Integrated representations of Euler-type for functions related To Kampe' de Fe'rietfunction of the fourth order (III)

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Abstract

In this work, we obtain certain integral representations for functions related to Kampe' de Fe'rietfunction of the fourth order,which are the sufficiently general in nature and are capable of yielding a large number of simpler and useful results merely by specializing the parameters in them.

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Introduction

It is often convenient to identify the various functions with integral along certain paths in the real or complex plane. These integrals provide recursion formulas, asymptotic forms and analytic continuations of the special functions. Inthis paper we consider Eulerian integral formulas of first kindand obtained a number of integral representations for functions related to Kampe' de Fe'riet function of the fourth order.A great interest in the theory ofhypergeometric functions (that is, hyper-geometric functions of one , two and several variables) ismotivated essentially by the fact that the solutions of many applied problems involving (for example) partial differentialequations are obtainable with the help of such hypergeometric functions (see, for details, [10, p. 47-48]).Also, in this regard, it is noticed that the general sextic equation can be solved in terms of Kampe' de Fe'rietfunction (see [2] and [8]).Although the integrals involving and representing hypergeometric functions have numerous applications in pure and applied mathematics (see, for example, [4]-[7]), not all such integrals have beencollected in tables or are readily available in the mathematical literature.It is noted that a few integrals involving functions related to Kampe' de Fe'riet function of two variables annexed those mathematical literature.

The Kampe' de Fe'riets hyper-geometric series of two variables $F_{l;i;j}^{p;q;k}$ (see [10] and [9]) is defined as follows:

$$F_{l;i;j}^{p;q;k}\left[\begin{matrix}(a_p);(b_q);(c_k);\\(\alpha_l);(\beta_m);(\gamma_n);\end{matrix}x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\, r!s!}x^r y^s, \tag{1.1}$$

where for convergence

$$\left.\begin{array}{l} p+q<l+m+1,\ p+k<l+n+1,\ |x|<\infty\ ,\ |y|\ <\infty \\ \text{or} \\ p+q<l+m+1,\ p+k<l+n+1,\ |x|,<\infty,\ |y|<\infty,\text{and} \\ |x|^{\frac{1}{p-1}}+|y|^{\wedge\frac{1}{p-1}}<1\ ,\text{if}\ p>l;\ \max\{\ |x|,|y|\}<1,\ if\ p\ <\ l \end{array}\right\} \quad (1.2)$$

where

$\prod_{j=1}^{p}\left(a_j\right)_{r+s}=(a_1)_{r+s}(a_2)_{r+s}......(a_p)_{r+s}$, with similar interpretations for $\prod_{j=1}^{l}\left(\alpha_j\right)_{r+s}$, , et.cetera and $(a)_n$ denotes the Pochhammer symbol given by $(a)_0=1,(a)_n=\Gamma(a+n)/\Gamma(a)$

and Γ being the well-known Gamma function.The Kampé de Fériet's function (1.1) being the most general hypergeometric function of two variables, this is because the Kampé de Fériet function reduces to the product of two generalized hypergeometric functions of one variable by choosing parameters suitably.Here, in this present work, we aim atinvestigating 83 integral representations for functions related to Kampe' de Fe'rietfunction of the fourth order.

Integral Representations

First, we recall the definition ofEuler integral of the first kind (the Beta function)[3]:

$$B\left(a,b\right)=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\qquad \text{Re}a>0,\ \text{Re}b>0.\ (2.1)$$

By making a simple application of (2.1), we begin by presenting each of thefollowing integral representations (2.2)–(2.83).

Theorem. Each of the following integral representations for Kampe' de Fe'riet functions holdstrue.

$$F^{0;4,4}_{1;2,2}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e+e_1;\quad f,g;\quad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma\left(e+e_1\right)}{\Gamma\left(e\right)\Gamma\left(e_1\right)}\int_0^1 \xi^{e-1}\left(1-\xi\right)^{e_1-1} \quad (2.2)$$
$$\times{}_4F_3\left(a,b,c,d;e,f,g;x\xi\right){}_4F_3\left(a_1,b_1,c_1,d_1;e_1,f_1,g_1;y\left(1-\xi\right)\right)d\xi,$$
$$\text{Re}\left(e\right)>0,\text{Re}\left(e_1\right)>0,$$

$$F^{0;4,4}_{1;2,2}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\quad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma\left(e\right)}{\Gamma\left(a\right)\Gamma\left(e-a\right)}\int_0^1 \xi^{a-1}\left(1-\xi\right)^{e-a-1} \quad (2.3)$$
$$\times{}_3F_2\left(b,c,d;f,g;x\xi\right){}_4F_3\left(a_1,b_1,c_1,d_1;e-a,f_1,g_1;y\left(1-\xi\right)\right)d\xi,$$
$$\text{Re}\left(e\right)>0,\text{Re}\left(a\right)>0,$$

