Double and triple generating functions of Laguerre polynomials of two variables

Mubarak Abood Al-Qufail , Ahmed Ali Atash and Salem Saleh Barahmah
Department of Mathematics, Faculty of Education, Aden University, Yemen
DOI: https://doi.org/10.47372/uajnas.2017.n1.a17

Abstract

The aim of the present paper is to obtain some double and triple generating functions for Laguerre polynomials of two variables. A number of interesting generating functions (known or new) are also derived as special cases of our main results.

Key words: Generating functions, Laguerre polynomials, Generalized Lauricella function, Hypergeometric functions.

1. Introduction

In 1991, Ragab [3] defined the Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\ \Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r\ L_{n-r}^{(\alpha)}(x)}{r!\ \Gamma(\alpha+n-r+1)\ \Gamma(\beta+r+1)}, \quad (1.1)$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomials of one variable [4,p.200(1)]

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\ {}_1F_1\left[\begin{matrix}-n\ ;\\ 1+\alpha\ ;\end{matrix}\ x\right]. \quad (1.2)$$

The definition (1.1) is equivalent to the following explicit representation of $L_n^{(\alpha,\beta)}(x,y)$:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n\,(\beta+1)_n}{(n!)^2}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}\ y^r\ x^s}{(\alpha+1)_s\,(\beta+1)_r\ r!s!}. \quad (1.3)$$

It may be noted that the definition (1.3) can be written as

$$L_n^{(\alpha,\beta)}(x,y)=\frac{(\alpha+1)_n(\beta+1)_n}{(n!)^2}\ \Psi_2\left[-n;\ \alpha+1,\ \beta+1;x,y\right], \quad (1.4)$$

where Ψ_2 is the confluent hypergeometric function of two variables [7,p,59(42)]

$$\Psi_2\left[a;b,c;x,y\right]=\sum_{r,s=0}^{\infty}\frac{(a)_{r+s}}{(b)_r\ (c)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}\ . \quad (1.5)$$

The general triple hypergeometric series $F^{(3)}[x,y,z]$ is defined as follows [7; p.69 (39)]:

$$F^{(3)}[x,y,z]=F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (e)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\ x,y,z\right]$$

$$=\sum_{m,n,p=0}^{\infty}\Lambda(m,n,p)\frac{x^m}{m!}\ \frac{y^n}{n!}\ \frac{z^p}{p!}, \quad (1.6)$$

Where

$$\Lambda(m,n,p)=\frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}$$

$$\frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p}. \tag{1.7}$$

(a) abbreviates the array of A parameters $a_1,\cdots,a_A$ with similar interpretation for (b), (b'), (b'') et cetera.

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined as follows [7,p.63(16)] :

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{matrix}x,y\right]$$

$$=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}\ . \tag{1.8}$$

The confluent hypergeometric functions $\Phi_2,\Phi_3,\ \Xi_1$ and Ξ_2 are defined as follows [7, p. 58-59]:

$$\Phi_2[\alpha,\beta;\gamma;x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.9}$$

$$\Phi_3[\beta;\gamma;x,y]=\sum_{m,n=0}^{\infty}\frac{(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\ , \tag{1.10}$$

$$\Xi_1[\alpha,\beta,\gamma;\delta;x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n(\gamma)_m}{(\delta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{1.11}$$

and

$$\Xi_2[\alpha,\beta;\gamma;x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_m(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\ .$$

(1.12)

2. Generating functions

In this section, we shall prove the following double and triple generating functions of Laguerre polynomials of two variables :

