Study of certain types of K^h-birecurrent Finsler spaces (II)

Fahmi Yaseen Abdo Qasem, Gamal Abobakar Abdallah Bawazeir and Meqdad Ahmed Abdullah Ali
Dept. of Maths., Faculty of Edu.-Aden,Univ.of Aden, Khormaksar, Aden-Yemen.
fahmi.yaseen@yahoo.com
DOI: https://doi.org/10.47372/uajnas.2017.n2.a13

Abstract

Qasem and Ali [15] defined the k^h– BR- affinely connected space. In thin space, they obtained the condition for some tensors to be birecurrent, proved some tensors are birecurrent, gave new definition for some tensors and found some identities. Also they defined for k^h– BR –Lands bergspace and obtained various identities in such space.
In this paper we have used the property of K^h-*BR*-$F_n$in P2-like space and P^*-Finsler space. We have obtained different theorems for some tensors to be satisfying the conditions of the above spaces and various identities in such spaces were also obtained.

Keywords: K^h-Birecurrent Space, P2-like K^h-Birecurrent Space and P^*-K^h-Birecurrent Space.

1.Introduction

A P2-like space has been introduced by Matsumoto [8]. Verma [17] obtained some results whenR^h-recurrent and C-concircularly spaces are P2-like spaces. Dikshit [3] obtained certain identities in a P2-like R^h-birecurrent space. Qasem [13] obtained certain identities in a P2-like R^h-generalized and P2-likeR^h-special generalized birecurrent spaces of the first and the second kind. Qasem and Muhib [14] obtained certain identities in a P2-like R^h-trirecurrent space. Muhib [10] established a different identity concerning P2-like R^h-generalized and P2-like R^h-special generalized trirecurrent spaces. Saleem [16] discussed P2-like C^h-generalized and P2-like C^h-special generalized birecurrent spaces and he obtained the necessary and sufficient condition of some tensors to be generalized birecurrent and special generalized birecurrent, as well as some identities in such spaces.
Izumi ([5], [6]) gave the concept of P^*-Finsler space. Matsumoto ([7],[8]) defined a C-concircularly flat spaces and tried to correlate it with a P^*-Finsler space of constant curvature. Dwivedi [4] worked out the role of P^*-reducible space in P^*-Finsler space. Verma [17] obtained some results whenR^h-recurrent and C-concircularly spaces are P^*-Finsler spaces. Mishra and Lodhi [9] discussed C^h-recurrent Finsler space of second order and obtained different theorems regarding this space when it is P^*-Finsler space. Saleem [16] obtained different theorems in C^h-generalized birecurrent and C^h-special generalized birecurrent spaces when C^h-recurrent space is P^*-Finsler space.
Let us consider an n-dimensional Finsler space F_n equipped with a metric function F(x^i, y^i)satisfying the requestic conditions of a Finslerian metric [13]. The relation between the metric function F and the set of quantities $g_{ij}(x,y)$(corresponding metric tensor g_{ij}) is defined by
(1.1)$g_{ij}(x,y) := \frac{1}{2}\dot{\partial}_i\dot{\partial}_j\ F^2\ (x,y)$.
These quantities constitute the components of a covariant tensor of second order . This tensor is called *the metric tensor* of the space F_n. From (1.1) and, due to the the homogeneity of $F(x,y)$ in y^i it is obvious that the metric tensor $g_{ij}(x,y)$ is positively homogenous of degree zero in y^i and symmetric in i and j.
Corresponding to each arbitrary contra variant vector$y^i of\ T_n(p)$, there is a covariant vector$y_i$such that
(1.2) $y_i = g_{ij}(x,y)y^j$.

Analogous to the metric tensor $g_{ij}(x,y)$, we define a tensor $g^{ij}(x^k,y_k)$ as follows:

(1.3) $g^{ij}(x^k,y_k)=\frac{1}{2}\bar{\partial}_i\bar{\partial}_j H^2(x^k,y_k)$,

where$\bar{\partial}_i$ denoted the partial differentiation with respect to the covariant vector y_i . The quantities $g^{ij}(x^k,y_k)$ constitute the components of a contravariant tensor of second order .

Transvecting (1.1) byy^j and using Euler's theorem on homogenous function, we get

(1.4) $g_{ij}\,y^j=\frac{1}{2}\dot{\partial}_i\,F^2\;=F\dot{\partial}_i\,F$.

From (1.2) and (1.4), we have

(1.5) $y_i=\;F\,\dot{\partial}_i\,F.$

The vector$y_i$also satisfies the following relation:

(1.6) $g_{ij}=\dot{\partial}_i\,y_j=\dot{\partial}_j\,y_i.$

From the metric tensor, we construct a new tensor C_{jik} by differentiating (1.1) partially with respect to y^k, we get

(1.7) $C_{ijk}\coloneqq\frac{1}{2}\dot{\partial}_i g_{jk}=\frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$,

whereC_{ijk} is known as *(h)hv-torsion tensor* [8]**,**it is positively homogenous of degree -1 in y^i and symmetric in all its indices. By Euler's theorem on homogenous functions, this tensor satisfies the following identities:

(1.8)$C_{ijk}\,y^i=C_{kij}\,y^i=C_{jki}\,y^i=0$.

The (v)hv-torsion tensor C^i_{jk}which is the associate tensor of the tensor C_{ijk} is defined by

(1.9) a) $C^h_{ik}:=\;g^{hj}C_{ijk}$ and b) $C_{ijk}:=g_{hj}\,C^h_{ik}$

which is positively homogenous of degree -1 in y^i and symmetric in its lower indices .

