q-Analogue modified Laguerre and generalized modified Laguerre Polynomials of One Variable

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Abstract

The q-Laguerre polynomials are important q-orthogonal polynomials whose applications and generalizations arise in many applications such as quantumgroup (oscillator algebra, etc.), q-harmonic oscillator and coding theory.
In this paper, we introduce the q-analogue modified Laguerre polynomials and generalized modified Laguerre polynomials of one variable. Some recurrence relations and q-Laplace transforms for these polynomials are derived.

Keywords: q-analogue modified Laguerre polynomials, generating functions, recurrence relations, q-Laplace transforms.

1. Introduction, Definitions and Notations

In this section, a summary of the mathematical notations and definitions required in this paper for the convenience of the reader is will be given.

The basic hyper geometric or q-hyper geometric function ${}_r\phi_s$ is defined by the series

$${}_r\phi_s\left[\begin{matrix} a_1,\dots,a_r; & \\ & q,z \\ b_1,\dots,b_s; & \end{matrix}\right]=\sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r,q)_n}{(b_1,\dots,b_s,q)_n}(-1)^{(1+s-r)n}q^{(1+s-r)\binom{n}{2}}\frac{z^n}{(q,q)_n}, \quad (1.1)$$

where $(a_1,\dots,a_r,q)_n=(a_1;q)_n\dots(a_r;q)_n$ and $(a;q)_n$ denotes the q-analogues of Pochhammer symbol, or q-shifted factorial is defined by [2]

$$(a;q)_n=\begin{cases}1 & ,\ n=0\\ \prod\limits_{0\le j\le n-1}(1-aq^j) & ,\ n=1,2,3,\dots\end{cases} ,(1.2)$$

$$\text{where } (q^{-n};q)_k=\begin{cases}0 & k>n\\ \dfrac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}, & k\le n\end{cases}, \quad (1.3)$$

$(0;q)_n=1$.

Also

$$(a;q)_{n+k}=(a;q)_n(aq^n;q)_k, \quad (1.4)$$

where $\lim\limits_{q\to1}\dfrac{(q^z;q)_k}{(1-q)^k}=(z)_k$.

The q-binomial coefficient is defined by [2]

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, \quad 0\le k\le n, \quad k,n\in N ,(1.5)$$

$$\begin{bmatrix} -n \\ k \end{bmatrix}_q = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q \left(-q^{-n}\right)^k q^{-\binom{k}{2}}, \quad n \in C; k \in N_0. \quad (1.6)$$

The q-exponential function $e_q(x)$ is defined by [2]

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \qquad e_1(x) = e\ (x), \qquad (1.7)$$

and

$$E_q(x) = \sum_{n\geq 0} (-1)^n q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} = (x;q)_\infty. \qquad (1.8)$$

The q-derivative with index α is defined by [7]

$$D_\alpha f(x) = \frac{f(q^\alpha x) - f(x)}{(q^\alpha - 1)x}, \qquad D_1 = D \qquad (1.9)$$

which for q-derivative of the pair of functions are valid

$$D(\lambda a(x) + \mu b(x)) = \lambda Da(x) + \mu Db(x), \quad (1.10)$$

$$D(a(x).b(x)) = a(qx)Db(x) + Da(x)b(x), \quad (1.11)$$

$$D\left(\frac{a(x)}{b(x)}\right) = \frac{Da(x)b(x) - a(x)Db(x)}{b(x)b(qx)}. \quad (1.12)$$

Exton [1] presented the following q-exponential functions:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} \frac{q^{\mu n(n-1)}}{[n]_q!} z^n,$$

where $[n]_q! = \dfrac{(q;q)_n}{(1-q)^n}$.

