Integral Inequalities for the Polar derivative and the generalized Polar derivative of complex Polynomials

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Abstract

For a polynomial $P(z)$ of degree n, having all zeros in $|z| \leq 1$, Malik [11] proved that for each $q > 0$,

$$n\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\, d\theta\right]^{1/q} \leq \left[\int_0^{2\pi}\left|1+e^{i\theta}\right|^q\, d\theta\right]^{1/q} \max_{|z|=1}\left|P'(z)\right|.$$

In this paper we generalize the above inequality to polar derivative and generalized polar derivative, which as special cases include several known results in this area.

Keywords: Polynomials, Restricted Zeros, Inequalities in the Complex Domain.

1. Introduction

The main of this paper improves and refineds some well known results concerning the polynomials due to Turàn [19], Al-Saeedi [1] , Rather, Ali ,Shafi and Dar [15] and others.

If $P(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n , then it is well known Bernstein´s inequality [2], on the derivative of a polynomial, we have

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{|z|=1}|P(z)| \quad . \tag{1.1}$$

This result is the best possible and equitable holding for a polynomial that has all zeros at the origin.

If the polynomial $P(z)$ of degree n not vanishing in $|z| < 1$, then Erdös [5] conjectured and Lax [10] proved that

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1}|P(z)| \,. \tag{1.2}$$

If we restrict ourselves to the class of polynomials which have all its zeros in $|z| \leq 1$, then it was proved
by Turán [19], that

$$\max_{|z|=1}\left|P'(z)\right| \geq \frac{n}{2} \max_{|z|=1}|P(z)| \quad . \tag{1.3}$$

The inequalities (1.2) and (1.3) are also the best possible, and become equality for polynomials which have all their zeros on $|z| = 1$.

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial of degree n with respect to a complex number α, then

$$D_\alpha P(z) = n\,P(z) + (\alpha - z)\,P'(z)\,, \quad \text{(see [12])}\,.$$

The polynomial $D_\alpha P(z)$ is of degree at most $(n-1)$, and it generalizes the ordinary $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha\to\infty}\left(\frac{D_\alpha P(z)}{\alpha}\right) = P'(z) \quad \text{uniformly with respect } z \text{ for } |z| \leq R\,,\ \ R > 0\,.$$

For each positive integer n , let $\mathcal{P}_n$ denote the set of all polynomials of degree n over the field $\mathbb{C}$ of complex number , $\partial\mathcal{P}_n$ denote the collection of all monic polynomials in $\mathcal{P}_n$ and $\mathbb{R}_+^n$ be the set of all $n-$tuples
$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ of non –negative real numbers (not all zeros) with $\gamma_1 + \gamma_2 + \cdots + \gamma_n = \Lambda$.

Let $D_{\alpha}^{\gamma}\,[P](z)$ denote the generalized polar derivative of the polynomial $P(z)$ as

$$D_{\alpha}^{\gamma}[P](z) = \Lambda\, P(z) + (\alpha - z)\, P^{\gamma}(z)$$

where $\Lambda = \sum_{j=1}^{n} \gamma_j$, for all $\gamma \in \mathbb{R}_{+}^{n}$, (see [15]) .

Note that for $\gamma = (1,1,1,\dots,1)$, $D_{\alpha}^{\gamma}[P](z) = D_{\alpha}P(z)$.

Zygmund [20] extended Bernstein´s inequality (1.1) to L^P norm as

$$\left[\int_0^{2\pi} |P'(e^{i\theta})|^q \, d\theta\right]^{1/q} \le n \left[\int_0^{2\pi} \left|P(e^{i\theta})\right|^q \, d\theta\right]^{1/q} , \qquad (1.4)$$

for any polynomial $P(z)$ of degree n and for any $q \ge 1$.

Malik [11] obtained the L^P extension of (1.3) due to Turán [19] by proving that , if $P(z)$ has all its zeros in $|z| \le 1$,then for any $q > 0$,

$$n \left[\int_0^{2\pi} \left|P(e^{i\theta})\right|^q \, d\theta\right]^{1/q} \le \left[\int_0^{2\pi} |\, 1 + e^{i\theta}\,|^q \, d\theta\right]^{1/q} \max_{|z|=1} \left|P'(z)\right| . \qquad (1.5)$$

In this paper we will extend and generalize the above inequality(1.5) to the class of polar derivative and generalized polar derivative of polynomials.

2.Lemmas

We need the following Lemmas for the proof of our theorems .The first Lemma is due to Govil [6].