$$F^{0;4,4}_{1;2,2}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\quad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma\left(a+a_1\right)}{\Gamma\left(a\right)\Gamma\left(a_1\right)}\int_0^1 \xi^{a-1}\left(1-\xi\right)^{a_1-1} \quad (2.4)$$
$$\times F^{1;3,3}_{1;2,2}\left[\begin{matrix}a+a_1;b,c,d;b_1,c_1,d_1;\\ e;\ f,g;\ f_1,g_1;\end{matrix}x\xi,y\left(1-\xi\right)\right]d\xi,$$
$$\text{Re}\left(a\right)>0,\text{Re}\left(a_1\right)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(a+a_1)\Gamma(b+b_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)}\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{a_1-1}$$
$$(1-\eta)^{b_1-1}\times F_{1;2,2}^{2;2,2}\left[\begin{matrix}a+a_1,b+b_1;c,d;c_1,d_1;\\ e;f,g;f_1,g_1;\end{matrix}x\xi\eta,y(1-\xi)(1-\eta)\right]d\xi d\eta, \tag{2.5}$$
$$\mathrm{Re}(a)>0,\mathrm{Re}(a_1)>0,\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1} \tag{2.6}$$
$$\times F_{1;2,2}^{3;1,1}\left[\begin{matrix}a+a_1,b+b_1,b+b_1;\quad d;\quad d_1;\\ e;f,g;f_1,g_1;\end{matrix}x\xi\eta\zeta,y(1-\xi)(1-\eta)(1-\zeta)\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(a)>0,\mathrm{Re}(a_1)>0,\mathrm{Re}(b)>0,\mathrm{Re}[b_1]>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}\tau^{d-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1}(1-\tau)^{d_1-1} \tag{2.7}$$
$$\times F_{1;2,2}^{4;0,0}\left[\begin{matrix}a+a_1,b+b_1,c+c_1,d+d_1;\quad -;\quad -;\\ e;\qquad f,g;f_1,g_1;\end{matrix}x\xi\eta\zeta\tau,y(1-\xi)(1-\eta)(1-\zeta)(1-\tau)\right]d\xi d\eta d\zeta d\tau,$$
$$\mathrm{Re}(a)>0,\mathrm{Re}(a_1)>0,\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\mathrm{Re}(d)>0,\mathrm{Re}d_1>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}\int_0^1 \xi^{c-1}(1-\xi)^{f-c-1}$$
$$\times F_{1;1,2}^{0;3,4}\left[\begin{matrix}-;a,b,d;a_1,b_1,c_1,d_1;\\ e;\quad g;\qquad f_1,g_1;\end{matrix}x\xi,y\right]d\xi, \tag{2.8}$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(c)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(f)\Gamma(f_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)}$$
$$\times\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}F_{1;1,1}^{0;3,3}\left[\begin{matrix}-;a,b,d;a_1,b_1,d_1;\\ e;\quad g;\qquad g_1;\end{matrix}x\xi,y\eta\right]d\xi d\eta, \tag{2.9}$$
$$\mathrm{Re}(f)>\mathrm{Re}c>(0),\mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\qquad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1}F_{1;0,1}^{0;2,3}\left[\begin{matrix}-;a,b;a_1,b_1,d_1;\\ e;\quad -;\qquad g_1;\end{matrix}x\xi\zeta,y\eta\right]d\xi d\eta d\zeta, \tag{2.10}$$
$$\mathrm{Re}(f)>\mathrm{Re}(c)>0,\mathrm{Re}(f)_1>\mathrm{Re}(c_1)>0,\ \mathrm{Re}(g)>\mathrm{Re}(d)>0,$$

$$F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e;\quad f,g;\quad f_1,g_1;\end{matrix}x,y\right]=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)\Gamma(g_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}\tau^{d_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1}(1-\tau)^{g_1-d_1-1} \tag{2.11}$$
$$\times F_3(a,b,a_1,b_1;e;x\xi\zeta,y\eta\tau)d\xi d\eta d\zeta d\tau,$$
$$\mathrm{Re}(f)>\mathrm{Re}(c)>0,\mathrm{Re}(f)_1>\mathrm{Re}(c_1)>0,\ \mathrm{Re}(g)>\mathrm{Re}(d)>0,\mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e+e_1,f;\quad g;\quad g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1}F_{1;2,2}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ f;\quad e,g;\quad e_1,g_1;\end{matrix}x\xi,y(1-\xi)\right]d\xi, \tag{2.12}$$
$$\mathrm{Re}(e)>0,\ \mathrm{Re}(e_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e+e_1,f+f_1;\quad g;\quad g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)\Gamma(f+f_1)}{\Gamma(e)\Gamma(e_1)\Gamma(f)\Gamma(f_1)}\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1}(1-\xi)^{e_1-1}(1-\eta)^{f_1-1} \tag{2.13}$$
$$\times{}_4F_3(a,b,c,d;e,f,g;x\xi\eta)\,{}_4F_3(a_1,b_1,c_1,d_1;e_1,f_1,g_1;y(1-\xi)(1-\eta))d\xi d\eta,$$
$$\mathrm{Re}(e)>0,\ \mathrm{Re}(e_1)>0,\ \mathrm{Re}(f)>0,\ \mathrm{Re}(f)_1>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e,f;\quad g;\quad g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1}F_{1;1,2}^{0;3,4}\left[\begin{matrix}-;b,c,d;a_1,b_1,c_1,d_1;\\ f;\quad g;\quad e-a,g_1;\end{matrix}x\xi,y(1-\xi)\right]d\xi, \tag{2.14}$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e,f;\quad g;\quad g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(e-a)\Gamma(a_1)\Gamma(f-a_1)}\int_0^1\int_0^1 \xi^{a-1}\eta^{a_1-1}(1-\xi)^{e-a-1}(1-\eta)^{f-a_1-1} \tag{2.15}$$
$$\times{}_3F_2(b,c,d;f-a_1,g;x\xi(1-\eta))\,{}_3F_2(b_1,c_1,d_1;e-a,g_1;y(1-\xi)\eta)d\xi d\eta,$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0,\ \mathrm{Re}(f)>\mathrm{Re}(a_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix}-;a,b,c,d;a_1,b_1,c_1,d_1;\\ e,f;\quad g;\quad g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(a+a_1)}{\Gamma(a)\Gamma(a_1)}\int_0^1 \xi^{a-1}(1-\xi)^{a_1-1}F_{2;1,1}^{1;3,3}\left[\begin{matrix}a+a_1;b,c,d;b_1,c_1,d_1;\\ e,f;\quad g;\quad g_1;\end{matrix}x\xi,y(1-\xi)\right]d\xi, \tag{2.16}$$
$$\mathrm{Re}(a)>0,\ \mathrm{Re}(a_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f; \quad g; \quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)}\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1} \tag{2.17}$$
$$\times F_{2;1,1}^{2;2,2}\left[\begin{matrix} a+a_1,b+b_1;c,d;c_1,d_1; \\ e,f; \quad g; \quad g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\operatorname{Re}(a)>0,\operatorname{Re}(a_1)>0,\operatorname{Re}(b)>0,\operatorname{Re}(b_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f; \quad g; \quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1} \tag{2.18}$$
$$\times F_{2;1,1}^{3;1,1}\left[\begin{matrix} a+a_1,b+b_1,c+c_1;d;d_1; \\ e,f; \ g; g_1; \end{matrix} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta)\right]d\xi d\eta d\zeta,$$
$$\operatorname{Re}(a)>0,\operatorname{Re}(a_1)>0,\operatorname{Re}(b)>0,\operatorname{Re}(b_1)>0,\ \operatorname{Re}(c)>0,\operatorname{Re}(c_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f; \quad g; \quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}\tau^{d-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1}(1-\tau)^{d_1-1} \tag{2.19}$$
$$\times F_{2;1,1}^{4;0,0}\left[\begin{matrix} a+a_1,b+b_1,c+c_1,d+d_1;-;-; \\ e,f; \ g; g_1; \end{matrix} x\xi\eta\zeta\tau, y(1-\xi)(1-\eta)(1-\zeta)(1-\tau)\right]d\xi d\eta d\zeta d\tau,$$
$$\operatorname{Re}a>0,\operatorname{Re}(a_1)>0,\operatorname{Re}(b)>0,\operatorname{Re}(b_1)>0,\ \operatorname{Re}(c)>0,\operatorname{Re}(c_1)>0,\ \operatorname{Re}(d)>0,\operatorname{Re}(d_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f; \quad g; \quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g)}{\Gamma(c)\Gamma(g-c)}\int_0^1 \xi^{c-1}(1-\xi)^{g-c-1}F_{2;0,1}^{0;3,4}\left[\begin{matrix} -;a,b,d;a_1,b_1,c_1,d_1; \\ e,f; \quad -; \quad g_1; \end{matrix} x\xi,y\right]d\xi, \tag{2.20}$$
$$\operatorname{Re}a>0,\operatorname{Re}(a_1)>0,\operatorname{Re}(b)>0,\operatorname{Re}(b_1)>0,\ \operatorname{Re}(c)>0,\operatorname{Re}(c_1)>0,\ \operatorname{Re}(d)>0,\operatorname{Re}(d_1)>0,$$