In Exton's formula, if we replace z by $\dfrac{x}{1-q}$ and μ by $2a$, we get

$$E\left(2a, \frac{x}{1-q}; q\right) = E_q(x, a),$$

where

$$E_q(x,a) = \sum_{n=0}^{\infty} \frac{q^{a\binom{n}{2}}}{(q;q)_n} x^n, \qquad (1.13)$$

which satisfies the functional relation

$$E_q(x,a) - E_q(qx,a) = xE_q(q^a x, a).$$

The above q-function can be rewritten by the formula

$$D_q E_q(x,a) = \frac{1}{1-q} E_q(q^a x, a). \quad (1.14)$$

Also, the q-analogue of $(x \pm y)^n$ is given by [6]

$$(x \pm y)^n = (x \pm y)_n = x^n \left(\mp y/x; q\right)_n = x^n \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} \left(\pm y/x\right)^k. \quad (1.15)$$

The Laguerre polynomials $L_n(x)$ of order n are defined by means of a generating relation [8]

$$(1-t)^{-1}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty} L_n(x)t^n, \quad |t|<1,\ 0<x<\infty \ (1.16)$$

and the following series definition:

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^k n!\, x^k}{(k!)^2(n-k)!}. \quad (1.17)$$

Also, the associated Laguerre polynomials are defined by the generating function [8]

$$(1-t)^{-1-\alpha}\exp\left(\frac{-xt}{1-t}\right)=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n, \quad (1.18)$$

and the series definition

$$L_n^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n\, x^k}{k!(n-k)!(1+\alpha)_k}. \quad (1.19)$$

Goyal[3] defines the generating relation for the one variable modified Laguerre polynomials (MLP)by

$$\sum_{n=0}^{\infty} L_{a,b,c,n}(x)t^n=(1-bt)^{-c}\exp\left(\frac{-axt}{1-bt}\right), (1.20)$$

Where the MLP $L_{a,b,c,n}(x)$ is given by

$$L_{a,b,c,n}(x)=\frac{(c)_n(b)^n}{n!}\,{}_1F_1\left(-n;c;\frac{ax}{b}\right). (1.21)$$

The q-Laguerre polynomials are defined by [4,5,7]

$$L_n^{(\alpha)}(x;q)=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix}q^{-n};\\ & q;-(1-q)q^{\alpha+n+1}x\\ q^{\alpha+1};\end{matrix}\right)$$

$$=\frac{(q^{\alpha+1};q)_n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(q^{-n};q)_k\, q^{\binom{k}{2}}(1-q)^k\,(q^{n+\alpha+1}x)^k}{(q^{\alpha+1};q)_k(q;q)_k} \quad (1.22)$$

where $\alpha>-1,\ \ 0<q<1$ and $n=0,1,2,3,\ldots$.

The q-Laguerre polynomials are specified by the following generating function:

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x;q)t^n=\frac{1}{(t;q)_\infty}\,{}_1\phi_1(-x;0;q,q^{\alpha+1}t). \quad (1.23)$$

The q-Laplace transforms is defined by[9]

$${}_qL_s\{f(t)\}=\frac{1}{1-q}\int_0^\infty e_q(-st)f(t)\,d(t;q); \qquad R(s)>0 \ (1.24)$$

and

$${}_qL_s\{t^n\}=\frac{(q;q)_n}{s^{n+1}}; \quad n>0. \quad (1.25)$$

2. q-Analogue Modified Laguerre Polynomials of One Variable

In this section, q-analogue modified Laguerre polynomial of one variable is introduced by the followinggenerating function:

$$[1-\beta t]_q^{-m}\exp_q\left[\frac{-\alpha xt}{1-\beta t}\right]=\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n. \quad (2.1)$$

Now, we get hyper geometric representation of q-analogue Laguerre polynomials $L_{\alpha,\beta,m,n}(x;q)$ in the form of the following theorem:

Theorem2.1. The following hyper geomatric representation for the q-analogue Laguerre polynomials $L_{\alpha,\beta,m,n}(x;q)$ holds true:

$$L_{\alpha,\beta,m,n}(x;q)=\frac{(q^m;q)_n(b)^n}{q^{mn}(q;q)_n}{}_1\phi_1\left(q^{-n};q^m;q,-q^{m+1}\frac{\alpha x}{\beta}\right), \qquad (2.2)$$

where $|t|<1,\ |q|<1$.