Lemma 2.1. If $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, then on $|z| = 1$

$$\left|q'(z)\right| \le k^n \left|P'(z)\right| \qquad (2.1)$$

Lemma 2.2. If $P(z)$ is a polynomial of degree at most n having all its zeros in $|z| \ge 1$, then on $|z| = 1$,

$$\left|Q'(z)\right| \ge \left|P'(z)\right| + n \min_{|z|=1} |P(z)| , \quad \text{where} \quad Q(z) = z^n\, \overline{P(1/\bar{z})}.$$

A proof of this Lemma is contained in the proof of Theorem 1 in [7].

Lemma 2.3. If $P(z)$ is a polynomial of degree n , then for $R \ge 1$

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)| . \qquad (2.2)$$

This lemma is due to [13]. The following lemma is due to Rahman and Schmeisser [14] (see also [3]).

Lemma 2.4. If $P(z)$ is a polynomial of degree n which dose not vanish in $|z| < 1$, then for every $R \ge 1$ and $q > 0$, we have

$$\int_0^{2\pi} \left|P(R\, e^{i\theta})\right|^q d\theta \le \frac{\left\{\int_0^{2\pi} \left|1 + R^n e^{i\theta}\right|^q \, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta\right\}} \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \qquad (2.3)$$

Lemma 2.5. If $P(z)$ is a polynomial of degree n , then for $R \ge 1$

$$\max_{|z|=R} |P(z)| \le R^n \max_{|z|=1} |P(z)| - \frac{2(R^n - 1)}{(n+2)} |P(0)| - \left[\frac{R^n - 1}{n} - \frac{R^{n-2} - 1}{(n-2)}\right] \left|P'(0)\right| ; \text{ provided } n > 2 . \qquad (2.4)$$

And

$$\max_{|z|=R} |P(z)| \le R^2 \max_{|z|=1} |P(z)| - \frac{(R-1)}{2} \left[(R+1)|P(0)| + (R-1)\left|P'(0)\right|\right] ; \text{ provided } n = 2 . \qquad (2.5)$$

This lemma is due to Dewan el at. [4].

Lemma 2.6. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k > 0$, then

$$\max_{|z|=R}|P(z)| \geq R^s \left(\frac{R+k}{1+k}\right) \max_{|z|=1}|P(z)| \quad , \quad \text{for} \quad k>1 \quad \text{and} \quad k<R<k^2 \qquad (2.6)$$

where s is the order of possible zeros of $P(z)$ at $z=0$.
This lemma is due to Jain [9]. We also need the following Lemma.

Lemma 2.7. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every
$\alpha \in \mathbb{C}$ with $|\alpha| \geq k^n$ and on $|z|=1$

$$|D_\alpha P(z)| \geq (|\alpha| - k^n)\, |P'(z)| \qquad (2.7)$$

Proof. Let $Q(z) = z^n \overline{P(1/\bar{z})}$, then $|Q'(z)| = |n\,P(z) - z\,P'(z)|$ on $|z|=1$, we have for $|z| = 1$,

$$|D_\alpha P(z)| = |n\,P(z) + (\alpha - z)P'(z)| = |\alpha\,P'(z) + n\,P(z) - zP'(z)|$$
$$\geq |\alpha\,P'(z)| - |Q'(z)| \qquad (2.8)$$

Using inequality (2.1) of Lemma 2.1 in (2.8), we get on $|z|=1$

$$|D_\alpha P(z)| \geq (|\alpha| - k^n)\,|P'(z)|$$

The following Lemmas is due to Rather el at. [15].

Lemma 2.8. If $P(z)$ is a polynomial of degree n, then for $|z|=1$
$|Q^\gamma(z)| = |\Lambda\,P(z) - z\,P^\gamma(z)|$ and $|P^\gamma(z)| = |\Lambda\,Q(z) - z\,Q^\gamma(z)|$, where $Q(z) = z^n \overline{P(1/\bar{z})}$.

Lemma 2.9. If $P(z)$ is a polynomial of degree n which dose not vanish in $|z| < k$, $k \geq 1$, then

$$|Q^\gamma(z)| \leq k\, |P^\gamma(z)| \quad , \text{ for } \quad |z|=1 \quad , \quad \text{where} \quad Q(z) = z^n \overline{P(1/\bar{z})} \,. \qquad (2.9)$$

3. Main Results and Proofs

Theorem 3.1. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then for every
$\alpha \in \mathbb{C}$ with $|\alpha| \geq k^n$, $q > 1$ and $k < R < k^2$

$$n\,(|\alpha| - k^n)\, R^s \left(\frac{R+k}{1+k}\right) \left[\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right]^{1/q}$$
$$\leq \left[\int_0^{2\pi} |1 + k^n e^{i\theta}|^q \, d\theta\right]^{1/q} \max_{|z|=R} |D_\alpha\,P(z)| \qquad (3.1)$$

where s is the order of possible zeros of $P(z)$ at $z=0$.