The expression for the variation of an arbitrary vector field X^i under the infinitesimal change of its line- element $(x,y)\;to\;(x+dx,y+dy)$ by means of covariant (absolute) differential given by Cartan in his second postulate [2]

(1.10) D$X^i=dX^i+X^j(\,C^i_{jk}dy^k+\Gamma^i_{jk}dx^k\,)$,

where

(1.11) $\Gamma^i_{jk}=\gamma^i_{jk}-C^i_{mk}\,G^m_j+g^{ih}C_{jkm}\,G^m_h$

together with

(1.12)$G^i:=\frac{1}{2}\gamma^i_{jk}\,y^j y^k$

and

(1.13) $G^i_j\coloneqq\dot{\partial}_j\,G^i$.

The function G^i is positively homogenous of degree two in y^i. Eliminating dy^k from (1.10) and the absolute differential of l^i, Cartan deduced [2]

(1.14) $X^i_{|k}\coloneqq\partial_k X^i-(\dot{\partial}_r X^i)\,G^r_k+X^r\,\Gamma^{*i}_{rk}$,

where the function Γ^{*i}_{rk} is defined by

(1.15) $\Gamma^{*i}_{rk}\coloneqq\Gamma^i_{rk}-C^i_{mr}\,\Gamma^m_{sk}y^s$.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constant with respect to the above process which is defined in (1.14), i.e.

(1.16) a)$g_{ij|k}\;=0$ and b) $g^{ij}_{|k}=0$.

The vectory^i vanish as under the h-covariant differentiation, i.e.

(1.17) $y^i_{|k}\;=0.$

The tensor$K^i_{rkh}$is defined by

(1.18)$K^i_{rkh}\coloneqq\partial_h\Gamma^{*i}_{kr}+(\dot{\partial}_l\,\Gamma^{*i}_{rh})\,G^l_k+\;\Gamma^{*i}_{mh}\,\Gamma^{*m}_{kr}-\text{h/k}$

And is called Cartan's fourth curvature tensor, which is skew-symmetric in its last two lower indices k and h and positively homogenous of degree zero in y^i. The curvature tensor K^i_{jkh}satisfies the following identities known as *Bianchi identities*:

(1.19) $K^r_{hkj} + K^r_{jhk} + K^r_{kjh} = 0$

and

(1.20)$K^r_{ihk|j} + K^r_{ijh|k} + K^r_{ikj|h} + (\dot{\partial}_s \Gamma^{*r}_{ij}) K^s_{thk} y^t + (\dot{\partial}_s \Gamma^{*r}_{ik}) K^s_{tjh} y^t$
$+(\dot{\partial}_s \Gamma^{*r}_{ih}) K^s_{tkj} y^t = 0$.

An obvious consequence of the identity (1.19) is

(1.21) $K_{jikh} + K_{hijk} + K_{kihj} = 0.$

The curvature tensor K^i_{jkh}satisfies the following relations too:

(1.22)$K^i_{jkh} y^j = H^i_{kh}.$

The curvature tensor $R^i_{jkh}$is defined as

(1.23)$R^i_{jkh} := \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_l \Gamma^{*i}_{jh}) G^l_k + G^i_{jm}(\partial_h G^m_h - G^m_{hl} G^l_k) + \Gamma^{*i}_{mh} \Gamma^{*m}_{jk} - h/k$

and is called *h- curvature tensor (Cartan's third curvature tensor),* this tensor is positively homogenous of degree -1 in y^i and skew-symmetric in its last two lower indices k and h and is satisfies the relations:

(1.24)$R^i_{jkh} = K^i_{jkh} + C^i_{jm} H^m_{kh}$,

(1.25) a)$R^i_{jkh} y^j = H^i_{kh} = K^i_{jkh} y^j$

and

b) $R_{jkhm} y^j = H_{kh.m} = K_{jkhm} y^j$,

where the associate curvature tensorR_{jkhm} of the curvature tensorR^i_{jkh} is given by

(1.26) $R_{ijkh} := g_{rj} R^r_{ikh}$

which is skew-symmetric in the first two lower indices i and j and satisfies

(1.27)$R_{ijkh} = K_{ijkh} + C_{ijm} H^m_{kh}$.

Also, the curvature tensor R^i_{jkh}satisfies the following identity known as *Bianchi identity*

(1.28) $R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + y^m(R^s_{mkh} P^r_{ijs} + R^s_{mjk} P^r_{ihs} + R^s_{mhj} P^r_{iks}) = 0$,

whereP^i_{jkh} is called*hv-curvature tensor (Cartan's second curvature tensor)* and defined by

(1.29)$P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm} P^m_{kh} - C^i_{jh|k}$.

The hv-curvature tensor P^i_{jkh} is positively homogenous of degree -1 in y^i and satisfies the relation

(1.30) $P^i_{jkh} y^j = \dot{\partial}_j \Gamma^{*r}_{jhk} y^j = P^i_{kh} = C^i_{kh|r} y^r$,

whereP^i_{jk} is called *v(hv)-torsion tensor* and its associate tensor P_{rkh} is given by:

(1.31) a)$P^i_{jk} := \dot{\partial}_j(\Gamma^{*r}_{hk}) y^h = P^i_{kh} = \Gamma^{*r}_{jhk} y^h$

and

b)$g_{ir} P^i_{kh} = P_{rkh}$.

Ricci-tensor P_{jk} is given by:

(1.32)$P^i_{jki} = P_{jk}$.

The associate curvature tensor $P_{ijkh}$of the hv-curvature tensor P^i_{jkh} is given by Qasem and Ali [15]

(1.33) $P_{ijkh} = g_{ir} P^r_{jkh}$

which is skew-symmetric in the last two lower indicesi and j .

The tensor ($P_{ij} - P_{ji}$)is given by

(1.34) $P_{ijkh} g^{kh} := P_{ij} - P_{ji}.$

The curvature vector $P_k$is given by

(1.35) $P^i_{ki} = P_k.$

Berwald curvature tensorH^i_{jkh} and the h(v)-torsion tensor H^i_{kh} are defined by

(1.36)$H^i_{jkh} := \partial_h G^i_{jk} + G^s_{jk} G^i_{sh} + G^i_{sjh} G^s_k - h/k$

and

(1.37) $H^i_{kh} := \partial_h G^i_k + G^r_k G^i_{rh} - h/k$,

Respectively and connected by

(1.38)$H^i_{rkh} = \dot{\partial}_r H^i_{kh}$.