Proof. From (2.1) and using (1.7), we obtain

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha xt)^r}{(q,q)_r}[1-\beta t]_q^{-m-r},$$

by using relation (1.15),we get

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n=\sum_{r=0}^{\infty}\frac{(-1)^r(\alpha xt)^r}{(q,q)_r}\sum_{n=0}^{\infty}\begin{bmatrix}-m-r\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta t)^n,$$

on using relation (1.6) the above equation becomes

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n=\sum_{n,r=0}^{\infty}\frac{(-1)^r}{(q,q)_r}\begin{bmatrix}m+r+n-1\\ n\end{bmatrix}_q\left(q^{-m-r}\right)^n(\alpha xt)^r(\beta t)^n,$$

Using relation (1.5), we get

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n=\sum_{n,r=0}^{\infty}\frac{(-1)^r\left(q^{-m-r}\right)^n(q;q)_{m+n+r-1}}{(q;q)_r(q;q)_{m+r-1}(q;q)_n}(\alpha x)^r(\beta)^n t^{n+r}$$

$$=\sum_{n=0}^{\infty}\frac{(q^m;q)_n(\beta)^n}{q^{mn}(q;q)_n}\sum_{r=0}^{n}\frac{(-1)^r q^{\binom{r}{2}-nr}(q;q)_n q^{\binom{r}{2}}}{(q;q)_r(q^m;q)_r(q;q)_{n-r}}\left(q^{1+m}\frac{\alpha x}{\beta}\right)^r t^n.$$

From relation (1.3), we get

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x;q)t^n=\sum_{n=0}^{\infty}\frac{(q^m;q)_n(\beta)^n}{q^{mn}(q;q)_n}\sum_{r=0}^{n}q^{\binom{r}{2}}\frac{(q^{-n};q)_r}{(q;q)_r(q^m;q)_r}\left(q^{1+m}\frac{\alpha x}{\beta}\right)^r t^n.$$

By using definition (1.1) and the nequating the coefficients of t^n, we get the required relation (2.2).

Next, we derive some recurrence relations for the polynomials $L_{\alpha,\beta,m,n}(x;q)$ in the form of the following theorems:

Theorem2.2. The q-analogue modified Laguerre polynomials of one variable $L_{\alpha,\beta,m,n}(x;q)$ satisfy the following relation:

$$\frac{\partial}{\partial x}L_{\alpha,\beta,m,n}(x;q)=-\alpha\, L_{\alpha,\beta,m+1,n-1}(x;q).\ (2.3)$$

Proof. Differentiating both sides of (2.1) with respect to x, we get

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{\alpha,\beta,m,n}(x;q)t^n=[1-\beta t]_q^{-m}\left[\frac{-\alpha t}{1-\beta t}\right]_q\exp_q\left[\frac{-\alpha xt}{1-\beta t}\right]$$

$$=-\alpha t\sum_{r=0}^{\infty}(-1)^r\frac{(\alpha xt)^r}{(q;q)_r}[1-\beta t]_q^{-m-r-1},$$

by using relation (1.15), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x;q)t^n = -\alpha t \sum_{r=0}^{\infty} (-1)^r \frac{(\alpha x t)^r}{(q;q)_r} \sum_{n=0}^{\infty} \begin{bmatrix} -m-r-1 \\ n \end{bmatrix} q^{\binom{n}{2}} (-\beta t)^n .$$

From relation (1.6), we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x;q)t^n = -\alpha t \sum_{r=0}^{\infty} (-1)^r \frac{(\alpha x t)^r}{(q;q)_r} \sum_{n=0}^{\infty} \begin{bmatrix} m+r+n \\ n \end{bmatrix} \left(q^{-m-r-1}\right)^n (\beta t)^n ,$$

which on using relation (1.5) becomes

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x;q)t^n = -\alpha \sum_{n,r=0}^{\infty} (-1)^r \frac{\left(q^{-m-r-1}\right)^n (q;q)_{m+r+n} (\alpha x)^r (\beta)^n}{(q;q)_r (q;q)_{m+r} (q;q)_n} t^{n+r+1}$$

$$= -\alpha \sum_{n=0}^{\infty} \frac{\left(q^{m+1};q\right)_{n-1} (\beta)^{n-1}}{q^{(m+1)(n-1)} (q;q)_{n-1}} \sum_{r=0}^{n-1} (-1)^r \frac{q^{\binom{r}{2} - (n-1)r} (q;q)_{n-1} q^{\binom{r}{2}}}{(q;q)_r \left(q^{m+1};q\right)_r (q;q)_{n-r-1}} \left(q^{m+2} \frac{\alpha x}{\beta} \right)^r t^n ,$$