$$F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g)\Gamma(g_1)}{\Gamma(c)\Gamma(g-c)\Gamma(c_1)\Gamma(g_1-c_1)} \tag{2.21}$$
$$\times\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}(1-\xi)^{g-c-1}(1-\eta)^{g_1-c_1-1} F_{2;0,0}^{0;3,3}\left[\begin{matrix} -;a,b,d;a_1,b_1,d_1; \\ e,f;\quad -;\quad -; \end{matrix} x\xi,y\eta\right]d\xi d\eta,$$
$$\mathrm{Re}(g)>\mathrm{Re}(c)>0,\mathrm{Re}(g_1)>\mathrm{Re}(c_1)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;\quad a,b,c,d;a_1,b_1,c_1,d_1; \\ e+e_1,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1} F_{2;1,1}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ f,g;\quad e;\quad e_1; \end{matrix} x\xi,y(1-\xi)\right]d\xi, \tag{2.22}$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e+e_1,f+f_1,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e+e_1)\Gamma(f+f_1)}{\Gamma(e)\Gamma(e_1)\Gamma(f)\Gamma(f_1)} \tag{2.23}$$
$$\times\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1}(1-\xi)^{e_1-1}(1-\eta)^{f_1-1} F_{1;2,2}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ g;\quad e,f;\quad e_1,f_1; \end{matrix} x\xi\eta,y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,\mathrm{Re}(f)>0,\mathrm{Re}(f)_1>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;\quad a,b,c,d;a_1,b_1,c_1,d_1; \\ e+e_1,f+f_1,g+g_1;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e+e_1)\Gamma(f+f_1)\Gamma(g+g_1)}{\Gamma(e)\Gamma(e_1)\Gamma(f)\Gamma(f_1)\Gamma(g)\Gamma(g_1)}\int_0^1\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1}\zeta^{g-1}(1-\xi)^{e_1-1}(1-\eta)^{f_1-1}(1-\zeta)^{g_1-1} \tag{2.24}$$
$$\times{}_4F_3(a,b,c,d;e,f,g;x\xi\eta\zeta)\,{}_4F_3(a_1,b_1,c_1,d_1;e_1,f_1,g_1;y(1-\xi)(1-\eta)(1-\zeta))d\xi d\eta d\zeta,$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,\mathrm{Re}(f)>0,\mathrm{Re}(f_1)>0,\mathrm{Re}(g)>0,\mathrm{Re}(g_1)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1} F_{2;0,1}^{0;3,3}\left[\begin{matrix} -;b,c,d;a_1,b_1,c_1,d_1; \\ f,g;\quad -;\quad e-a; \end{matrix} x\xi,y(1-\xi)\right]d\xi, \tag{2.25}$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(e-a)\Gamma(b)\Gamma(f-b)} \tag{2.26}$$
$$\times\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{e-a-1}(1-\eta)^{f-b-1} F_{1;0,2}^{0;2,4}\left[\begin{matrix} -;c,d;a_1,b_1,c_1,d_1; \\ g;\quad -;e-a,f-b; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(a)>\mathrm{Re}(e)>0, \mathrm{Re}(f)>\mathrm{Re}(b)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)\Gamma(f)\Gamma(g)}{\Gamma(a)\Gamma(e-a)\Gamma(b)\Gamma(f-b)\Gamma(c)\Gamma(g-c)}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}(1-\xi)^{e-a-1}(1-\eta)^{f-b-1}(1-\zeta)^{g-c-1}(1-x\xi\eta\zeta)^{-d} \tag{2.27}$$
$$\times {}_4F_3\left(a_1,b_1,c_1,d_1;e-a,f-b,g-c;y(1-\xi)(1-\eta)(1-\zeta)\right)d\xi d\eta d\zeta,$$
$$\mathrm{Re}(e)>\mathrm{Re}(a)>0, \mathrm{Re}(f)>\mathrm{Re}(b)>0, \mathrm{Re}(g)>\mathrm{Re}(c)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)}{\Gamma(a)\Gamma(a_1)}\int_0^1 \xi^{a-1}(1-\xi)^{a_1-1} F_{3;0,0}^{1;3,3}\left[\begin{matrix} a+a_1;b,c,d;b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \tag{2.28}$$
$$\mathrm{Re}(a)>0, \mathrm{Re}(a_1)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)}\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1} \tag{2.29}$$
$$\times F_{3;0,0}^{2;2,2}\left[\begin{matrix} a+a_1,b+b_1;c,d;c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(a)>0, \mathrm{Re}(a_1)>0, \mathrm{Re}(b)>0, \mathrm{Re}(b_1)>0,$$