Berwald curvature tensor H^i_{jkh} also satisfies the following :

(1.39) $H^i_{jkh} + H^i_{jkh} + H^i_{jkh} = 0$

which is one of the Bianchi identities for Berwald curvature tensor.

The tensor$H^i_{jh.k}$is defined by

(1.40) $H^i_{jh.k} = g_{ih}\, H^i_{jk}$

2.P2- LikeK^h-Birecurrent Space

Let us consider a K^h-birecurrent space which is characterized by the condition [1,(2.1)]

$$K^i_{jkh|m|\ell} = a_{\ell m} K^i_{jkh} \quad , \quad K^i_{jkh} \neq 0 \,,$$

where$a_{\ell m}$ is a non-zero covariant tensor field of second order will be called *h-birecurrence tensor.* We shall denote such space and tensor briefly by K^h-*BR*-$F_n$and *h-BR* respectively .

A P2-like space [9] is characterized by the condition

(2.1) $P^i_{jkh} = \emptyset_j\, C^i_{kh} - \emptyset^i C_{jkh}$,

where$\emptyset_j$and $\emptyset^i$ are non-zero covariant and contravariant vectors field, respectively.

Definition 2.1. The K^h-birecurrent space which is a P2-like space [satisfies the condition (2.1)]is called it a *P2-like K^h-birecurrent space* and denoted briefly by *P2-like K^h-BR-F_n*.

Let us consider a P2-like K^h- BR-F_n , then, necessarily, we have the condition (2.1).

Taking the h-covariant derivative for (2.1) twice with respect to x^mandx^l, successively, we get

$$P^i_{jkh|m|\ell} = \emptyset_{j|m|\ell}\, C^i_{kh} + \emptyset_{j|m}\, C^i_{kh|\ell} + \emptyset_{j|\ell} C^i_{kh|m} + \emptyset_j C^i_{kh|m|\ell} - \emptyset^i_{|m|\ell}\, C_{jkh} - \emptyset^i_{|m}\, C_{jkh|\ell} - \emptyset^i_{|\ell}\, C_{jkh|m} - \emptyset^i C_{jkh|m|\ell}$$

which can be written as:

(2.2) $$P^i_{jkh|m|\ell} = \emptyset_j C^i_{kh|m|\ell} - \emptyset^i\, C_{jkh|m|\ell} + (\, \emptyset_{j|m|\ell}\, C^i_{kh} + \emptyset_{j|m} C^i_{kh|\ell} + \emptyset_{j|\ell}\, C^i_{kh|m} - \emptyset^i_{|m|\ell}\, C_{jkh} - \emptyset^i_{|m} C_{jkh|\ell} - \emptyset^i_{|\ell}\, C_{jkh|m}\,) \,.$$

In view of the condition[1,(3.9)]

$C^i_{jr|m|\ell} = b_{\ell m}\, C^i_{jr}$,

Provided the h(v)-torsion tensor$H^r_{kh}$and the(v)hv- torsion tensor C^i_{jr} are h-recurrent and in view of the condition[1,(3.22)]

$C_{pjr|m|\ell} = b_{\ell m}\, C_{pjr}$,

provided the h(v)- torsion tensor$H^r_{kh}$and the (h)hv-torsion tensor$C_{pjr}$are h-recurrent the equation (2.2) can be written as

(2.3) $$P^i_{jkh|m|\ell} = a_{\ell m}\big(\emptyset_j C^i_{kh} - \emptyset^i\, C_{jkh}\big) + (\, \emptyset_{j|m|\ell}\, C^i_{kh} + \emptyset_{j|m} C^i_{kh|\ell} + \emptyset_{j|\ell}\, C^i_{kh|m} - \emptyset^i_{|m|\ell}\, C_{jkh} - \emptyset^i_{|m}\, C_{jkh|\ell} - \emptyset^i_{|\ell}\, C_{jkh|m}\,) \,.$$

Using (2.1) in (2.3), we get

(2.4) $$P^i_{jkh|m|\ell} = a_{\ell m} P^i_{jkh} + (\, \emptyset_{j|m|\ell}\, C^i_{kh} + \emptyset_{j|m}\, C^i_{kh|\ell} + \emptyset_{j|\ell} C^i_{kh|m} - \emptyset^i_{|m|\ell}\, C_{jkh} - \emptyset^i_{|m}\, C_{jkh|\ell} - \emptyset^i_{|\ell}\, C_{jkh|m}\,)$$

which can be written as:

(2.5) $P^i_{jkh|m|\ell} = a_{\ell m}\, P^i_{jkh}$

if and only if

(2.6) $$\emptyset_{j|m|\ell} C^i_{kh} + \emptyset_{j|m} C^i_{kh|\ell} + \emptyset_{j|\ell} C^i_{kh|m} - \emptyset^i_{|m|\ell} C_{jkh} - \emptyset^i_{|m} C_{jkh|\ell} - \emptyset^i_{|\ell} C_{jkh|m} = 0 \,.$$

Thus, we conclude

Theorem 2.1.*Ina P2-like K^h-BR-F_n,thehv-curvature tensorP^i_{jkh} is h-BR*[*providedtheh(v)-torsion tensorH^r_{kh}, the* (v)*hv-torsion tensorC^i_{kh}andthe (h)hv-torsion tensorC_{jkh} are h-recurrent*] *if and only if (2.6) holds good.*