Proof. The polynomial $G(z) = P(kz)$ has all its zeros in $|z| \leq 1$ and $H(z)$ has all its zeros in $|z| \geq 1$,
$H(z) = z^n \overline{G(1/\bar{z})}$ will has all its zeros in $|z| \geq 1$.

$$|H'(z)| \leq |n\,H(z) - z\,H'(z)|, \quad \text{for} \quad |z| = 1\,. \qquad (3.2)$$

Also since $G(z)$ has all its zeros in $|z| \leq 1$, by Gauss-Lucas theorem all the zeros of $G'(z)$ also lie in $|z| \leq 1$. This implies that the polynomial $z^{n-1}\,\overline{G'(1/\bar{z})} \equiv n\,H(z) - z\,H'(z)$ dose not vanish in $|z| < 1$. Therefore, it follows from (3.2) that the function

Let $w(z) = \dfrac{z\,H'(z)}{n\,H(z) - z\,H'(z)}$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + w(z)$ is subordination to the function $1 + z$. Hence by a well-known property of subordination [8], we have for each $q > 0$,

$$\int_0^{2\pi} |1 + w(e^{i\theta})|^q \, d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta \qquad (3.3)$$

Now,

$$1 + w(z) = \frac{n\,H(z)}{n\,H(z) - z\,H'(z)} \tag{3.4}$$

And

$$|G'(z)| = |\,z^{n-1}\,\overline{G'(1/\bar{z})}\,| = |n\,H(z) - z\,H'(z)|\,, \qquad \text{for} \quad |z| = 1\,. \tag{3.5}$$

Therefore, by (3.4), for $|z| = 1$, we have

$$n\,|H(z)| = |1 + w(z)|\,|n\,H(z) - z\,H'(z)|$$

Which, by employing (3.5), yields the following

$$n\,|H(z)| = |1 + w(z)|\,|G'(z)| \qquad \text{for} \quad |z| = 1\,. \tag{3.6}$$

From (3.3) and (3.6), we deduce, for $q > 0$, that

$$n^q \int_0^{2\pi} |H(e^{i\theta})|^q d\theta \le \int_0^{2\pi} |1 + e^{i\theta}|^q\, d\theta \left\{\max_{|z|=1} |G'(z)|\right\}^q. \tag{3.7}$$

Since $H(z)$ is a polynomial of degree n which dose not vanish in $|z| < 1$, using Lemma 2.4 with $R = k \ge 1$ to $H(z)$, we obtain

$$\int_0^{2\pi} |H(k\,e^{i\theta})|^q\, d\theta \le \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q\, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q\, d\theta\right\}} \int_0^{2\pi} |H(e^{i\theta})|^q\, d\theta$$

From the fact that $|H(k\,e^{i\theta})| = k^n\,|P(e^{i\theta})|$ it follows that

$$n^q\,k^{nq} \int_0^{2\pi} |P(e^{i\theta})|^q\, d\theta$$

$$\le \frac{\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q\, d\theta\right\}}{\left\{\int_0^{2\pi} |1 + e^{i\theta}|^q\, d\theta\right\}}\; n^q \int_0^{2\pi} |H(e^{i\theta})|^q\, d\theta\,, \quad k \ge 1\,. \tag{3.8}$$

Now, employing (3.7), this implies

$$n^q\; k^{nq} \int_0^{2\pi} |P(e^{i\theta})|^q\, d\theta \le \left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^q\, d\theta\right\} \left\{\max_{|z|=1} |G'(z)|\right\}^q\,. \tag{3.9}$$

From Lemma 2.3, we have

$$\max_{|z|=R\ge1} |P(z)| \le R^n \max_{|z|=1} |P(z)|$$

Since $G(z) = P(kz)$, then $G'(z) = k\,P'(kz)$ which is of degree $(n-1)$, we get

$$\max_{|z|=1} |G'(z)| = k\, \max_{|z|=k} |P'(z)|$$

Which using inequality (2.2) implies

$$\max_{|z|=1} |G'(z)| \le k^n \max_{|z|=1} |P'(z)| \tag{3.10}$$

Now of Lemma 2.6, we have

$$\max_{|z|=1} |P(z)| \le \frac{1}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P(z)| \tag{3.11}$$

where s is the order of possible zeros of $P(z)$ at $z = 0$.