using relation (1.3) and the equating the coefficients of t^n , we obtain

$$\frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x;q) = -a \frac{\left(q^{m+1};q\right)_{n-1} (\beta)^{n-1}}{q^{(m+1)(n-1)} (q;q)_{n-1}} \sum_{r=0}^{n-1} q^{\binom{r}{2}} \frac{\left(q^{1-n};q\right)_r}{(q;q)_r \left(q^{m+1};q\right)_r} \left(q^{m+2} \frac{\alpha x}{\beta} \right)^r$$

$$= -a \frac{\left(q^{m+1};q\right)_{n-1} (\beta)^{n-1}}{q^{(m+1)(n-1)} (q;q)_{n-1}} {}_1\phi_1\left(q^{1-n}; q^{m+1}; q, -q^{m+2} \frac{\alpha x}{\beta} \right),$$

which on using definition (1.1) yields the required relation (2.3).

Theorem2.3. The q-analogue modified Laguerre polynomials of one variable $L_{\alpha,\beta,m,n}(x;q)$ satisfy the following relation:

$$[n+1]_q L_{\alpha,\beta,m,n+1}(x;q) = m\beta L_{\alpha,\beta,m+1,n}(x;q)$$

$$-\alpha x \sum_{k=0}^{n} \sum_{r=0}^{n-k} (-1)^k \frac{q^{(-k-1)(n-k-1)} \left(q^{k+1};q\right)_{n-k-r} \left(q^{m+1};q\right)_r (\alpha x)^k (\beta)^{n-k}}{q^{(m+1)r} (q;q)_r (q;q)_k (q;q)_{n-k-r}} . \quad (2.4)$$

Proof. Differentiating both sides of (2.1) with respect to t, we get

$$\sum_{n=1}^{\infty} [n]_q L_{\alpha,\beta,m,n}(x;q)t^{n-1} = m\beta [1-\beta t]_q^{-m-1} \exp_q\left[-\frac{\alpha x t}{1-\beta t} \right]$$

$$+ \left\{ \frac{-\alpha x}{[1-\beta t]_q [1-q\beta t]_q} \right\} [1-q\beta t]_q^{-m} \exp_q\left[-\frac{\alpha x t}{1-\beta t} \right],$$

by using relations (1.7) and (1.15), we get

$$\sum_{n=0}^{\infty} [n+1]_q L_{\alpha,\beta,m,n+1}(x;q)t^n$$

$$= m\beta \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha x t)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -m-k-1 \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-\beta t)^n$$

$$-\alpha x \sum_{r=0}^{\infty} \begin{bmatrix} -m-1 \\ r \end{bmatrix}_q q^{\binom{r}{2}} (-q\beta t)^r \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha x t)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -k-1 \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-\beta t)^n ,$$

applying relation (1.6), we find

$$\sum_{n=0}^{\infty} [n+1]_q L_{\alpha,\beta,m,n+1}(x;q)t^n = m\beta \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha x)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} m+k+n \\ n \end{bmatrix}_q q^{(-m-k-1)n} (\beta)^n t^{n+k}$$

$$-\alpha x \sum_{r=0}^{\infty} \begin{bmatrix} m+r \\ r \end{bmatrix}_q q^{(-m-1)r} (q\beta)^r \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{(-k-1)n} (q^{k+1};q)_n (\alpha x)^k (\beta)^n}{(q;q)_k (q;q)_n} t^{n+k+r},$$

on using relations (1.5) and (1.3), we find

$$\sum_{n=0}^{\infty} [n+1]_q L_{\alpha,\beta,m,n+1}(x;q)t^n = m\beta \sum_{n=0}^{\infty} \frac{(q^{m+1};q)_n (\beta)^n}{q^{(m+1)n} (q;q)_n} \sum_{k=0}^{n} q^{\binom{k}{2}} \frac{(q^{-n};q)_k}{(q;q)_k (q^{m+1};q)_k} \left(q^{m+2} \frac{\alpha x}{\beta} \right)^k t^n$$

$$-\alpha x \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{r=0}^{n-k} (-1)^k \frac{q^{(-k-1)(n-k-1)} (q^{k+1};q)_{n-k-r} (q^{m+1};q)_r (\alpha x)^k (\beta)^{n-k}}{q^{(m+1)r} (q;q)_r (q;q)_k (q;q)_{n-k-r}} t^n$$

By equating the coefficients of t^n , we get the required relation(2.4).