Transvecting (2.4) byg_{ir},using(1.16a), (1.33) and (1.9b), we get

(2.7)$P_{rjkh|m|\ell} = a_{\ell m} P_{rjkh} + \{(\emptyset_{j|m|\ell} C_{rkh} + \emptyset_{j|m} C_{rkh|\ell} + \emptyset_{j|\ell} C_{rkh|m}) - g_{ir}(\emptyset^i_{|m|\ell} C_{jkh} + \emptyset^i_{|m} C_{jkh|\ell} + \emptyset^i_{|\ell} C_{jkh|m})\}$

which can be written as:

(2.8)$P_{rjkh|m|\ell} = a_{\ell m} P_{rjkh}$

if and only if

(2.9)$\emptyset_{j|m|\ell} C_{rkh} + \emptyset_{j|m} C_{rkh|\ell} + \emptyset_{j|\ell} C_{rkh|m} - g_{ir}(\emptyset^i_{|m|\ell} C_{jkh} + \emptyset^i_{|m} C_{jkh|\ell} + \emptyset^i_{|\ell} C_{jkh|m}) = 0$.

Thus, we conclude

Theorem 2.2.*In a P2-like K^h-BR-F_n , the associatecurvature tensor P_{rjkh} of the hv-curvature tensorP^r_{jkh} is h-BR [providedtheh(v)-torsion tensorH^r_{kh}, the* (v)*hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor $C_{jkh}$are h-recurrent] if and only if (2.9) holds good.*

Transvecting(2.4) byy^j, using (1.17), (1.8) and (1.30), we get

(2.10)$P^i_{kh|m|\ell} = a_{\ell m} P^i_{kh} + y^j(\emptyset_{j|m|\ell} C^i_{kh} + \emptyset_{j|m} C^i_{kh|\ell} + \emptyset_{j|\ell} C^i_{kh|m})$

which can be written as:

(2.11) $P^i_{kh|m|\ell} = a_{\ell m} P^i_{kh}$

if and only if

$$y^j(\emptyset_{j|m|\ell} C^i_{kh} + \emptyset_{j|m} C^i_{kh|\ell} + \emptyset_{j|\ell} C^i_{kh|m}) = 0$$

or

(2.12) $\emptyset_{|m|\ell} C^i_{kh} + \emptyset_{|m} C^i_{kh|\ell} + \emptyset_{|\ell} C^i_{kh|m} = 0$,

where $\emptyset_j\, y^j = \emptyset$ and since(1.17) holds good.

Thus, we conclude

Theorem 2.3.*In a P2-like K^h-BR-F_n , the* v(*hv*)*-torsion tensorP^i_{kh} is h-BR [provided the h(v)-torsion tensorH^r_{kh},the* (v)*hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor C_{jkh} are h-recurrent] if and only if (2.12) holds good.*

Transvecting(2.10) byg_{ir}, using(1.16a), (1.9b) and(1.31b), we get

(2.13)$P_{jkh|m|\ell} = a_{\ell m} P_{jkh} + y^j(\emptyset_{j|m|\ell} C_{jkh} + \emptyset_{j|m} C_{jkh|\ell} + \emptyset_{j|\ell} C_{jkh|m})$

which can be written as:

(2.14) $P_{jkh|m|\ell} = a_{\ell m} P_{jkh}$

if and only if

$$y^j(\emptyset_{j|m|\ell} C_{jkh} + \emptyset_{j|m} C_{jkh|\ell} + \emptyset_{j|\ell} C_{jkh|m}) = 0$$

or

(2.15) $\emptyset_{j|m|\ell} C_{jkh} + \emptyset_{j|m} C_{jkh|\ell} + \emptyset_{j|\ell} C_{jkh|m} = 0$,

where$\emptyset_j\, y^j = \emptyset$ and since(1.17) holds good.

Thus, we conclude

Theorem 2.4.*In a P2-like K^h-BR-F_n,thetensorP_{jkh} is h-BR[provided the h(v)-torsion tensorH^r_{kh},the* (v)*hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor C_{jkh}areh-recurrent] if and only if (2.15) holds good.*

Transvecting (2.8) byg^{kh}, using(1.16b)and (1.34), we get

(2.16)$(P_{rj} - P_{jr})_{|m|\ell} = a_{\ell m}(P_{rj} - P_{jr}) + g^{kh}\{(\emptyset_{j|m|\ell} C_{rkh} + \emptyset_{j|m} C_{rkh|\ell} + \emptyset_{j|\ell} C_{rkh|m}) - g_{ir}(\emptyset^i_{|m|\ell} C_{jkh} + \emptyset^i_{|m} C_{jkh|\ell} + \emptyset^i_{|\ell} C_{jkh|m})\}$

which can be written as:

(2.17)$(P_{rj} - P_{jr})_{|m|\ell} = a_{\ell m}(P_{rj} - P_{jr})$

if and only if

(2.18)$g^{kh}\{(\emptyset_{j|m|\ell} C_{rkh} + \emptyset_{j|m} C_{rkh|\ell} + \emptyset_{j|\ell} C_{rkh|m}) - g_{ir}(\emptyset^i_{|m|\ell} C_{jkh} + \emptyset^i_{|m} C_{jkh|\ell} + \emptyset^i_{|\ell} C_{jkh|m})\}$.

Thus, we conclude

Theorem 2.5. *In a P2-like K^h-BR-F_n , the tensor $P_{rj} - P_{jr}$ is h-BR [provided the h(v)-torsion tensor H^r_{kh}, the (v)hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor C_{jkh} are h-recurrent] if and only if (2.18) holds good.*

Contracting the indices i and h in (2.4) and using (1.32), we get

(2.19) $P_{jk|m|\ell} = a_{\ell m} P_{jk} + (\emptyset_{j|m|\ell}\, C_k + \emptyset_{j|m}\, C_{k|\ell} + \emptyset_{j|\ell}\, C_{k|m} - \emptyset^i_{|m|\ell}\, C_{jki} - \emptyset^i_{|m}\, C_{jki|\ell} - \emptyset^i_{|\ell}\, C_{jki|m}),$

where

(2.20) $C^j_{ji} = C_i$.