On applying (3.11) to the polynomial $P'(z)$, we obtain

$$\max_{|z|=1} |P'(z)| \le \frac{1}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P'(z)| \tag{3.12}$$

Therefore, now using (3.12) in (3.10), we get

$$\max_{|z|=1} |G'(z)| \le \frac{k^n}{R^s} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |P'(z)| \tag{3.13}$$

Appling Lemma 2.7 in (3.13), we get

$$\max_{|z|=1} |G'(z)| \le \frac{k^n}{R^s\,(|\alpha| - k^n)} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |D_\alpha\, P(z)| \tag{3.14}$$

Finally, on using (3.14) in (3.9), we obtain for each $q > 1$

$$n^q\, k^{nq} \int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^q d\theta \;\le\; \left\{\int_0^{2\pi} \left|1+k^n e^{i\theta}\right|^q \,d\theta\right\} \left\{ \frac{k^n}{R^s\,(|\alpha|-k^n)} \left(\frac{1+k}{R+k}\right) \max_{|z|=R} |D_\alpha\, P(z)|\right\}^q$$

This implies to

$$n\,(|\alpha|-k^n)\, R^s \left(\frac{R+k}{1+k}\right) \left[\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^q \,d\theta\right]^{1/q}$$

$$\le \left[\int_0^{2\pi} \left|\,1+k^n e^{i\theta}\,\right|^q \,d\theta\right]^{1/q} \max_{|z|=R} |D_\alpha\, P(z)|\,.$$

Hence, the theorem is completely proved.

Remark 3.1. Dividing both sides of inequality (3.1) by $|\alpha|$ and letting $|\alpha| \to \infty$, we obtain

$$n\; R^s \left(\frac{R+k}{1+k}\right) \left[\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^q \,d\theta\right]^{1/q}$$

$$\le \left[\int_0^{2\pi} \left|\,1+k^n e^{i\theta}\,\right|^q \,d\theta\right]^{1/q} \max_{|z|=R} \left|\,P'(z)\right|. \qquad (3.15)$$

This inequality is due to Al-Saeedi [1].

Theorem 3.2. If $P(z)$ is a polynomial of degree $n \ge 3$ having all its zeros in $|z| \le k$, $k \ge 1$, then for every
$\alpha \in \mathbb{C}$ with $|\alpha| \ge k$ and $q > 0$

$$n\,(|\alpha|-k) \left[\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^q \,d\theta\right]^{1/q} \le \left[\int_0^{2\pi} \left|\,1+k^n e^{i\theta}\,\right|^q \,d\theta\right]^{1/q} \Big[\, \max_{|z|=1} |D_\alpha P(z)| - \frac{2\,(k^{n-1}-1)}{k^{n-1}(n+1)}\, |n\, a_0 + \alpha\, a_1\,| - \frac{1}{k^{n-1}} \left\{\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right\} |(n-1)a_1 + 2\,\alpha\, a_2\,| - \frac{n\,m}{k^{n-1}}\,\Big], \qquad \text{provided} \quad n > 3 \qquad (3.16)$$

And

$$n\,(|\alpha|-k) \left[\int_0^{2\pi} \left|P\left(e^{i\theta}\right)\right|^q \,d\theta\right]^{1/q} \le \left[\int_0^{2\pi} \left|\,1+k^n e^{i\theta}\,\right|^q \,d\theta\right]^{1/q} \left[\max_{|z|=1} |D_\alpha P(z)| - \frac{(k-1)}{2\,k^{n-1}} \{\,(k+1)|n\,a_0+\alpha\,a_1\,| + (k-1)|(n-1)a_1 + 2\,\alpha\,a_2\,|\} - \frac{n\,m}{k^{n-1}}\right], \qquad \text{provided} \quad n = 3 \qquad (3.17)$$

where $\quad m = \min_{|z|=k} |P(z)|$.