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_{m+k+n}}{m!k!(\alpha+1)_{m+k+n}(\beta+1)_{m+k+n}}L_n^{(\alpha+m+k,\beta+m+k)}(x,y)u^mv^kt^n$$

$$=(1-t)^{-\lambda}F^{1:0;0;0;0}_{2:0;0;0;0}\left[\begin{matrix}(\lambda:1,1,1,1)&:-;-;-;-;\\(\alpha+1:1,1,1,0),(\beta+1:1,1,0,1)&:-;-;-;-;\end{matrix}\frac{u}{1-t},\frac{v}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right], \tag{2.1}$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_{m+k+n}}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^mv^kt^n$$

$$=(1-t)^{-\lambda} F\begin{matrix}1:0;0;0;0\\2:0;0;0;0\end{matrix}\left[\begin{matrix}(\lambda:1,1,1,1) & :-;-;-;-;\\(\alpha+1:1,0,1,0),(\beta+1:0,1,0,1):-;-;-;-;\end{matrix}\frac{u}{1-t},\frac{v}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right], \tag{2.2}$$

where we have used a special case of generalized Lauricella function of Srivastava - Daoust [5, p.454] (see also [6, p.37 (21)]).

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda_1)_m\cdots(\lambda_p)_m(\delta_1)_k\cdots(\delta_q)_k}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^m v^k t^n$$

$$=e^t F\begin{matrix}0:p;0\\1:0;0\end{matrix}\left[\begin{matrix}- & :\lambda_1,\cdots,\lambda_p;- ;\\1+\alpha: & - \quad ;- ;\end{matrix}u,-xt\right]F\begin{matrix}0:q;0\\1:0;0\end{matrix}\left[\begin{matrix}- & :\delta_1,\cdots,\delta_q;- ;\\1+\beta: & - \quad ;- ;\end{matrix}v,-yt\right], \tag{2.3}$$

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\mu_1)_k\cdots(\mu_p)_k}{k!(\beta+1)_{k+n}}L_n^{(\alpha-n,\beta+k)}(x,y)v^k t^n$$

$$=e^{-xt}(1+t)^{\alpha} F\begin{matrix}0:p;1\\1:0;0\end{matrix}\left[\begin{matrix}- & :\mu_1,\cdots,\mu_p;-\alpha ;\\1+\beta: & - \quad ; - ;\end{matrix}v,\frac{yt}{1+t}\right]. \tag{2.4}$$

In view of the hypergeometric reduction formula [6,p.28(30)]

$$F\begin{matrix}p:0;0\\q:0;0\end{matrix}\left[\begin{matrix}a_1,...,a_p: -;-;\\b_1,...,b_q: -;-;\end{matrix}x,y\right]={}_pF_q\left[\begin{matrix}a_1,...,a_p ;\\b_1,...,b_q ;\end{matrix}x+y\right], \tag{2.5}$$

the generating function (2.2) can be written in the form

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_{m+k+n}}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^m v^k t^n$$

$$=(1-t)^{-\lambda}\Psi_2\left[\lambda;\alpha+1,\beta+1;\frac{u-xt}{1-t},\frac{v-yt}{1-t}\right]. \tag{2.6}$$

Proof of (2.1)

Denoting the left hand side of (2.1) by L and using the definition (1.3), we get

$$L=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(\lambda)_{m+k+n}u^m v^k t^n(-x)^r(-y)^s}{(n-r-s)!\,r!s!\,m!\,k!(\alpha+1)_{m+k+r}(\beta+1)_{m+k+s}}$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{m+k+n+r+s}u^m v^k t^n(-xt)^r(-yt)^s}{m!\,k!n!\,r!s!\,(\alpha+1)_{m+k+r}(\beta+1)_{m+k+s}}$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{m+k+r+s}u^m v^k(-xt)^r(-yt)^s}{m!\,k!r!s!\,(\alpha+1)_{m+k+r}(\beta+1)_{m+k+s}}\sum_{n=0}^{\infty}\frac{(\lambda+m+k+r+s)_n t^n}{n!}$$

$$=(1-t)^{-\lambda}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\lambda)_{m+k+r+s}}{m!\,k!r!s!\,(\alpha+1)_{m+k+r}(\beta+1)_{m+k+s}}\left(\frac{u}{1-t}\right)^m\left(\frac{v}{1-t}\right)^k\left(\frac{-xt}{1-t}\right)^r\left(\frac{-yt}{1-t}\right)^s$$

$$=(1-t)^{-\lambda} F\begin{matrix}1:0;0;0;0\\2:0;0;0;0\end{matrix}\left[\begin{matrix}(\lambda:1,1,1,1) & :-;-;-;-;\\(\alpha+1:1,1,1,0),(\beta+1:1,1,0,1):-;-;-;-;\end{matrix}\frac{u}{1-t},\frac{v}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right],$$

This completes the proof of (2.1).The results (2.2), (2.3) and (2.4) can be established similarly.