The equation (2.19) can be written as:

(2.21) $P_{jk|m|\ell} = a_{\ell m} P_{jk}$

if and only if

(2.22) $\emptyset_{j|m|\ell}\, C_k + \emptyset_{j|m}\, C_{k|\ell} + \emptyset_{j|\ell} C_{k|m} - \emptyset^i_{|m|\ell}\, C_{jki} - \emptyset^i_{|m} C_{jki|\ell} - \emptyset^i_{|\ell}\, C_{jki|m} = 0.$

Thus, we conclude

Theorem 2.6. *In a P2-like K^h-BR-F_n , Riccitensor P_{jk} is h-BR [provided the h(v)-torsion tensor H^r_{kh}, the* (v)*hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor C_{jkh} are h-recurrent] if and only if (2.22) holds good.*

Contracting the indices i and h in (2.9), using (2.20) and (1.35), we get

(2.23) $P_{k|m|\ell} = a_{\ell m} P_k + \left(\emptyset_{j|m|\ell} C_k + \emptyset_{j|m}\, C_{k|\ell} + \emptyset_{j|\ell}\, C_{k|m}\right) y^j$

which can be written as

(2.24) $P_{k|m|\ell} = a_{\ell m} P_k$

if and only if

$$\left(\emptyset_{j|m|\ell}\, C_k + \emptyset_{j|m}\, C_{k|\ell} + \emptyset_{j|\ell} C_{k|m}\right) y^j = 0$$

or

(2.25) $\emptyset_{|m|\ell} C_k + \emptyset_{|m} C_{k|\ell} + \emptyset_{|\ell} C_{k|m} = 0$,

where $\emptyset_j\, y^j = \emptyset$ and since (1.17) holds good.

Thus, we conclude

Theorem 2.7. *In a P2-like K^h-BR-F_n , the tensor P_k is h-BR [provided the h(v)-torsion tensor H^r_{kh}, the* (v)*hv-torsion tensor C^i_{kh} and the (h)hv-torsion tensor C_{jkh} are h-recurrent] if and only if (2.25) holds good.*

Let us consider a *P2* -like K^h-*BR*-F_n, then necessarily, we have the condition (2.1) and, if we take Bianchi identity for Cartan's fourth curvature tensor K^i_{jkh} which is given by (1.25).

Transvecting (1.25) by y^j, using (1.17),(1.22),(1.31a) and (1.30),we get

(2.26) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + H^s_{hm}\left(P^i_{jsk}\, y^j\right) + H^s_{kh}\left(P^i_{jsm}\, y^j\right) + H^s_{mk}\left(P^i_{jsh}\, y^j\right) = 0$.

Using the condition (2.1) in (2.26), we get

$$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + H^s_{hm}\left(\emptyset_j\, C^i_{sk} - \emptyset^i\, C_{jsk}\right) y^j + H^s_{kh}\left(\emptyset_j C^i_{sm} - \emptyset^i C_{jsm}\right) y^j + H^s_{mk}\left(\emptyset_j\, C^i_{sh} - \emptyset^i\, C_{jsh}\right) y^j = 0$$

which can be written as:

(2.27) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset_j\left(H^s_{hm} C^i_{sk} + H^s_{kh} C^i_{sm} + H^s_{mk}\, C^i_{sh}\right) y^j - \emptyset^i\left(H^s_{hm}\, C_{jsk} + H^s_{kh}\, C_{jsm} + H^s_{mk}\, C_{jsh}\right) y^j = 0$.

Using (1.24) and (1.27) in (2.27), we get

(2.28) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset_j\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) y^j - \emptyset_j\left(K^i_{khm} + K^i_{mkh} + K^i_{hmk}\right) y^j - \emptyset^i\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) y^j - \emptyset^i\left(K_{jkhm} + K_{jmkh} + K_{jhmk}\right) y^j = 0$.

Using (1.19) and (1.21) in (2.28), we get

(2.29) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset_j\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) y^j$
$- \emptyset^i\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) y^j = 0 .$

Using (1.25b) in (2.29), we get

(2.30) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) - \emptyset^i(H_{kh.m} + H_{mk.h} + H_{hm.k}) = 0 ,$

where $\emptyset_j\, y^j = \emptyset$.

Transvecting (2.30) by g_{ij}, using (1.16a), (1.26) and (1.40), we get

$$H_{kj.h|m} + H_{mj.k|h} + H_{hj.m|k} + \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) - \emptyset^i\left(H_{kh.m} + H_{mk.h} + H_{hm.k}\right) = 0$$

which can be written as

(2.31) $H_{kj.h|m} + H_{mj.k|h} + H_{hj.m|k} = \emptyset^i\left(H_{kh.m} + H_{mk.h} + H_{hm.k}\right) - \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) .$

Now, differentiating $y_i\, H^i_{kh} = 0$ [13] partially with respect to y^j , using (1.39) and (1.6), we get

(2.32) $g_{ij}\, H^i_{kh} + y_i\, H^i_{jkh} = 0$.

Taking skew-symmetric part of (2.32) with respect tothe indices j , k and h,using (1.39), we get

(2.33) $g_{ij}\, H^i_{kh} + g_{ih}\, H^i_{jk} + g_{ik}\, H^i_{hj} = 0$.

Using (1.40) in (2.33), we get

(2.34) $H_{jk.h} + H_{hj.k} + H_{kh.j} = 0$.

Putting (2.34) in (2.31), we get

(2.35) $H_{jk.h|m} + H_{jm.k|h} + H_{jh.m|k} = -\emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right)$.

Taking the h-covariant derivative for (2.35) with respect to x^l and using [1,(2.10)]

$H_{kj.h|m|\ell} = a_{\ell m} H_{kj.h}$,

we get

(2.36) $a_{\ell m}\, H_{jk.h} + a_{\ell h}\, H_{jm.k} + a_{\ell k} H_{jh.m} = -\emptyset_{|\ell}\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) - \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right)_{|\ell}$.