Proof. The polynomial $G(z) = P(kz)$ has all its zeros in $|z| \le 1$ and hence its conjugate reciprocal polynomial $H(z)$ has all its zeros in $|z| \ge 1$. But then from Lemma 2.2, we have

$$\left|H'(z)\right| \le \left|\,G'(z)\right| - n\, \min_{|z|=1} |P(z)| \qquad \text{on} \quad |z| = 1\,,$$

which is on $|z| = 1$ equivalent to

$$\left|H'(z)\right| \le \left|\,G'(z)\right| - n\;m \qquad (3.18)$$

Now on $|z| = 1$

$$\left|D_{\alpha/k} G(z)\right| = \left|n\,G(z) - \left(\frac{\alpha}{k} - z\right) G'(z)\right| \ge \left|\frac{\alpha}{k}\right| \left|G'(z)\right| - \left|\,H'(z)\right|$$

Using (3.18) we get $\left|D_{\alpha/k} G(z)\right| \ge \left(\frac{|\alpha|}{k} - 1\right) \left|G'(z)\right| + n\,m \quad$ on $\quad |z| = 1$

$$\max_{|z|=1} |D_{\alpha/k} G(z)| \geq \left(\frac{|\alpha|}{k} - 1\right) \max_{|z|=1} |G'(z)| + n\,m$$

This implies to

$$k \max_{|z|=k} |D_\alpha P(z)| \geq (|\alpha| - k) \max_{|z|=1} |G'(z)| + k\,n\,m \qquad (3.19)$$

which can be expressed after applying (2.4) of Lemma 2.5 on $|D_\alpha P(z)|$ for any polynomial $P(z)$ of degree
$n > 3$,

$$k\left\{ k^{n-1}\,|D_\alpha P(z)| - \frac{2(k^{n-1}-1)}{(n+1)}\,|D_\alpha P(0)| - \left[\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right]\left|D_\alpha P'(0)\right|\right\}$$
$$\geq (|\alpha| - k) \max_{|z|=1} |G'(z)| + k\,n\,m,$$

for each $q > 0$

$$\left\{k^n \max_{|z|=1} |D_\alpha P(z)| - \frac{2k\,(k^{n-1}-1)}{(n+1)}\,|D_\alpha P(0)| - k\left[\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right]\left|D_\alpha P'(0)\right| - k\,n\,m\right\}^q$$
$$\geq (|\alpha| - k)^q \left\{\max_{|z|=1} |G'(z)|\right\}^q \qquad (3.20)$$

Using (3.7) in (3.20), we get

$$\left\{k^n \max_{|z|=1} |D_\alpha P(z)| - \frac{2k\,(k^{n-1}-1)}{(n+1)}\,|D_\alpha P(0)| - k\left[\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right]\left|D_\alpha P'(0)\right| - k\,n\,m\right\}^q$$
$$\geq n^q\,(|\alpha| - k)^q\,\frac{\int_0^{2\pi}\left|H\left(e^{i\theta}\right)\right|^q d\theta}{\int_0^{2\pi}|1+e^{i\theta}|^q\,d\theta} \qquad (3.21)$$

Now, from inequalities (3.8) and (3.21), we get for each $q > 0$

$$n\,(|\alpha| - k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q d\theta\right]^{1/q} \leq \left[\int_0^{2\pi}\left|1 + k^n e^{i\theta}\right|^q d\theta\right]^{1/q}$$
$$\left[\max_{|z|=1} |D_\alpha P(z)| - \frac{2\,(k^{n-1}-1)}{k^{n-1}(n+1)}\,|n\,a_0 + \alpha\,a_1|\right.$$
$$\left. - \frac{1}{k^{n-1}}\left\{\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right\}|(n-1)a_1 + 2\,\alpha\,a_2| - \frac{n\,m}{k^{n-1}}\right].$$

That proves the inequality (3.16) for $n > 3$. For the case $n = 3$, the result follows on similar lines by applying inequality (2.5) of Lemma 2.5 on $|D_\alpha P(z)|$ in the inequality (3.19). This completes the proof of Theorem 3.2.

Taking $q \to \infty$ in Theorem 3.2, we get

Corollary 3.1. If $P(z)$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$

$$n\,\frac{(|\alpha| - k)}{1 + k^n} \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |D_\alpha P(z)| - \frac{2\,(k^{n-1}-1)}{k^{n-1}(n+1)}\,|n\,a_0 + \alpha\,a_1|$$
$$- \frac{1}{k^{n-1}}\left\{\frac{k^{n-1}-1}{(n-1)} - \frac{k^{n-3}-1}{(n-3)}\right\}|(n-1)a_1 + 2\,\alpha\,a_2| - \frac{n\,m}{k^{n-1}}, \text{ provided } n > 3. \qquad (3.22)$$

And

$$n\,\frac{(|\alpha| - k)}{1 + k^n} \max_{|z|=1} |P(z)| \leq \max_{|z|=1} |D_\alpha P(z)| - \frac{(k-1)}{2\,k^{n-1}}\{(k+1)\,|n\,a_0 + \alpha\,a_1|$$
$$+ (k-1)|(n-1)a_1 + 2\,\alpha\,a_2|\} - \frac{n\,m}{k^{n-1}}, \quad \text{provided } n = 3. \qquad (3.23)$$

where $m = \min_{|z|=k} |P(z)|$.