$$F_{3;0,0}^{0;4,4}\left[\begin{matrix} -;a,b,c,d;a_1,b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1} \tag{2.30}$$
$$\times F_{3;0,0}^{3;1,1}\left[\begin{matrix} a+a_1,b+b_1,c+c_1;d;d_1; \\ e,f,g;\ -;\ -; \end{matrix} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta)\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(a)>0, \mathrm{Re}(a_1)>0, \mathrm{Re}(b)>0, \mathrm{Re}(b_1)>0, \mathrm{Re}(c)>0, \mathrm{Re}(c_1)>0,$$

$$F^{0;4,4}_{3;0,0}\left[\begin{matrix} -; a,b,c,d; a_1,b_1,c_1,d_1; \\ e,f,g; \quad -; \quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(a+a_1)\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(a)\Gamma(a_1)\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}\zeta^{c-1}\tau^{d-1}(1-\xi)^{a_1-1}(1-\eta)^{b_1-1}(1-\zeta)^{c_1-1}(1-\tau)^{d_1-1} \tag{2.31}$$
$$\times{}_4F_3\left(a+a_1,b+b_1,c+c_1,d+d_1;e,f,g;x\xi\eta\zeta\tau+y(1-\xi)(1-\eta)(1-\zeta)(1-\tau)\right)d\xi d\eta d\zeta d\tau,$$
$$\mathrm{Re}(a)>0,\mathrm{Re}(a_1)>0,\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\ \mathrm{Re}(d)>0,\mathrm{Re}(d_1)>0,$$

$$F^{1;3,3}_{0;3,3}\left[\begin{matrix} a; b,c,d; b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)}{\Gamma(b)\Gamma(b_1)}\int_0^1 \xi^{b-1}(1-\xi)^{b_1-1}F^{2;2,2}_{0;3,3}\left[\begin{matrix} a,b+b_1; \quad c,d; \quad c_1,d_1; \\ -; e,f,g;e_1,f_1,g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \tag{2.32}$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,$$

$$F^{1;3,3}_{0;3,3}\left[\begin{matrix} a; b,c,d; b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}\zeta^{d-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1}(1-\zeta)^{d_1-1} \tag{2.33}$$
$$\times F^{4;0,0}_{0;3,3}\left[\begin{matrix} a,b+b_1,c+c_1,d+d_1; \quad -; \quad -; \\ -; e,f,g;e_1,f_1,g_1; \end{matrix} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta)\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\ \mathrm{Re}(d)>0,\mathrm{Re}(d_1)>0,$$

$$F^{1;3,3}_{0;3,3}\left[\begin{matrix} a; b,c,d; b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}\int_0^1 \xi^{c-1}(1-\xi)^{f-c-1}F^{1;2,3}_{0;2,3}\left[\begin{matrix} a;b,d;b_1,c_1,d_1; \\ -;e,g;e_1,f_1,g_1; \end{matrix} x\xi, y\right]d\xi, \tag{2.34}$$
$$\mathrm{Re}(f)>\mathrm{Re}(c)>0,$$

$$F^{1;3,3}_{0;3,3}\left[\begin{matrix} a; b,c,d; b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(f)\Gamma(f_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)} \tag{2.35}$$
$$\times\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}F^{1;2,2}_{0;2,2}\left[\begin{matrix} a;b,d;b_1,d_1; \\ -;e,g;e_1,g_1; \end{matrix} x\xi, y\eta\right]d\xi d\eta,$$
$$\mathrm{Re}(f)>\mathrm{Re}(c)>0,\mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0,$$

$$F_{0;3,3}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)} \tag{2.36}$$

$$\times\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1} F_{0;1,2}^{1;1,2}\left[\begin{matrix} a;b;b_1,d_1; \\ -;e;e_1,g_1; \end{matrix} x\xi\zeta, y\eta\right]d\xi d\eta d\zeta,$$

$$\mathrm{Re}(f)>\mathrm{Re}(c)>0, \mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0, \mathrm{Re}(g)>\mathrm{Re}(d)>0,$$

$$F_{0;3,3}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)\Gamma(g_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)}$$