Thus, we conclude

Theorem 2.8. *In a P2-like K^h-BR-F_n , the identities (2.34), (2.35) and (2.36) hold good.*

Transvecting (2.35) by g^{ij}, using (1.16b), (1.40) and (1.26), we get

(2.37) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} = -\emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) .$

Taking the h-covariant derivative for (2.37) with respect to x^l and using and using [1,(2.6)]

$H^i_{kh|m|\ell} = a_{\ell m}\, H^i_{kh}$,

we get

(2.38) $a_{\ell m}\, H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k}\, H^i_{hm} = -\emptyset_{|\ell}\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) - \emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right)_{|\ell}$.

Thus, we conclude

Theorem 2.9. *In a P2-like K^h-BR-F_n , the identities (2.36), (2.37) and (2.38) hold good.*

Let usconsider*aP2-like K^h-BR-F_n* , then the necessarily we have the condition (2.1).

Using the condition (2.1) in (1.28), we get

(2.39) $R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + y^s\{R^r_{shm}\left(\emptyset_j C^i_{kr} - \emptyset^i C_{jkr}\right) + R^r_{skh}\left(\emptyset_j C^i_{mr} - \emptyset^i C_{jmr}\right) + R^r_{smk}\left(\emptyset_j C^i_{hr} - \emptyset^i C_{jhr}\right)\} = 0 .$

Using (1.25a) in (2.39), we get

$$R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + H^r_{hm}\left(\emptyset_j C^i_{kr} - \emptyset^i C_{jkr}\right) + H^r_{kh}\left(\emptyset_j\, C^i_{mr} - \emptyset^i\, C_{jmr}\right) + H^r_{mk}\left(\emptyset_j C^i_{hr} - \emptyset^i C_{jhr}\right) = 0$$

or

(2.40) $R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \emptyset_j\left(H^r_{hm}\, C^i_{kr} + H^r_{kh} C^i_{mr} + H^r_{mk}\, C^i_{hr}\right) - \emptyset^i\left(H^r_{hm} C_{jkr} + H^r_{kh} C_{jmr} + H^r_{mk} C_{jhr}\right) = 0$.

Using (1.24) and (1.21) in (2.40), we get

(2.41) $R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \emptyset_j\{\left(R^i_{khm} - K^i_{khm}\right) + \left(R^i_{mkh} - K^i_{mkh}\right) + \left(R^i_{hmk} - K^i_{hmk}\right)\} - \emptyset^i\{\left(R_{jkhm} - K_{jkhm}\right) + \left(R_{jmkh} - K_{jmkh}\right) + \left(R_{jhmk} - K_{jhmk}\right)\} = 0$.

Using (1.19) and (1.21) in (2.41), we get

(2.42) $R^i_{jkh|m} + R^i_{jmk|h} + R^i_{jhm|k} + \emptyset_j\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) - \emptyset^i\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) = 0.$

Transvecting(2.42) by g_{ip}, using(1.16a) and (1.26), we get

(2.43) $-\left(R_{pjkh|m} + R_{pjmk|h} + R_{pjhm|k}\right) = \emptyset_j\left(R_{pkhm} + R_{pmkh} + R_{phmk}\right) + \emptyset_p\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right),$

where $g_{ip}\ \emptyset^i = \emptyset_p$.

Transvecting(2.43) by y^p, using (1.17) and (1.25b), we get

(2.44) $-\left(H_{jk.h|m} + H_{jm.k|h} + H_{jh.m|k}\right) = \emptyset_j\left(H_{kh.m} + H_{mk.h} + H_{hm.k}\right) + \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right),$

where $\emptyset_p\, y^p = \emptyset$.

In view of (2.35),the equation (2.44) can be written as:

(2.45) $H_{jk.h|m} + H_{jm.k|h} + H_{jh.m|k} = -\emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right).$

Taking the h-covariant derivative for (2.45) with respect to x^l and using [1,(2.7)]

$H^i_{h|m|\ell} = a_{\ell m}\, H^i_h,$

we get

(2.46) $a_{\ell m} H_{jk.h} + a_{\ell h} H_{jm.k} + a_{\ell k} H_{jh.m} = -\emptyset_{|\ell}\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right) - \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right)_{|\ell}$.

Thus, we conclude

Theorem 2.10. *In a P2-like K^h-BR-F_n , the identities (2.44), (2.45) and (2.46) hold good.*

Transvecting(2.45) by g^{ij}, using (1.16b), (1.40) and (1.26), we get

(2.47) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} = -\emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right).$

Taking the h-covariant derivative for (2.47) with respect to x^l and using [1,(2.7)],we get

(2.48) $a_{\ell m}\, H^i_{kh} + a_{\ell h}\, H^i_{mk} + a_{\ell k} H^i_{hm} = -\emptyset_{|\ell}\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right) - \emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right)_{|\ell}$.

Thus, we conclude

Theorem 2.11. *In a P2-like K^h-BR-F_n , the identities (2.47) and (2.48) hold good .*

Remark 2.1. The different determents of Bianchi identity for Cartan's fourth curvature tensor K^i_{jkh} and Cartan's third curvature tensor R^i_{jkh} in a *P2-like K^h-BR-F_n* gives the same results.

3. P^*-K^h-Birecurrent Space

A P^*-Finsler space is characterized by the condition ([6],[7])

(3.1) $P^i_{kh} = C^i_{kh|j}\ y^j = \emptyset\, C^i_{kh},\ \emptyset \neq 0$.

H.Izumi([6],[7]) denoted $\emptyset$ by λ .