Theorem 3.3. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \leq 1$, then for every
$\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and $q > 0$

$$\Lambda\,(|\alpha| - k)\left[\int_0^{2\pi} \left|P(e^{i\theta})\right|^q \, d\theta\right]^{1/q} \leq \left[\int_0^{2\pi} \left| 1 + k\,e^{i\theta} \right|^q \, d\theta\right]^{1/q} \max_{|z|=1}\left|D_\alpha^\gamma[\,P](z)\right| . \qquad (3.24)$$

Proof. Since $P(z)$ is a polynomial of degree n , from Lemma 2.8 , we have

$$|Q^\gamma(z)| = |\Lambda\, P(z) - z\, P^\gamma(z)| \text{ and } |P^\gamma(z)| = |\Lambda\, Q(z) - z\, Q^\gamma(z)| , \text{ for } |z| = 1 \qquad (3.25)$$

From Lemma 2.9, we have

$$|Q^\gamma(z)| \leq k\,|P^\gamma(z)| , \qquad \text{for } |z| = 1 \qquad (3.26)$$

Using (3.25) in (3.26), we get for $|z| = 1$,

$$|Q^\gamma(z)| \leq k\,|\Lambda\, Q(z) - z\, Q^\gamma(z)| \qquad (3.27)$$

From (3.26) for every real or complex number α with $|\alpha| \geq k$ and $|z| = 1$, we have

$$\left|D_\alpha^\gamma\,[P](z)\right| \geq |\alpha|\,|P^\gamma(z)| - |\Lambda\, P(z) - z\, P^\gamma(z)| = |\alpha|\,|P^\gamma(z)| - |Q^\gamma(z)| \geq (|\alpha| - k)\,|P^\gamma(z)| \qquad (3.28)$$

Since $P(z)$ has all its zeros in $|z| \leq k$, it follows from Rather el at. [18], we have
Every convex set containing all the zeros of $P(z)$ also contains the zeros of $P^\gamma(z)$ for all $\gamma \in \mathbb{R}_+^n$. Therefor,
from (3.26) that the function

$$w(z) = \frac{z\, Q^\gamma(z)}{k(\,\Lambda\, Q(z) - z\, Q^\gamma(z)\,)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| = 1$. Furthermore , $w(0) = 0$. Thus the function $1 + k\,w(z)$ is subordination to the function $1 + k\,z$. Hence by a well-known property of subordination [8] ,we have for each $q > 0$,

$$\int_0^{2\pi} \left| 1 + k\,w(e^{i\theta}) \right|^q \, d\theta \leq \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^q \, d\theta \qquad (3.29)$$

Now

$$1 + k\,w(z) = \frac{\Lambda\, Q(z)}{\Lambda\, Q(z) - z\, Q^\gamma(z)}$$

and

$$|P^\gamma(z)| = |\Lambda\, Q(z) - z\, Q^\gamma(z)| \qquad \text{for} \quad |z| = 1 ,$$

therefore for $|z| = 1$,

$$\Lambda\,|Q(z)| = |1 + k\,w(z)|\,|\,\Lambda\, Q(z) - z\, Q^\gamma(z)\,| = |1 + k\,w(z)|\,|P^\gamma(z)|$$

Equivalent,

$$\Lambda\,\left|z^n\,\overline{P(1/\bar{z})}\right| = |1 + k\,w(z)|\,|P^\gamma(z)|.$$ This implies

$$\Lambda\,|P(z)| = |1 + k\,w(z)|\,|P^\gamma(z)| , \qquad \text{for } |z| = 1 . \qquad (3.30)$$

From (3.28) and (3.30), we deduce that for $q > 0$,

$$\Lambda^q\,(|\alpha| - k)^q \int_0^{2\pi} \left|P(e^{i\theta})\right|^q d\theta \leq \int_0^{2\pi} \left|1 + k\,w(e^{i\theta})\right|^q \left|D_\alpha^\gamma\,[P](e^{i\theta})\right|^q \, d\theta \leq \int_0^{2\pi} \left|1 + k\,w(e^{i\theta})\right|^q \, d\theta \left(\max_{|z|=1}\left| D_\alpha^\gamma\,[P](z)\right|\right)^q .$$

Which is equivalent to for each $q > 0$,

$$\Lambda\,(|\alpha| - k)\left[\int_0^{2\pi} \left|P(e^{i\theta})\right|^q \, d\theta\right]^{1/q} \leq \left[\int_0^{2\pi} \left| 1 + k\,e^{i\theta} \right|^q \, d\theta\right]^{1/q} \max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right| .$$

This proves Theorem 3.3.