3. Special cases

In this section, we mention the following special cases :

(1) Setting $v=0$ in (2.1), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_{m+n}}{m!(\alpha+1)_{m+n}(\beta+1)_{m+n}}L_n^{(\alpha+m,\beta+m)}(x,y)u^m t^n$$
$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}\lambda::\ -\ ;-;\ -\ :-;-;-; \\ -::1+\alpha\ ;-;1+\beta:-\ ;-;-;\end{matrix}\frac{u}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right]. \quad (3.1)$$

(2) Setting $v=0$ in (2.2), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_{m+n}}{m!(\alpha+1)_{m+n}(\beta+1)_{n}}L_n^{(\alpha+m,\beta)}(x,y)u^m t^n$$
$$=(1-t)^{-\lambda}F^{(3)}\left[\begin{matrix}\lambda::\ -\ ;-;-:-;-;\ -\ ; \\ -::1+\alpha\ ;-;-:-\ ;-\ ;1+\beta;\end{matrix}\frac{u}{1-t},\frac{-xt}{1-t},\frac{-yt}{1-t}\right]. \quad (3.2)$$

(3) In (2.3) Setting

(I) $p=q=1$

(II) $p=q=2$

(III) $p=1,\ q=2$,

we get respectively the following triple generating functions:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\delta)_k}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^m v^k t^n$$
$$=e^t\,\Phi_3\left[\lambda;1+\alpha;u,-xt\right]\Phi_3\left[\delta;1+\beta;v,-yt\right], \quad (3.3)$$

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda_1)_m(\lambda_2)_m(\delta_1)_k(\delta_2)_k}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^m v^k t^n$$
$$=e^t\,\Xi_2\left[\lambda_1,\lambda_2;1+\alpha;u,-xt\right]\Xi_2\left[\delta_1,\delta_2;1+\beta;v,-yt\right] \quad (3.4)$$

and

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\lambda)_m(\delta_1)_k(\delta_2)_k}{m!k!(\alpha+1)_{m+n}(\beta+1)_{k+n}}L_n^{(\alpha+m,\beta+k)}(x,y)u^m v^k t^n$$
$$=e^t\,\Phi_3\left[\lambda;1+\alpha;u,-xt\right]\Xi_2\left[\delta_1,\delta_2;1+\beta;v,-yt\right]. \quad (3.5)$$

(4) Setting $u=0$ in (2.3), we get

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\delta_1)_k\cdots(\delta_p)_k}{k!(\alpha+1)_{n}(\beta+1)_{k+n}}L_n^{(\alpha,\beta+k)}(x,y)v^k t^n$$
$$=e^t\ {}_0F_1(-;1+\alpha\ ;-xt)\,F\begin{matrix}0:q;0\\1:0;0\end{matrix}\left[\begin{matrix}-\ :\delta_1,\cdots,\delta_p;-\ ; \\ 1+\beta:\ \ -\ \ ;-\ ;\end{matrix}v,-yt\right]. \quad (3.6)$$

(5) Further, for $q=1$ and $q=2$ (3.6) reduces respectively to the following generating functions:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\delta)_k}{k!(\alpha+1)_{n}(\beta+1)_{k+n}}L_n^{(\alpha,\beta+k)}(x,y)v^k t^n$$
$$=e^t\ {}_0F_1(-;1+\alpha\ ;-xt)\ \Phi_3\left[\delta;1+\beta;v,-yt\right] \quad (3.7)$$

and

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\delta_1)_k(\delta_2)_k}{k!(\alpha+1)_{n}(\beta+1)_{k+n}}L_n^{(\alpha,\beta+k)}(x,y)v^k t^n$$
$$=e^t\ {}_0F_1(-;1+\alpha\ ;-xt)\ \Xi_2\left[\delta_1,\delta_2;1+\beta;v,-yt\right]. \quad (3.8)$$

