{jkh} + \varphi^i_{r} C_{rjm}) H^m_{kh} \right) = \lambda_m \left( (P^i_{jkh} + \varphi^i_{r} C_{rjm}) H^m_{kh} + R^i_{jkh} B_m \varphi_{r} \right) - K^i_{jkh} B_m \varphi_{r}, \]
since $\varphi_r \neq 0$. Assuming $B_m \varphi_{r} = 0$, i.e. covariant constant, then
\[ B_m \left( (P^i_{jkh} + \varphi^i_{r} C_{rjm}) H^m_{kh} \right) = \lambda_m \left( (P^i_{jkh} + \varphi^i_{r} C_{rjm}) H^m_{kh} \right). \]
Thus, we conclude that

**Theorem 2.2.** In $P_2$ – like – $GBK – RF_n$, the curvature tensor $R^i_{jkh}$ behaves as generalized recurrent if and only if the tensor $(P^i_{jkh} + \varphi^i_{r} C_{rjm}) H^m_{kh}$ behaves as recurrent, provided that $B_m \varphi_{r} = 0$.

Transvecting (2.5) by $y^j$, using (1.6), (1.9), (1.4), (1.2) and (1.15b), we get
\[ (2.10) \quad B_m H^i_{kh} = \lambda_m H^i_{kh} + \mu_m (\delta^i_{h} Y_k - \delta^i_{k} Y_h), \]
if and only if
\[ (2.11) \quad B_m \varphi_{r} = 0, \]
since $H^i_{kh} \neq 0$. Thus, we conclude that

**Theorem 2.3.** In $P_2$ – Like – $GBK – RF_n$, Berwald’s covariant derivative of first order for the hv-torsion tensor $H^i_{kh}$ is given by (2.10) if and only if $B_m \varphi_{r} = 0$ holds good.

Transvecting (1.12) by $\varphi_m$, we get
\[ (2.12) \quad \varphi_m R_j = \varphi_m K_j + \varphi_m C^i_{jih} H^i_{h}. \]
Using (2.1) in (2.12), we get
\[ (2.13) \quad \varphi_m R_j = \varphi_m K_j + (P^i_{mjh} + \varphi^i_{r} C^i_{mjh}) H^i_{h}. \]
Transvecting (2.13) by $y^m$, using (1.15a) and (1.4), we get
\[ (2.14) \quad \varphi_m y^m R_j = \varphi_m y^m K_j + P^i_{jih} H^i_{h} \]
which can be written as
\[ (2.15) \quad P^i_{jih} H^i_{h} = \varphi (R_j - K_j), \]
where $\varphi = \varphi_m y^m$.

Using (1.12) in (2.15), we get
\[ (2.16) \quad P^i_{jih} H^i_{h} = \varphi C^i_{jih} H^i_{h} \]
or
\[ (2.17) \quad P^i_{jih} = \varphi C^i_{jih}, \quad \text{since } H^i_{h} \neq 0. \]
Transvecting (2.17) by $y^h$ or $y^j$ and using (1.5), we get
\[ (2.18) \quad a) \quad P^i_{jih} y^h = 0 \quad \text{and} \quad b) \quad P^i_{jih} y^j = 0. \]
Thus, we conclude that

**Theorem 2.4.** In $P_2$ – Like – $GBK – RF_n$, we have the identities (2.15), (2.17), (2.18a) and (2.18b).

3. An $P^* – GBK – RF_n$

**Definition 3.1.** A $P^*$ – Finsler space is characterized by the condition Izumi ([7], [8], [9])
\[ (3.1) \quad p^i_{kh} = C^i_{khj} y^j = \varphi C^i_{kh}, \quad \varphi \neq 0. \]

**Definition 3.2.** The $GBK – RF_n$, which is a $P^*$ – Finsler space [satisfies the condition (3.1)] will be called $P^* – GBK – recurrent space and is denoted briefly by $P^* – GBK – RF_n$.

Transvecting (1.23) by $\varphi$ and using (3.1), we get
\[ (3.2) \quad \varphi B_m R^i_{jkh} = \lambda_m \varphi R^i_{jkh} + \varphi \mu_m (\delta^i_{h} g_{jk} - \delta^i_{k} g_{jh}) + \varphi \mu_m (P^i_{jkh} y_k - P^i_{jk} y_h) + (B_m p^i_{tr}) H^i_{kh}. \]
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This shows that
\[
\mathcal{B}_m R_{jk}^i = \lambda_m R_{jk}^i + \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh})
\]
if and only if
\[
(3.3) \quad \varphi \mu_m (\delta_h^i g_{jk} - \delta_k^i g_{jh}) + (\mathcal{B}_m P_{jr}^i) H_{kh}^r = 0.
\]
Thus, we conclude that

**Theorem 3.1.** In \( P^* - GBK - R F_h \), Cartan's third curvature tensor \( R_{jkh}^i \) behaves as generalized recurrent if and only if the condition (3.3) holds good.