Remark 3.2. Taking $q \to \infty$ in inequality (3.24) reduces to

$$\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right| \geq \frac{\Lambda\,(|\alpha|-k)}{1+k}\max_{|z|=1}|P(z)| \quad . \tag{3.31}$$

The above inequality is due to Rather el at. [15].

Remark 3.3. For the $n-$ tuple $\gamma=(1,1,1,\dots,1)$, the inequality (3.24) reduces to

$$n\,(|\alpha|-k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q}$$
$$\leq\left[\int_0^{2\pi}\left|\,1+k\,e^{i\theta}\,\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}|D_\alpha\,P(z)|\ . \tag{3.32}$$

This result is due to Rather el at. [17].

Theorem 3.4. If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then for every
$\alpha\in\mathbb{C}$ with $|\alpha|\geq k$, $q>0$

$$\Lambda\,(|\alpha|-k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q}$$
$$\leq\left[\int_0^{2\pi}\left|\,1+k^n\,e^{i\theta}\,\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right|. \tag{3.33}$$

Proof. Since all the zeros of $P(z)$ lie in $|z|\leq k$,therefore, all the zeros of $F(z)=P(kz)$ lie in $|z|\leq 1$.
Applying inequality (3.24) with $k=1$ to the polynomial $F(z)$, it follows for each $q>0$ and $|\beta|\geq 1$,

$$\Lambda\,(|\beta|-1)\left[\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q}\leq\left[\int_0^{2\pi}\left|\,1+\,e^{i\theta}\,\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}\left|D_\beta^\gamma\,[F](z)\right|.$$

Setting $\beta=\frac{\alpha}{k}$ in above inequality and noting that $|\beta|=\left|\frac{\alpha}{k}\right|\geq 1$, we get

$$\Lambda\,\left(\left|\frac{\alpha}{k}\right|-1\right)\left[\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q}$$
$$\leq\left[\int_0^{2\pi}\left|\,1+\,e^{i\theta}\,\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^\gamma\,[F](z)\right|. \tag{3.34}$$

Let $G(z)=\,z^n\,\overline{F\left(1/\bar{z}\right)}$. Then $|G(z)|=|F(z)|$ for $|z|=1$ and $G(z)$ dose not vanish in $|z|<1$. Therefore, by Lemma 2.4 applied to the polynomial $G(z)$ with $R=k\geq 1$, it follows that for each $q>0$,

$$\int_0^{2\pi}\left|G\left(k\,e^{i\theta}\right)\right|^q\,d\theta\leq\frac{\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q\,d\theta\right\}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^q\,d\theta\right\}}\int_0^{2\pi}\left|G\left(e^{i\theta}\right)\right|^q d\theta$$
$$=\frac{\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q\,d\theta\right\}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^q\,d\theta\right\}}\int_0^{2\pi}\left|F\left(e^{i\theta}\right)\right|^q\,d\theta \tag{3.35}$$

Combining (3.34) and (3.35), we get for each $q>0$,

$$\Lambda\,(|\alpha|-k)\left[\int_0^{2\pi}\left|G\left(k\,e^{i\theta}\right)\right|^q d\theta\right]^{1/q}$$

$$\le k\,\frac{\left\{\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q d\theta\right\}^{1/q}}{\left\{\int_0^{2\pi}|1+e^{i\theta}|^q\,d\theta\right\}^{1/q}}\left[\int_0^{2\pi}\left|\,1+e^{i\theta}\right|^q d\theta\right]^{1/q}\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[F](z)\right|$$

$$= k\,\left[\int_0^{2\pi}\left|\,1+k^n\,e^{i\theta}\,\right|^q\;d\theta\right]^{1/q}\;\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[F](z)\right|. \qquad (3.36)$$

Also since

$$G(z)=z^n\,\overline{F(1/\bar{z})}\;=\;z^n\,\overline{P(k/\bar{z})}\,,$$

we see that for $0\le\theta<2\pi$,

$$\left|G\left(k\,e^{i\theta}\right)\right|=\left|\,k^n\,e^{in\theta}\;\overline{P(\,e^{i\theta})}\;\right|=\;k^n\left|P(\,e^{i\theta})\right|.$$