Theorem 2.4. The q-Laplace transform of the q-analogue modified Laguerre polynomials $L_{\alpha,\beta,m,n}(t;q)$is given by

$${}_qL_s\{L_{\alpha,\beta,m,n}(t;q)\} = \frac{(q^m,q)_n (\beta)^n}{sq^{mn}(1-q)(q,q)_n} \sum_{r=0}^{n} q^{\binom{r}{2}} \frac{(q^{-n},q)_r}{(q^m,q)_r} \left(q^{1+m} \frac{\alpha}{\beta s} \right)^r ,(2.5)$$

where $t \geq 0$.

Proof. By using the definition of the q-Laplace transform (1.24), we get

$${}_qL_s\{L_{\alpha,\beta,m,n}(t;q)\} = {}_qL_s\left\{ \frac{(q^m,q)_n (\beta)^n}{q^{mn}(q,q)_n} \sum_{r=0}^{n} q^{\binom{r}{2}} \frac{(q^{-n},q)_r}{(q,q)_r (q^m,q)_r} \left(q^{1+m} \frac{\alpha t}{\beta} \right)^r \right\}$$

$$= \frac{1}{1-q} \int_0^{\infty} e_q(-st) \left\{ \frac{(q^m,q)_n (\beta)^n}{q^{mn}(q,q)_n} \sum_{r=0}^{n} q^{\binom{r}{2}} \frac{(q^{-n},q)_r}{(q,q)_r (q^m,q)_r} \left(q^{1+m} \frac{\alpha t}{\beta} \right)^r \right\} d(t;q)$$

$$= \frac{(q^m,q)_n (\beta)^n}{q^{mn}(1-q)(q,q)_n} \sum_{r=0}^{n} q^{\binom{r}{2}} \frac{(q^{-n},q)_r}{(q,q)_r (q^m,q)_r} \left(q^{1+m} \frac{\alpha}{\beta} \right)^r \int_0^{\infty} e_q(-st) t^r dt,$$

on using relation (1.25), we get

$${}_qL_s\{L_{\alpha,\beta,m,n}(t;q)\} = \frac{(q^m,q)_n (\beta)^n}{q^{mn}(1-q)(q,q)_n} \sum_{r=0}^{n} q^{\binom{r}{2}} \frac{(q^{-n},q)_r}{(q,q)_r (q^m,q)_r} \left(q^{1+m} \frac{\alpha}{\beta} \right)^r \left\{ \frac{(q,q)_r}{s^{r+1}} \right\},$$

which is the required relation (2.5).

3. The Generalized q-Analogue Modified Laguerre Polynomials of One Variable

In this section, a generalized q-analogue modified Laguerre polynomial of one variable by is introduced means of the following generating function:

$$(1-\beta t)_q^{-m} E_q\left(\frac{-\alpha x t}{1-\beta t}, a \right) = \sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n, \quad |t|<1, \ |q|<1 .(3.1)$$

Now, we get hyper geometric representation of the generalized q-analogue Laguerre polynomials in the form of the following theorem:

Theorem 3.1. The following hyper geometric representation of the generalized q-analogue Laguerre polynomials $L_{\alpha,\beta,m,n}(x,a;q)$ holds true:

$$L_{\alpha,\beta,m,n}(x,a;q) = \frac{(q^m;q)_n (\beta)^n}{q^{mn}(q;q)_n} {}_1\phi_1\left(q^{-n}, q^m; q^{(a+1)}, -q^{m+1} \frac{\alpha x}{\beta} \right) .(3.2)$$