Transvecting (3.2) by \( g_{it} \), using (1.3b), (1.13), (1.6) and (1.16), we get
\[
\varphi \mathcal{B}_m R_{jkh} = \varphi \lambda_m R_{jkh} + \varphi \mu_m (g_{jk} g_{ht} - g_{jh} g_{kt} + P_{tjh} y_k - P_{tjk} y_h)
\]
\[
+ (\mathcal{B}_m P_{tjr}^i) H_{kh}^r - \mathcal{B}_m g_{it} (g_{jr} H_{kh}^r - \varphi R_{jkh}^i) = 0.
\]
This shows that
\[
\mathcal{B}_m R_{jkh} = \lambda_m R_{jkh}
\]
if and only if
\[
(3.4) \quad \varphi \mu_m (g_{jk} g_{ht} - g_{jh} g_{kt} + P_{tjh} y_k - P_{tjk} y_h)
\]
\[
+ (\mathcal{B}_m P_{tjr}^i) H_{kh}^r - \mathcal{B}_m g_{it} (g_{jr} H_{kh}^r - \varphi R_{jkh}^i) = 0.
\]
Contracting the indices \( i \) and \( h \) in the condition (3.2), using (1.3b), (1.10a) and (1.17), we get
\[
\varphi \mathcal{B}_m R_{jk} = \varphi \lambda_m R_{jk} + \varphi \mu_m [(n - 1) g_{jk} + P_j y_k - P_j^s y_3] + (\mathcal{B}_m P_{jr}^i) H_{ks}^r.
\]
This shows that
\[
\mathcal{B}_m R_{jk} = \lambda_m R_{jk}
\]
if and only if
\[
(3.5) \quad \varphi \mu_m [(n - 1) g_{jk} + P_j y_k - P_j^s y_3] + (\mathcal{B}_m P_{jr}^i) H_{ks}^r = 0.
\]
Transvecting (3.2) by \( g^{jk} \), using (1.3a) , (1.10b) and in view of (1.3), we get
\[
\varphi \mathcal{B}_m R_{h} = \lambda_m \varphi R_{h} + \varphi \mu_m [(n - 1) \delta_h^i + g^{jk} (P_j y_k - P_j^s y_3)]
\]
\[
+ g^{jk} (\mathcal{B}_m P_{jr}^i) H_{kh}^r + \varphi (\mathcal{B}_m g^{jk}) R_{jkh}.
\]
This shows that
\[
\mathcal{B}_m R_{h} = \lambda_m R_{h}
\]
if and only if
\[
(3.6) \quad \varphi \mu_m [(n - 1) \delta_h^i + g^{jk} (P_j y_k - P_j^s y_3)] + g^{jk} (\mathcal{B}_m P_{jr}^i) H_{kh}^r + \varphi (\mathcal{B}_m g^{jk}) R_{jkh} = 0.
\]
Transvecting (3.5) by \( g^{jk} \), using (1.3a) and (1.11) we get
\[
\varphi \mathcal{B}_m R = \lambda_m \varphi R + \varphi \mu_m [n (n - 1) + g^{jk} (P_j y_k - P_j^s y_3)]
\]
\[
+ \varphi (\mathcal{B}_m g^{jk}) R_{jkh} + g^{jk} (\mathcal{B}_m P_{jr}^i) H_{kh}^r.
\]
This shows that
\[
\mathcal{B}_m R = \lambda_m R
\]
if and only if
\[
(3.7) \quad \varphi \mu_m [n (n - 1) + g^{jk} (P_j y_k - P_j^s y_3)] + \varphi (\mathcal{B}_m g^{jk}) R_{jkh} + g^{jk} (\mathcal{B}_m P_{jr}^i) H_{kh}^r = 0.
\]
Thus, we conclude that

**Theorem 3.2.** In \( P^* - GBK - R F_h \), the associatve curvature tensor \( R_{jkh} \), Ricci tensor \( R_{jk} \), the deviation tensor \( R_{h} \) and the curvature scalar \( R \) all behave as recurrent if and only if (3.4), (3.6), (3.7) and (3.8), respectively hold good.
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فضاء فنسلر، فضاء أحادي المعاودة

احادي المعاودة $BK$ – المعمم $P^*$ – $P_2$– like

في هذه الورقة تم دراسة بعض الأنواع لفضاء فنسلر المعمم على التوالي، $P^*$، $P_2$– like space

تم التسميته بتسمية $BK$ – recurrent space $P^*$ – recurrent space $P_2$ – generalized $BK$ – recurrent space $P^*$ –

على الترتيب.

وتم الحصول على مبرهنات مختلفة، والعديد من المطابقات التي تتحقق في هذه الفضاءات.

الكلمات المفتاحية: فضاء فنسلر، فضاء أحادي المعاودة $BK$ – المعمم