Using this in (3.36), we get

$$\Lambda\,k^n(|\alpha|-k)\left[\int_0^{2\pi}\left|P(\,e^{i\theta})\right|^q\;d\theta\right]^{1/q}$$

$$\le\;k\;\left[\int_0^{2\pi}\left|\,1+k^n\,e^{i\theta}\,\right|^q\;d\theta\right]^{1/q}\;\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[F](z)\right|\;. \qquad (3.37)$$

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[P](kz)\right|\;=\;\max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha}{k}-z\right)\;P^{\gamma}(k\,z)\,\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha-k\,z}{k}\right)\;P(k\,z)\;\sum_{j=1}^{n}\frac{\gamma_j}{z-\frac{z_j}{k}}\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha-k\,z}{k}\right)k\;P(k\,z)\;\sum_{j=1}^{n}\frac{\gamma_j}{k\,z-z_j}\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+(\alpha-k\,z)\;\;P(k\,z)\;\sum_{j=1}^{n}\frac{\gamma_j}{k\,z-z_j}\right|$$

$$=\;\max_{|z|=1}|G(kz)|\;=\;\max_{|z|=k}|G(z)|\;\;,$$

where $\quad \mathrm{G(z)}=\Lambda\,P(\,z)+(\alpha-\,z)\;\,P(\,z)\;\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}$ is a polynomial of degree at most $(n-1)$. On using

inequality (2.3), this gives

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[P](kz)\right|=\max_{|z|=k}|G(z)|\;\;\le\;k^{n-1}\;\max_{|z|=1}|G(z)|$$

$$=\;k^{n-1}\;\max_{|z|=1}\left|\Lambda\,P(\,z)+(\alpha-z)\;\;P(\,z)\;\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}\right|\;=\;k^{n-1}\;\max_{|z|=1}\left|D^{\gamma}_{\alpha}\,[P](z)\right|$$

Hence,

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}\,[P](kz)\right|\le\,k^{n-1}\;\max_{|z|=1}\left|D^{\gamma}_{\alpha}\,[P](z)\right| \qquad (3.38)$$

On combining (3.38) and (3.37), we get

$$\Lambda\,k^n\;(|\alpha|-k)\left[\int_0^{2\pi}\left|P(\,e^{i\theta})\right|^q\;d\theta\right]^{1/q}\;\le\;k^n\;\left[\int_0^{2\pi}\left|\,1+k^n\,e^{i\theta}\,\right|^q\;d\theta\right]^{1/q}\;\max_{|z|=1}\left|D^{\gamma}_{\alpha}\,[P](z)\right|$$

$$\Lambda\,(|\alpha|-k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q} \le \left[\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right|.$$

Which proves the Theorem 3.4.

Remark 3.4. Taking $q\to\infty$ in inequality (3.33) reduces to

$$\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right| \ge \frac{\Lambda\,(|\alpha|-k)}{1+k^n}\max_{|z|=1}|P(z)| \quad . \tag{3.39}$$

The above result is due to Rather el at. [15].

Remark 3.5. For the $n-$tuple $\gamma=(1,1,1,\ldots,1)$,the inequality (3.33) reduces to

$$n\,(|\alpha|-k)\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q}$$

$$\le \left[\int_0^{2\pi}\left|1+k^n e^{i\theta}\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}|D_\alpha\,P(z)| \quad . \tag{3.40}$$

This result is due to Rather el at. [16].

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المتباينات التكامل للمشتقة القطبية والمشتقة القطبية المعممة من متعددات الحدود المركبة

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الملخص

لمتعددة الحدود $P(z)$ من الدرجة n، تملك جميع الأصفار في $|z| \leq 1$، مالك [11] بشرط أن $q > 0$،

$$n\left[\int_0^{2\pi}\left|P\left(e^{i\theta}\right)\right|^q\,d\theta\right]^{1/q} \leq \left[\int_0^{2\pi}\left|1+e^{i\theta}\right|^q\,d\theta\right]^{1/q}\max_{|z|=1}\left|P'(z)\right|.$$

في هذه الورقة البحثية ، نقوم بتعميم المتباينة أعلاه على المشتقة القطبية و المشتقة القطبية المعممة، والتي تتضمن – بوصفها حالات خاصة العديد من النتائج المعروفة في هذا المجال.

الكلمات المفتاحية: متعددات الحدود ، أصفار مقيدة ، متباينات في المجال المركب.