$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}\tau^{d_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1}(1-\tau)^{g_1-d_1-1} \tag{2.37}$$

$$\times F_2(a,b,b_1;e,e_1;x\xi\zeta, y\eta\tau)d\xi d\eta d\zeta d\tau,$$

$$\mathrm{Re}(f)>\mathrm{Re}(c)>0, \mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0, \mathrm{Re}(g)>\mathrm{Re}(d)>0, \mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\ f_1,g_1; \end{matrix} x,y\right]=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}$$

$$\times\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1} {}_3F_2(b,c,d;f,g;x\xi)\, {}_3F_2(b_1,c_1,d_1;f_1,g_1;y\xi)d\xi, \tag{2.38}$$

$$\mathrm{Re}(e)>\mathrm{Re}(a),$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e+e_1;\ f,g;\ f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1} F_{0;3,3}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \tag{2.39}$$

$$\mathrm{Re}(e)>0, \mathrm{Re}(e_1)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\ f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(e)}{\Gamma(b)\Gamma(e-b)}\int_0^1 \xi^{b-1}(1-\xi)^{e-b-1} F_{0;2,3}^{1;2,3}\left[\begin{matrix} a;c,d;\quad b_1,c_1,d_1; \\ -;f,g;e-b,f_1,g_1; \end{matrix} x\xi, y(1-\xi)\right]d\xi, \tag{2.40}$$

$$\mathrm{Re}(e)>\mathrm{Re}(b),$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\quad f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)}{\Gamma(b)\Gamma(b_1)}\int_0^1 \xi^{b-1}(1-\xi)^{b_1-1}F_{1;2,2}^{2;2,2}\left[\begin{matrix} a,b+b_1;c,d;c_1,d_1; \\ e;f,g;f_1,g_1; \end{matrix} x\xi,y(1-\xi)\right]d\xi, \tag{2.41}$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\quad f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1} \tag{2.42}$$
$$\times F_{1;2,2}^{3;1,1}\left[\begin{matrix} a,b+b_1,c+c_1;\quad d;\quad d_1; \\ e;f,g;f_1,g_1; \end{matrix} x\xi\eta,y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\quad f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}\zeta^{d-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1}(1-\zeta)^{d_1-1} \tag{2.43}$$
$$\times F_{1;2,2}^{4;0,0}\left[\begin{matrix} a,b+b_1,c+c_1,d+d_1;\quad -;\quad -; \\ e;\ f,g;f_1,g_1; \end{matrix} x\xi\eta\zeta,y(1-\xi)(1-\eta)(1-\zeta)\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\mathrm{Re}(d)>0,\mathrm{Re}(d_1)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\quad f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}\int_0^1 \xi^{c-1}(1-\xi)^{f-c-1}F_{1;1,2}^{1;2,3}\left[\begin{matrix} a;b,d;b_1,c_1,d_1; \\ e;\quad g;\quad f_1,g_1; \end{matrix} x\xi,y\right]d\xi, \tag{2.44}$$
$$\mathrm{Re}(f)>Re(c)>0,$$

$$F_{1;2,2}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e;\ f,g;\quad f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(f)\Gamma(f_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)} \tag{2.45}$$
$$\times\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}F_{1;1,1}^{1;2,2}\left[\begin{matrix} a;b,d;b_1,d_1; \\ e;\quad g;\quad g_1; \end{matrix} x\xi,y\eta\right]d\xi d\eta,$$
$$\mathrm{Re}(f)>Re(c)>0,\mathrm{Re}(f_1)>Re(c_1)>0,$$

$$F^{1;3,3}_{1;2,2}\left[\begin{matrix}a;b,c,d;b_1,c_1,d_1;\\ e;\ f,g;\ \ f_1,g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)} \tag{2.46}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1}F^{1;1,2}_{1;0,1}\left[\begin{matrix}a;b;b_1,d_1;\\ e;-;\ \ g_1;\end{matrix}x\,\xi\zeta,y\,\eta\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(f)>Re(c)>0,\mathrm{Re}(f_1)>Re(c_1)>0,\ \mathrm{Re}(g)>\mathrm{Re}(d)>0,$$

$$F^{1;3,3}_{1;2,2}\left[\begin{matrix}a;b,c,d;b_1,c_1,d_1;\\ e;\ f,g;\ \ f_1,g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(f)\Gamma(f_1)\Gamma(g)\Gamma(g_1)}{\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{c-1}\eta^{c_1-1}\zeta^{d-1}\tau^{d_1-1}(1-\xi)^{f-c-1}(1-\eta)^{f_1-c_1-1}(1-\zeta)^{g-d-1}(1-\tau)^{g_1-d_1-1} \tag{2.47}$$
$$\times F_1(a;b,b_1;e;x\,\xi\zeta,y\,\eta\tau)d\xi d\eta d\zeta d\tau,$$
$$\mathrm{Re}(f)>Re(c)>0,\mathrm{Re}(f_1)>Re(c_1)>0,\ \mathrm{Re}(g)>\mathrm{Re}(d)>0,\ \mathrm{Re}(d_1)>Re(g_1)>0,$$

$$F^{1;3,3}_{2;1,1}\left[\begin{matrix}a;b,c,d;b_1,c_1,d_1;\\ e,f;\ \ \ g;\ \ \ \ g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1}F^{0;3,3}_{1;1,1}\left[\begin{matrix}a;b,c,d;b_1,c_1,d_1;\\ e,f;\ \ \ g;\ \ \ \ g_1;\end{matrix}x\,\xi,y\,\xi\right]d\xi, \tag{2.48}$$
$$\mathrm{Re}(e)>\mathrm{Re}(a),$$