Proof. From (3.1) and using (1.13), we obtain

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n = (1-\beta t)_q^{-m} \sum_{k=0}^{\infty} \frac{(-1)^k q^{a\binom{k}{2}} (\alpha\, xt)^k}{(q;q)_k} (1-\beta t)_q^{-k} ,$$

by using relation (1.15), we get

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n = \sum_{k=0}^{\infty} (-1)^k \frac{q^{a\binom{k}{2}} (\alpha xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -m-k \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-\beta t)^n ,$$

which by using relation (1.6) becomes

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{a\binom{k}{2}} (\alpha x)^k}{(q;q)_k} \begin{bmatrix} m+k+n-1 \\ n \end{bmatrix}_q \left(q^{-m-k}\right)^n (\beta)^n t^{n+k} ,$$

On using relation (1.5), we obtain

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{q^{a\binom{k}{2}+(-m-k)n} (\alpha x)^k}{(q;q)_k} \frac{(q;q)_{m+k+n-1}}{(q;q)_{m+k-1}(q;q)_n} (\beta)^n t^{n+k}$$

$$= \sum_{n=0}^{\infty} \frac{(q^m;q)_n (\beta)^n}{q^{mn}(q;q)_n} \sum_{k=0}^{n} (-1)^k \frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}}{(q;q)_k} \frac{(q;q)_n}{(q^m;q)_k (q;q)_{n-k}} \left(q^{m+1} \frac{\alpha x}{\beta} \right)^k t^n ,$$

now using relation (1.3), we get

$$\sum_{n=0}^{\infty} L_{\alpha,\beta,m,n}(x,a;q)t^n = \sum_{n=0}^{\infty} \frac{(q^m;q)_n (\beta)^n}{q^{mn}(q;q)_n} \sum_{k=0}^{n} \frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k} \frac{(q^{-n};q)_k}{(q^m;q)_k} \left(q^{m+1} \frac{\alpha x}{\beta} \right)^k t^n ,$$

Which, by equating the coefficients of t^n, yields the required relation (3.2).

Next, some recurrence relations are derived for the polynomials $L_{\alpha,\beta,m,n}(x,a;q)$ in the form of the following theorems:

Theorem3.2. The generalized q-analogue Laguerre polynomials of one variable $L_{\alpha,\beta,m,n}(x,a;q)$ satisfy the following relation:

$$\frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,a;q) = -\frac{\alpha}{(1-q)} L_{\alpha,\beta,m+1,n-1}(q^a x,a;q) . \quad (3.3)$$

Proof.Differentiating both sides of (3.1) with respect to x and using (1.14), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,a;q)t^n = (1-\beta t)^{-m} \left\{ \frac{-\alpha t}{(1-q)(1-\beta t)} \right\} E_q\left(-\frac{q^a \alpha xt}{1-\beta t}, a \right),$$

by using relations (1.13) and (1.15), we get

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,a;q)t^n = -\frac{\alpha t}{(1-q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{ak+a\binom{k}{2}} (\alpha xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} -1-m-k \\ n \end{bmatrix}_q q^{\binom{n}{2}} (-\beta t)^n ,$$

from relation (1.6), we find

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial x} L_{\alpha,\beta,m,n}(x,a;q)t^n = -\frac{\alpha t}{(1-q)} \sum_{k=0}^{\infty} (-1)^k \frac{q^{ak+a\binom{k}{2}} (\alpha xt)^k}{(q;q)_k} \sum_{n=0}^{\infty} \begin{bmatrix} m+k+n \\ n \end{bmatrix}_q \left(q^{-1-m-k}\right)^n (\beta t)^n ,$$

which by using relation (1.5), we find

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial x}L_{\alpha,\beta,m,n}(x,a;q)t^n=-\frac{\alpha}{(1-q)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-1-m-k)n}(q;q)_{m+k+n}(\alpha x)^k(\beta)^n}{(q;q)_k(q;q)_{m+k}(q;q)_n}t^{n+k+1}$$

$$=-\frac{\alpha}{(1-q)}\sum_{n=0}^{\infty}\frac{(q^{m+1};q)_{n-1}(\beta)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{n-1}(-1)^k\frac{q^{ak+(a+1)\binom{k}{2}+\binom{k}{2}-(n-1)k}(q;q)_{n-1}}{(q;q)_k(q^{m+1};q)_k(q;q)_{n-k-1}}\left(q^{m+2}\frac{\alpha x}{\beta}\right)^k t^n,$$