Remark 3.1. A P2-like space is necessarily a P^*-Finsler space which characterized by the condition (3.1) or in other word by

$P^i_{kh} = \emptyset\, C^i_{kh},$

where

$$P^i_{jkh}\ y^j = P^i_{kh} = C^i_{kh|s} y^s.$$

Definition 3.1. TheK^h-birecurrent space which is aP^*-Finsler space[satisfies the condition (3.1)] is called ,*a P^*-K^h-birecurrent space* and denoted briefly by P^*-K^h-*BR*- F_n.

Letus consider an P^*-K^h-*BR*-F_n, then necessarily we have the condition (3.1).

Remark 3.2. All results in *a P2- like*K^h-*BR*-$F_n$which were obtained in thepreviоussectionsatisfy in aP^*- K^h-*BR*- F_n .

Taking the h-covariant derivative for (3.1) twice with respect to x^mandx^l, successively, we get

(3.2) $$P^i_{kh|m|\ell} = \emptyset_{|m|\ell} C^i_{kh} + \emptyset_{|m} C^i_{kh|\ell} + \emptyset_{|\ell} C^i_{kh|m} + \phi\, C^i_{kh|m|\ell}.$$

In view of the condition{(3.19), [1]} the condition (3.2) can be written as

(3.3) $$P^i_{kh|m|\ell} = a_{\ell m}\, \phi\, C^i_{kh} + \left(\emptyset_{|m|\ell}\, C^i_{kh} + \emptyset_{|m}\, C^i_{kh|\ell} + \emptyset_{|\ell}\, C^i_{kh|m}\right).$$

Using the condition (3.1) in (3.3), we get

(3.4) $$P^i_{kh|m|\ell} = a_{\ell m}\, P^i_{kh} + \left(\emptyset_{|m|\ell}\, C^i_{kh} + \emptyset_{|m}\, C^i_{kh|\ell} + \emptyset_{|\ell}\, C^i_{kh|m}\right)$$

which can be written as:

(3.5) $$P^i_{kh|m|\ell} = a_{\ell m}\, P^i_{kh}$$

if and only if

(3.6) $$\emptyset_{|m|\ell}\, C^i_{kh} + \emptyset_{|m}\, C^i_{kh|\ell} + \emptyset_{|\ell}\, C^i_{kh|m} = 0.$$

Thus, we conclude

Theorem 3.1.*In P^*- K^h-BR- F_n, the $v(hv)$-torsion tensorP^i_{kh} is h-BR [providedthe h(v)-torsion tensor$H^r_{kh}$and the (v) hv-torsion tensor $C^i_{kh}$are h-recurrent] if and only if (3.6) holds good .*

Transvecting (3.4) byg_{ir},using(1.16a), (1.31b) and (1.9a), we get

(3.7) $$P_{rkh|m|\ell} = a_{\ell m}\, P_{rkh} + \left(\emptyset_{|m|\ell}\, C_{rkh} + \emptyset_{|m}\, C_{rkh|\ell} + \emptyset_{|\ell}\, C_{rkh|m}\right)$$

Which can be written as:

(3.8) $$P_{rkh|m|\ell} = a_{\ell m}\, P_{rkh}$$

if and only if

(3.9)$$\emptyset_{|m|\ell}\, C_{rkh} + \emptyset_{|m}\, C_{rkh|\ell} + \emptyset_{|\ell}\, C_{rkh|m} = 0\,.$$

Thus, we conclude

Theorem 3.2.*In P^*- K^h-BR- F_n, the associate tensor$P_{rkh}$of the$v(hv)$-torsion tensorP^i_{kh} is h-BR [provided the h(v)-torsion tensor$H^r_{kh}$and the* (v)*hv-torsion tensorC^i_{kh} areh-recurrent] if and only if (3.9)holds good.*

Contracting the indices i and$h$in (3.4), using (2.22) and (1.35), we get

(3.10)$$P_{k|m|\ell} = a_{\ell m} P_k + \left(\emptyset_{|m|\ell}\, C_k + \emptyset_{|m}\, C_{k|\ell} + \emptyset_{|\ell}\, C_{k|m}\right)$$

which can be written as:

(3.11) $$P_{k|m|\ell} = a_{\ell m}\, P_k$$

if and only if

(3.12)$$\emptyset_{|m|\ell} C_k + \emptyset_{|m}\, C_{k|\ell} + \emptyset_{|\ell}\, C_{k|m} = 0.$$

Thus, we conclude

Theorem 3.3. *In P^*-K^h-BR-F_n,the curvature vector$P_k$is h-BR [provided the h(v)-torsion tensor$H^r_{kh}$and the$(v)hv$-torsion tensorC^i_{kh} are h-recurrent] if and only if (3.12) holds good.*

Transvecting(1.20) byy^j, using (1.17), (1.22) and (1.31a), we get

(3.13)$$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + H^s_{hm}\, P^i_{sk} + H^s_{kh} P^i_{sm} + H^s_{mk}\, P^i_{sh} = 0\quad .$$

In view of (3.1),(3.13) can be written as:

(3.14)$$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset\left(H^s_{hm} C^i_{sk} + H^s_{kh} C^i_{sm} + H^s_{mk}\, C^i_{sh}\right) = 0\,.$$

Using (1.24) in (3.14), we get

(3.15) $$H^i_{kh|m}+ H^i_{mk|h}+ H^i_{hm|k}+ \emptyset\left\{\left(R^i_{hmk} + R^i_{khm} + R^i_{mkh}\right) - \left(K^i_{hmk} + K^i_{khm} + K^i_{mkh}\right)\right\} = 0.$$

Using (1.19) in (3.15), we get

(3.16) $$H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} + \emptyset\left(R^i_{hmk} + R^i_{khm} + R^i_{mkh}\right) = 0$$

which can be written as

(3.17) $H^i_{kh|m} + H^i_{mk|h} + H^i_{hm|k} = -\emptyset\left(R^i_{hmk} + R^i_{khm} + R^i_{mkh}\right)$.