$$F^{1;3,3}_{2;1,1}\left[\begin{matrix}a;\ b,c,d;b_1,c_1,d_1;\\ e+e_1,f;\ \ \ g;\ \ \ \ g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)}{\Gamma(e)\Gamma(e_1)}\int_0^1 \xi^{e-1}(1-\xi)^{e_1-1}F^{1;3,3}_{1;2,2}\left[\begin{matrix}a;b,c,d;b_1,c_1,d_1;\\ f;\ \ e,g;\ \ \ e_1,g_1;\end{matrix}x\,\xi,y\,(1-\xi)\right]d\xi, \tag{2.49}$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,$$

$$F^{1;3,3}_{2;1,1}\left[\begin{matrix}a;\ \ \ \ b,c,d;b_1,c_1,d_1;\\ e+e_1,f+f_1;\ \ \ g;\ \ \ \ g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(e+e_1)\Gamma(f+f_1)}{\Gamma(e)\Gamma(e_1)\Gamma(f)\Gamma(f_1)} \tag{2.50}$$
$$\times\int_0^1\int_0^1 \xi^{e-1}\eta^{f-1}(1-\xi)^{e_1-1}(1-\eta)^{f_1-1}F^{1;3,3}_{0;3,3}\left[\begin{matrix}a;b,c,d;\ b_1,c_1,d_1;\\ -;e,f,g;e_1,f_1,g_1;\end{matrix}x\,\xi\eta,y\,(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(e)>0,\mathrm{Re}(e_1)>0,\mathrm{Re}(f)>0,\mathrm{Re}(f_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(b)\Gamma(e-b)}\int_0^1 \xi^{b-1}(1-\xi)^{e-b-1} F_{1;1,2}^{1;2,3}\left[\begin{matrix} a;c,d;b_1,c_1,d_1; \\ f;\quad g;e-b,g_1; \end{matrix} x\xi, y(1-\xi)\right] d\xi, \qquad (2.51)$$
$$\operatorname{Re}(e)>\operatorname{Re}(b)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)\Gamma(f)}{\Gamma(b)\Gamma(e-b)\Gamma(c)\Gamma(f-c)}\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}(1-\xi)^{e-b-1}(1-\eta)^{f-c-1}$$
$$\times F_{0;1,3}^{1;1,3}\left[\begin{matrix} a;d;\quad b_1,c_1,d_1; \\ -;g;e-b,f-c,g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right] d\xi d\eta, \qquad (2.52)$$
$$\operatorname{Re}(e)>\operatorname{Re}(b)>0, \operatorname{Re}(f)>\operatorname{Re}(c)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)}{\Gamma(b)\Gamma(b_1)}\int_0^1 \xi^{b-1}(1-\xi)^{b_1-1} F_{2;1,1}^{2;2,2}\left[\begin{matrix} a,b+b_1;c,d;c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x\xi, y(1-\xi)\right] d\xi, \qquad (2.53)$$
$$\operatorname{Re}(b)>0, \operatorname{Re}(b_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1}$$
$$\times F_{2;1,1}^{3;1,1}\left[\begin{matrix} a,b+b_1,c+c_1;d;d_1; \\ e,f;g;g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right] d\xi d\eta \qquad (2.54)$$
$$\operatorname{Re}(b)>0, \operatorname{Re}(b_1)>0, \operatorname{Re}(c)>0, \operatorname{Re}(c_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}\zeta^{d-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1}(1-\zeta)^{d_1-1}$$
$$\times F_{2;1,1}^{4;0,0}\left[\begin{matrix} a,b+b_1,c+c_1,d+d_1;-;-; \\ e,f;\quad g;g_1; \end{matrix} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta)\right] d\xi d\eta d\zeta, \qquad (2.55)$$
$$\operatorname{Re}(b)>0, \operatorname{Re}(b_1)>0, \operatorname{Re}(c)>0, \operatorname{Re}(c_1)>0, \operatorname{Re}(d)>0, \operatorname{Re}(d_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\int_0^1 \xi^{d_1-1}(1-\xi)^{g_1-d_1-1} F_{2;1,0}^{1;3,2}\left[\begin{matrix} a;b,c,d;b_1,c_1; \\ e,f;\quad g;\quad -; \end{matrix} x,y\xi\right]d\xi, \qquad (2.56)$$
$$\operatorname{Re}(g_1)>\operatorname{Re}(d_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g)\Gamma(g_1)}{\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)} \qquad (2.57)$$
$$\times\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1} F_{2;0,0}^{1;2,2}\left[\begin{matrix} a;b,c;b_1,c_1; \\ e,f;\quad -;\quad -; \end{matrix} x\xi,y\eta\right]d\xi d\eta,$$
$$\operatorname{Re}(g)>\operatorname{Re}(d)>0,\ \operatorname{Re}(g_1)>\operatorname{Re}(d_1)>0,$$

$$F_{2;1,1}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f;\quad g;\quad g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(g)\Gamma(g_1)\Gamma(e)}{\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)\Gamma(a)\Gamma(e-a)}\int_0^1\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}\zeta^{a-1}(1-\xi)^{g-d-1} \qquad (2.58)$$
$$\times(1-\eta)^{g_1-d_1-1}(1-\zeta)^{e-a-1}F_3(b,b_1,c,c_1;f;x\xi\zeta,y\eta\zeta)d\xi d\eta d\zeta,$$
$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,\ \operatorname{Re}(g)>\operatorname{Re}(d)>0,\ \operatorname{Re}(g_1)>\operatorname{Re}(d_1)>0,$$