by equating the coefficient of t^n, we obtain

$$\frac{\partial}{\partial x}L_{\alpha,\beta,m,n}(x,a;q)=-\frac{\alpha}{(1-q)}\frac{(q^{m+1};q)_{n-1}(\beta)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\sum_{k=0}^{\infty}\frac{q^{ak+(a+1)\binom{k}{2}}(q^{1-n};q)_k}{(q;q)_k(q^{m+1};q)_k}\left(q^{m+2}\frac{\alpha x}{\beta}\right)^k$$

$$=-\frac{\alpha}{(1-q)}\frac{(q^{m+1};q)_{n-1}(\beta)^{n-1}}{q^{(m+1)(n-1)}(q;q)_{n-1}}\,{}_1\phi_1\left(q^{1-n},q^{m+1};q^{(a+1)},-q^{a+m+2}\frac{\alpha x}{\beta}\right),$$

which is the required result (3.3).

Theorem3.3. The generalized q-analogue Laguerre polynomials of one variable $L_{\alpha,\beta,m,n}(x,a;q)$ satisfy the following relations:

$$[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n=m\beta L_{\alpha,\beta,m+1,n}(x,a;q)$$

$$-\frac{\alpha x}{(1-q)}\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-k-1)(n-k-r)}(q^{m+1};q)_r(q^{k+1};q)_{n-k-r}}{q^{mr}(q;q)_r(q;q)_k(q;q)_{n-k-r}}(\alpha x)^k(\beta)^{n-k}.\quad(3.4)$$

Proof.Differentiating the both sides of (3.1) with respect to t, we get

$$\sum_{n=0}^{\infty}[n]_q L_{\alpha,\beta,m,n}(x,a;q)t^{n-1}=m\beta(1-\beta t)^{-m-1}E_q\left(-\frac{\alpha xt}{1-\beta t},a\right)$$

$$+\left\{\frac{-\alpha x}{(1-q)(1-\beta t)(1-q\beta t)}\right\}(1-q\beta t)^{-m}E_q\left(-\frac{q^a\alpha xt}{1-\beta t},a\right),$$

by using relation (1.13), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n=m\beta\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-m-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta t)^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}-m-1\\ r\end{bmatrix}_q q^{\binom{r}{2}}(-q\beta t)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha xt)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}-k-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}(-\beta t)^n,$$

Which, by using relation (1.6), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n=m\beta\sum_{k=0}^{\infty}(-1)^k\frac{q^{a\binom{k}{2}}(\alpha x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}m+k+n\\ n\end{bmatrix}_q(q^{-m-k-1}\beta)^n t^{n+k}$$

$$-\frac{\alpha x}{(1-q)}\sum_{r=0}^{\infty}\begin{bmatrix}m+r\\ r\end{bmatrix}_q(q^{-m-1}q\beta t)^r\sum_{k=0}^{\infty}(-1)^k\frac{q^{ak+a\binom{k}{2}}(\alpha x)^k}{(q;q)_k}\sum_{n=0}^{\infty}\begin{bmatrix}k+n\\ n\end{bmatrix}_q(q^{-k-1}\beta)^n t^{n+k},$$

Applying relation (1.5), we find

$$\sum_{n=0}^{\infty}[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n = m\beta\sum_{n,k=0}^{\infty}(-1)^k \frac{q^{a\binom{k}{2}+(-m-k-1)n}(q;q)_{m+k+n}(\alpha x)^k(\beta)^n}{(q;q)_k(q;q)_{m+k}(q;q)_n}t^{n+k}$$

$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}(-1)^k \frac{q^{ak+a\binom{k}{2}+(-k-1)n}(q;q)_{m+r}(q;q)_{m+k+n}(\alpha x)^k(\beta)^{n+r}}{q^{mr}(q;q)_m(q;q)_r(q;q)_k(q;q)_k(q;q)_n}t^{n+k+r},$$