Taking the h-covariant derivative for (3.17) with respect to x^l and using [1,(2.6)], we get

(3.18) $a_{\ell m} H^i_{kh} + a_{\ell h} H^i_{mk} + a_{\ell k} H^i_{hm} = -\emptyset_{|\ell}\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right)$
$- \emptyset\left(R^i_{khm} + R^i_{mkh} + R^i_{hmk}\right)_{|\ell}$.

Transvecting(3.18) by g_{ij}, using(1.16a), (1.26) and (1.40), we get

(3.19) $a_{\ell m} H_{jk.h} + a_{\ell h} H_{jm.k} + a_{\ell k} H_{jh.m} = -\emptyset_{|\ell}\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right)$
$- \emptyset\left(R_{jkhm} + R_{jmkh} + R_{jhmk}\right)_{|\ell}$.

Thus, we conclude

Theorem 3.4. *In P^*-K^h-BR- F_n, the identities (3.17), (3.18) and (3.19) hold good.*

Remark 4.3. If we take Bianchi identity for Cartan's third curvature tensor R^i_{jkh}, we get the same identities, i.e. the identities (4.17), (4.18) and (4.19).

References

1. Alqufail, M.A.H., Qasem, F.Y.A. and Ali,M.A.A. (2014):On study K^h-birecurrent Finsler space,the Scientific Journal of the Faculty of Education, Thamar University,1-18, (to be Published).
2. **Cartan, É.** (1933):*Sur les espaces de Finsler*, C.R. Acad. Sci. (Paris),196, 582-586.
3. **Dikshit, S.**(1992):*Certain types of recurrences in Finsler spaces*, D. Phil. Thesis, University of Allahabad, (Allahabad), (India), 45-59 and 84-99.
4. **Dwivedi, P. K.**(2011):P^*-reducible Finsler spaces and applications, Int. Journal of Math.Analysis, Vol. 5, No. 5, 223-229.
5. **Izumi, H.**(1976):On *P- Finsler space I, Memo. Defence Acad. Japan, 16, 133-138.
6. **Izumi, H.**(1977): On *P- Finsler space II, Memo. Defence Acad. Japan, 17,1 - 9.
7. **Matsumoto, M.**(1971):On h-isotropic and C^h-recurrent Finsler spaces, J. Math. Kyoto Univ., 11, 1- 9.
8. **Matsumoto, M.**(1971): On Finsler spaces with curvature tensors of some special forms, Tensor N.S., 22, 201-204.
9. **Mishra, C. K.** and **Lodhi, G.**(2008):On C^h-recurrent and C^v-recurrent Finslerspaces of second order, Int. J. Contemp. Math. Sciences, Vol. 3, No. 17, 801-810 .
10. **Muhib, A.A.A.**(2009):*On independent components of tensor, I-relative tensor and R^h-generalized trirecurrent Finsler spaces*, M.Sc. dissertation, University of Aden, (Aden), (Yemen),56 – 77.
11. **Pandey, P.N.**(1981): A note on recurrence vector, proc. Nat.Acad.Sci.(India), 51A, l, 6-8.
12. **Pandey, P.N.**(1993): *Some problems in Finsler space*, D.Sc., Thesis University of Allahabad, (Allahabad), (India), 115-142.
13. **Qasem, F.Y.A.**(2000): *On transformations in Finsler spaces*,D. Phil Thesis, university of Allahabad, (Allahabad), (India), 1 – 2 and 26 – 101.
14. **Qasem, F.Y.A.** and **Muhib, A.A.A.**(2008): On R^h-trirecurrent Finsler spaces, the Scientific Journal of the Faculty of Education, Thamar University, Vol.(L), No(5),59-75.
15. **Qasem, F.Y.A.** and **Ali, M.A.A.**(2015): Certain types of K^h- birecurrent space (I), Abhath Journal, the Faculty of Education, University of Alhodidah, (Hodidah), (Yemen), Volume (2), Issue (3), 21-37.
16. **Saleem, A.A.M.**(2011):*On certain generalized birecurrent and trirecurrent Finsler spaces,* M.Sc. Thesis, University of Aden, (Aden), (Yemen), 20 – 78.
17. **Verma, R.**(1991):*Some transformation in Finsler spaces,* D. Phil. Thesis University of Allahabad, (Allahabad), (India), 45 - 59.

دراسة لبعض أنواع فضاءات فنسلر K^h – ثُنائية المُعاودة (II)

فهمي ياسين عبده قاسم، جمال أبو بكر عبدالله باوزير و مقداد أحمد عبدالله علي
Fahmi.yaseen@yahoo.com
قسم الرياضيات، كلية التربية، عدن، جامعة عدن، خور مكسر، عدن، اليمن
DOI: https://doi.org/10.47372/uajnas.2017.n2.a13

المُلخص

M.A.A.Ali, F.Y.A. Qasem[16] عرفا فضاء K^h-BR- affinely connected. في هذا الفضاء تحصلا على الحالة لبعض الموترات لتكون ثُنائية المُعاودة وأثبتا بعض الموترات تكون ثُنائية المُعاودة، أعطا تعريف جديد لبعض الموترات وتحصلا بعض المُتطابقات أيضاً عرفا فضاء K^h-BR- affinely connected وتحصلا على مُتطابقات مُختلفة في هذا الفضاء.في هذه الورقة نستخدم الخاصية لفضاء K^h- ثُنائي المُعاودة، فضاء P2-like وفضاء فنسلر $-P^*$. تحصلنا على مُبرهنات مُختلفة لبعض الموترات لتحقق حالات الفضاء أعلاه وتحصلنا على مُتطابقات في هذه الفضاءات.

الكلمات المفتاحية: فضاء K^h – ثُنائي المُعاودة، فضاء K^hP2-like، ثُنائي المُعاودة وفضاء K^h،P^*- ثُنائي المُعاودة.