$$F_{3;0,0}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)}{\Gamma(a)\Gamma(e-a)}\int_0^1 \xi^{a-1}(1-\xi)^{e-a-1} F_{2;0,0}^{0;3,3}\left[\begin{matrix} -;b,c,d;b_1,c_1,d_1; \\ f,g;\quad -;\quad -; \end{matrix} x\xi,y\xi\right]d\xi, \qquad (2.59)$$
$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,$$

$$F_{3;0,0}^{1;3,3}\left[\begin{matrix} a;b,c,d;b_1,c_1,d_1; \\ e,f,g;\quad -;\quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(e-a)\Gamma(b)\Gamma(f-b)} \qquad (2.60)$$
$$\times\int_0^1\int_0^1 \xi^{a-1}\eta^{b-1}(1-\xi)^{e-a-1}(1-\eta)^{f-b-1} F_{1;0,1}^{0;2,3}\left[\begin{matrix} -;c,d;b_1,c_1,d_1; \\ g;\quad -;\quad f-b; \end{matrix} x\xi\eta,y\xi(1-\eta)\right]d\xi d\eta,$$
$$\operatorname{Re}(e)>\operatorname{Re}(a)>0,\operatorname{Re}(f)>\operatorname{Re}(b)>0,$$

$$F_{3;0,0}^{1;3,3}\left[\begin{matrix} a;\, b,c,d;b_1,c_1,d_1; \\ e,f,g; \quad -; \quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)}{\Gamma(b)\Gamma(b_1)}\int_0^1 \xi^{b-1}(1-\xi)^{b_1-1} F_{3;0,0}^{2;2,2}\left[\begin{matrix} a,b+b_1;c,d;c_1,d_1; \\ e,f,g; \quad -; \quad -; \end{matrix} x,y\right]d\xi, \tag{2.61}$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,$$

$$F_{3;0,0}^{1;3,3}\left[\begin{matrix} a;\, b,c,d;b_1,c_1,d_1; \\ e,f,g; \quad -; \quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)}\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1} \tag{2.62}$$
$$\times F_{3;0,0}^{3;1,1}\left[\begin{matrix} a,b+b_1,c+c_1;d;d_1; \\ e,f,g;-;-; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,$$

$$F_{3;0,0}^{1;3,3}\left[\begin{matrix} a;\, b,c,d;b_1,c_1,d_1; \\ e,f,g; \quad -; \quad -; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(b+b_1)\Gamma(c+c_1)\Gamma(d+d_1)}{\Gamma(b)\Gamma(b_1)\Gamma(c)\Gamma(c_1)\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1\int_0^1 \xi^{b-1}\eta^{c-1}\zeta^{d-1}(1-\xi)^{b_1-1}(1-\eta)^{c_1-1}(1-\zeta)^{d_1-1} \tag{2.63}$$
$$\times {}_4F_3\left(a,b+b_1,c+c_1,d+d_1;e,f,g;x\xi\eta\zeta+y(1-\xi)(1-\eta)(1-\zeta)\right)d\xi d\eta d\zeta$$
$$\mathrm{Re}(b)>0,\mathrm{Re}(b_1)>0,\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\mathrm{Re}(d)>0,\mathrm{Re}(d_1)>0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; \quad c,d; \quad c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(c+c_1)}{\Gamma(c)\Gamma(c_1)}\int_0^1 \xi^{c-1}(1-\xi)^{c_1-1} F_{0;3,3}^{3;1,1}\left[\begin{matrix} a,b,c+c_1; \quad d; \quad d_1; \\ -;\, e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]d\xi, \tag{2.64}$$
$$\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; \quad c,d; \quad c_1,d_1; \\ -;e,f,g;e_1,f_1,g_1; \end{matrix} x,y\right]$$
$$=\frac{\Gamma(c+c_1)}{\Gamma(c)\Gamma(c_1)}\frac{\Gamma(d+d_1)}{\Gamma(d)\Gamma(d_1)}\int_0^1\int_0^1 \xi^{c-1}\eta^{d-1}(1-\xi)^{c_1-1}(1-\eta)^{d_1-1} \tag{2.65}$$
$$\times F_{0;3,3}^{4;0,0}\left[\begin{matrix} a,b,c+c_1,d+d_1; \quad -; \quad -; \\ -;\, e,f,g;e_1,f_1,g_1; \end{matrix} x\xi\eta, y(1-\xi)(1-\eta)\right]d\xi d\eta,$$
$$\mathrm{Re}(c)>0,\mathrm{Re}(c_1)>0,\mathrm{Re}(d)>0,\mathrm{Re}(d_1)>0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; & c,d; & c_1,d_1; \\ -; e,f & ,g; e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\int_0^1 \xi^{d_1-1}(1-\xi)^{g_1-d_1-1} F_{0;3,2}^{2;2,1}\left[\begin{matrix} a,b; & c,d; & c_1; \\ -; e,f & ,g; e_1,f_1; \end{matrix} x\xi, y\right] d\xi, \quad (2.66)$$

$$\operatorname{Re}(g_1) > \operatorname{Re}(d_1) > 0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; & c,d; & c_1,d_1; \\ -; e,f & ,g; e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)} \quad (2.67)$$

$$\times\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1} F_{0;2,2}^{2;1,1}\left[\begin{matrix} a,b; & c; & c_1; \\ -; e,f & ; e_1,f_1; \end{matrix} x\xi, y\eta\right] d\xi d\eta,$$

$$\operatorname{Re}(g) > \operatorname{Re}(d) > 0, \operatorname{Re}(g_1) > \operatorname{Re}(d_1) > 0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; & c,d; & c_1,d_1; \\ -; e,f & ,g; e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\frac{\Gamma(f_1)}{\Gamma(c_1)\Gamma(f_1-c_1)}\int_0^1\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}\zeta^{c_1-1} \quad (2.68)$$