from relation (1.3), we obtain

$$\sum_{n=0}^{\infty}[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n$$

$$=m\beta\sum_{n=0}^{\infty}\frac{(q^{m+1};q)_n(\beta)^n}{q^{(m+1)n}(q;q)_n}\sum_{k=0}^{n}(-1)^k\frac{q^{(a+1)\binom{k}{2}+\binom{k}{2}-nk}(q;q)_n}{(q;q)_k(q^{m+1};q)_k(q;q)_{n-k}}\left(q^{m+2}\frac{\alpha x}{\beta}\right)^k t^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-k-1)(n-k-r)}(q^{m+1};q)_r(q^{k+1};q)_{n-k-r}(\alpha x)^k(\beta)^{n-k}}{q^{mr}(q;q)_r(q;q)_k(q;q)_{n-k-r}}t^n,$$

using definition (1.1), we get

$$\sum_{n=0}^{\infty}[n+1]_q L_{\alpha,\beta,m,n+1}(x,a;q)t^n = m\beta\sum_{n=0}^{\infty}\frac{(q^{m+1};q)_n(\beta)^n}{q^{(m+1)n}(q;q)_n}{}_1\phi_1\left(q^{-n},q^{m+1};q^{(a+1)},-q^{m+2}\frac{\alpha x}{\beta}\right)t^n$$

$$-\frac{\alpha x}{(1-q)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{r=0}^{n-k}(-1)^k\frac{q^{ak+a\binom{k}{2}+(-k-1)(n-k-r)}(q^{m+1};q)_r(q^{k+1};q)_{n-k-r}(\alpha x)^k(\beta)^{n-k}}{q^{mr}(q;q)_r(q;q)_k(q;q)_{n-k-r}}t^n,$$

by equating the coefficients of t^n , we get the required relation(3.4).

Theorem3.4. The q-Laplace transform of q-modified Laguerre polynomials $L_{\alpha,\beta,m,n}(t,a;q)$ is defined by

$${}_qL_s\{L_{\alpha,\beta,m,n}(t,a;q)\}=\frac{(q^m;q)_n(\beta)^n}{sq^{mn}(q;q)_n}\sum_{k=0}^{n}q^{(a+1)\binom{k}{2}}\frac{(q^{-n};q)_k}{(q^m;q)_k}\left(q^{m+1}\frac{\alpha}{s\beta}\right)^k, \tag{3.5}$$

where $t\geq 0,\quad R(s)>0$.

Proof. By using definition of the q-Laplace transform (1.24),then the relation (3.2) can be written

$${}_qL_s\{L_{\alpha,\beta,m,n}(t,a;q)\}=\int_0^\infty e_q(-st)\left\{\frac{(q^m;q)_n(\beta)^n}{q^{mn}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{(q^{-n};q)_k}{(q^m;q)_k}\left(q^{m+1}\frac{\alpha t}{\beta}\right)^k\right\}d(t;q)$$

as

$$=\frac{(q^m;q)_n(\beta)^n}{q^{mn}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{(q^{-n};q)_k}{(q^m;q)_k}\left(q^{m+1}\frac{\alpha}{\beta}\right)^k {}_qL_s\{t^k\},$$

by using relation (1.25), we obtain

$${}_qL_s\{L_{\alpha,\beta,m,n}(t,a;q)\}=\frac{(q^m;q)_n(\beta)^n}{q^{mn}(q;q)_n}\sum_{k=0}^{n}\frac{q^{(a+1)\binom{k}{2}}}{(q;q)_k}\frac{(q^{-n};q)_k}{(q^m;q)_k}\left(q^{m+1}\frac{\alpha}{\beta}\right)^k\left\{\frac{(q;q)_k}{s^{k+1}}\right\},$$

which is the required relation (3.5).

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q-نظير كثيرات حدود لأجير المعدلة ولاجير المعدلة المعممة بمتغير واحد

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الملخص

كثيرات حدود لاجير من النوع كيو هي مهمة في كثيرات حدود المتعامدة من النوع كيو حيث تظهر تطبيقاتها وتعميمها في العديد من التطبيقات مثل نظرية الكم (الجبر المتذبذب) والمذبذب المتناسق من النوع كيو ونظرية التشفير.
في هذه الورقة قدمنا كثيرات حدود معدلة للاجير و لاجير المعممة ذات متغير من النوع كيو وأيضاً أثبتنا العلاقات التكرارية و صيغة لابلاس لهما.

الكلمات المفتاحية: كثيرات حدود معدلة لاجير من النوع كيو, الدوال المولدة, العلاقات التكرارية, تحويل لابلاس من نوع كيو.