(6) Again, setting $u=v=0$ in (2.3), we get a known generating function of Chatterje [1,p.63 (2)]

$$\sum_{n=0}^{\infty}\frac{n!}{(\alpha+1)_n\,(\beta+1)_n}L_n^{(\alpha,\beta)}(x,y)t^n = e^t\ {}_0F_1(-;1+\alpha\,;-xt)\ {}_0F_1(-;1+\beta\,;-yt)\,. \tag{3.9}$$

(7) Setting $p=1$ and $p=2$ in (2.4), we get respectively the following double generating functions:

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\mu)_k}{k!(\beta+1)_{k+n}}L_n^{(\alpha-n,\beta+k)}(x,y)v^k t^n = e^{-xt}(1+t)^{\alpha}\,\Phi_2\left[\mu,-\alpha;\,1+\beta;v,\frac{yt}{1+t}\right] \tag{3.10}$$

and

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{n!(\mu_1)_k(\mu_2)_k}{k!(\beta+1)_{k+n}}L_n^{(\alpha-n,\beta+k)}(x,y)v^k t^n$$

$$= e^{-xt}(1+t)^{\alpha}\,\Xi_1\left[\mu_1,-\alpha,\mu_2\,;\,1+\beta;v,\frac{yt}{1+t}\right]. \tag{3.11}$$

(8) Setting $u=v=0$ in (2.6) , we get a known generating function of Khan and Shukla [2,p.163(7.4)]

$$\sum_{n=0}^{\infty}\frac{n!(\lambda)_n}{(\alpha+1)_n\,(\beta+1)_n}L_n^{(\alpha,\beta)}(x,y)t^n$$

$$= (1-t)^{-\lambda}\,\Psi_2\left[\lambda;\alpha+1,\beta+1;\,\frac{-xt}{1-t},\frac{-yt}{1-t}\right]. \tag{3.12}$$

References

1. Chatterjea, S.K. (1991). A note on Laguerre polynomials of two variables, Bull. Cal. Math. Sco. 82, 263-266.
2. Khan, M. A. and Shukla, A. K. (1997). On Laguerre polynomials of several variables, Bull. Cal. Math. Soc. 89,155-164.
3. Ragab, S. F. (1991). On Laguerre polynomials of two variables $L_n^{(\alpha,\beta)}(x,y)$, Bull. Cal. Math. Sco. 83, 253-262.
4. Rainville, E. D. (1960). Special Functions, Macmillan Company, New York . 365.
5. Srivastava, H. M. and Daoust, M. C. (1969). Certain generalized Neumann expansions associated with Kampé de Fériet function, Nederl. Akad. Wetensch. Indag. Math. 31, 449-457.
6. Srivastava, H. M. and Karlsson, P. W. (1985). Multiple Gaussian Hypergeometric Series, Halsted Press , New York. 425.
7. Srivastava, H. M. and Manocha, H. L. (1984). A Treatise on Generating Functions, Halsted Press, New York. 507.

دوال مولدة ثنائية وثلاثية لكثيرات حدود لأجير ذات متغيرين

مبارك عبود القفيل، أحمد علي عتش وسالم صالح بارحمة
قسم الرياضيات، كلية التربية، جامعة عدن
DOI: https://doi.org/10.47372/uajnas.2017.n1.a17

الملخص

يهدف هذا البحث إلى إيجاد دوال مولدة ثنائية وثلاثية لكثيرات حدود لأجير ذات متغيرين. عدد من الحالات الخاصة الجديدة والمعروفة تم اشتقاقها حالات خاصة لنتائجنا الرئيسية.

الكلمات المفتاحية: دوال مولدة ، كثيرات حدود لأجير، دالة لورسلا المعممة، دوال فوق هندسية.