$$\times(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1}(1-\zeta)^{f_1-c_1-1} F_{0;2,1}^{2;1,0}\left[\begin{matrix} a,b; & c; -; \\ -; e,f & ; e_1; \end{matrix} x\xi, y\zeta\eta\right] d\xi d\eta d\zeta,$$

$$\operatorname{Re}(g) > \operatorname{Re}(d) > 0, \operatorname{Re}(g_1) > \operatorname{Re}(d_1) > 0, \operatorname{Re}(c_1) > \operatorname{Re}(f_1) > 0,$$

$$F_{0;3,3}^{2;2,2}\left[\begin{matrix} a,b; & c,d; & c_1,d_1; \\ -; e,f & ,g; e_1,f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(g)\Gamma(g_1)\Gamma(f)\Gamma(f_1)}{\Gamma(d)\Gamma(g-d)\Gamma(d_1)\Gamma(g_1-d_1)\Gamma(c)\Gamma(f-c)\Gamma(c_1)\Gamma(f_1-c_1)}\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}\zeta^{c_1-1}\tau^{c-1} \quad (2.69)$$

$$\times(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1}(1-\zeta)^{f_1-c_1-1}(1-\tau)^{f-c-1} F_4(a,b;e,e_1;x\xi\tau, y\eta\zeta) d\xi d\eta d\zeta d\tau,$$

$$\operatorname{Re}(g) > \operatorname{Re}(d) > 0, \operatorname{Re}(g_1) > \operatorname{Re}(d_1) > 0, \operatorname{Re}(c) > \operatorname{Re}(f) > 0, \operatorname{Re}(c_1) > \operatorname{Re}(f_1) > 0,$$

$$F_{1;2,2}^{2;2,2}\left[\begin{matrix} a,b; c,d; c_1,d_1; \\ e; f,g; f_1,g_1; \end{matrix} x,y\right]$$

$$=\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\int_0^1 \xi^{d_1-1}(1-\xi)^{g_1-d_1-1} F_{1;2,1}^{2;2,1}\left[\begin{matrix} a,b; c,d; c_1; \\ e; f,g; f_1; \end{matrix} x, y\xi\right] d\xi, \quad (2.70)$$

$$\operatorname{Re}(g_1) > \operatorname{Re}(d_1) > 0,$$

$$F^{2;2,2}_{1;2,2}\left[\begin{matrix}a,b;c,d;c_1,d_1;\\ e;f,g;f_1,g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)} \tag{2.71}$$
$$\times\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1}F^{2;1,1}_{1;1,1}\left[\begin{matrix}a,b;c;c_1;\\ e;f;f_1;\end{matrix}x\xi,y\eta\right]d\xi d\eta,$$
$$\mathrm{Re}(g)>\mathrm{Re}(d)>0,\mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,$$

$$F^{2;2,2}_{1;2,2}\left[\begin{matrix}a,b;c,d;c_1,d_1;\\ e;f,g;f_1,g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\frac{\Gamma(f_1)}{\Gamma(c_1)\Gamma(f_1-c_1)} \tag{2.72}$$
$$\times\int_0^1\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}\zeta^{c_1-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1}(1-\zeta)^{f_1-c_1-1}F^{2;1,0}_{1;1,0}\left[\begin{matrix}a,b;c;-;\\ e;f;-;\end{matrix}x\xi,y\eta\zeta\right]d\xi d\eta d\zeta,$$
$$\mathrm{Re}(g)>\mathrm{Re}(d)>0,\mathrm{Re}(g_1)>\mathrm{R}(ed_1)>0,\mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0,$$

$$F^{2;2,2}_{1;2,2}\left[\begin{matrix}a,b;c,d;c_1,d_1;\\ e;f,g;f_1,g_1;\end{matrix}x,y\right]$$
$$=\frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)}\frac{\Gamma(g_1)}{\Gamma(d_1)\Gamma(g_1-d_1)}\frac{\Gamma(f_1)}{\Gamma(c_1)\Gamma(f_1-c_1)}\frac{\Gamma(f)}{\Gamma(c)\Gamma(f-c)}$$
$$\times\int_0^1\int_0^1\int_0^1\int_0^1 \xi^{d-1}\eta^{d_1-1}\zeta^{c_1-1}\tau^{c-1}(1-\xi)^{g-d-1}(1-\eta)^{g_1-d_1-1}(1-\zeta)^{f_1-c_1-1}(1-\tau)^{f-c-1} \tag{2.73}$$
$$\times F(a,b;e;x\xi\tau+y\eta\zeta)d\xi d\eta d\zeta d\tau,$$
$$\mathrm{Re}(g)>\mathrm{Re}(d)>0,\mathrm{Re}(g_1)>\mathrm{Re}(d_1)>0,\mathrm{Re}(f_1)>\mathrm{Re}(c_1)>0,\mathrm{Re}(f)>\mathrm{Re}(c)>0,$$

Proof. It is noted that each of the integral representations (2.2) to (2.83) can be proved directly by expressing the series definition of the involved special functionin each integrand and changing the order of the integral sign and the summation, and finally using the Beta function $B(a,b)$ defined by(2.1).

We conclude this paper by remarking that by assigning suitable special values to the coefficients in (2.2) to (2.83),we can derive integral representations for Appellfunctions of two variables F_1,F_2,F_3 and F_4 (see [10]). The detailsinvolved in these derivations are fairly straightforward and are being left as an exercisefor theinterested reader. Also, theEuler integral of the first kind (2.1) can be applied in order to establish other integral representations for more functions related to Kampe' de Fe'riet function of the fourth order.

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تمثيلات تكاملية من نوعا ويلرلد والذات صلة بدالة كامب ذي فرت من الرتبة الرابعة (III)

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الملخص

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