Some properties of the generalized Gamma and Beta functions

Fadhle B.F. Mohsen and Maisoon A. H. Kulib
Department of Mathematics, Faculty of Education , Aden University, Aden, Yemen

Abstract

In this paper, a new generalization of Gamma and Beta functions have been deduced Also for the generalized Beta function, an integral representation, a functional relation and a summation relation was given for the new generalized Gamma function established integral representation involving the product of two functions has been established , also, give a new generalization for the generalized and confluent hypergeometric functions.

Key word: Gamma function, Beta function, Hypergeometric functions, Confluent hypergeometric functions.

Introduction

In recent years, several extensions of the well known special functions have been considered by several authors (see. e. g., [1], [5-9] and [11,12]). In 1994, Chaudhry and Zubair [5] have introduced the following extension of Gamma function:

$$\Gamma_p(x)=\int_0^\infty t^{x-1}\exp\left(-t-\frac{p}{t}\right)dt, \qquad \left(\mathrm{Re}(p)>0,\mathrm{Re}(x)>0\right). \tag{1.1}$$

In 1997, Chaudhry et al. [6] presented the following extension of Euler's Beta function

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t(1-t)}\right)dt, \tag{1.2}$$

$$\left(\mathrm{Re}(p)>0,\mathrm{Re}(x)>0,\mathrm{Re}(y)>0\right).$$

Afterwards, Chaudhry et. al. [7] used $B_p(x,y)$ to extend the hypergeometric and confluent hypergeometric functions as follows;

$$F_p(a,b;c;z)=\sum_{n=0}^\infty (a)_n\frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.3}$$

$$\left(p\ge 0;|z|<1;\mathrm{Re}(c)>\mathrm{Re}(b)>0\right),$$

$$\phi_p(b;c;z)=\sum_{n=0}^\infty \frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}, \tag{1.4}$$

$$\left(p\ge 0;\mathrm{Re}(c)>\mathrm{Re}(b)>0\right).$$

Very recently Lee et al. [8] generalized the Beta, Gamma, hypergeometric and confluent hypergeometric function as

$$B_p^m(x,y)=B_p(x,y;m)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(-\frac{p}{t^m(1-t)^m}\right)dt, \tag{1.5}$$

$$\left(\text{Re}(p)>0,\text{Re}(x)>0,\text{Re}(y)>0,\text{Re}(\text{m})>0\right),$$

$$\Gamma_p^m(x)=\int_0^\infty t^{x-1}\exp\left(-t-\frac{p}{t^m}\right)dt,\quad \left(\text{Re}(p)>0,\text{Re}(x)>0,\text{Re}(\text{m})>0\right),$$

$$F_p^m(a,b;c;z)=F_p(a,b;c;z;m)=\sum_{n=0}^\infty (a)_n\frac{B_p^m(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

(1.6)

$$\left(\text{Re}(p)>0,\text{Re}(x)>0,\text{Re}(y)>0,\text{Re}(\text{m})>0\right);$$

and

$$\phi_p^m(b;c;z)=\phi_p(b;c;z;m)=\sum_{n=0}^\infty \frac{B_p^m(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!};$$

(1.7)

$$\left(p\geq 0;\text{Re}(c)>\text{Re}(b)>0,\text{Re}(m)>0\right),$$

Respectively and Ozergin et al. [11] generalized the Gamma, Beta, hypergeometric function and confluent hypergeometric function as

$$\Gamma_p^{(\alpha,\beta)}(x)=\int_0^\infty t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt,$$

(1.8)

$$\left(\text{Re}(\alpha)>0,\text{Re}(\beta)>0,\text{Re}(p)>0,\text{Re}(x)>0\right),$$

$$B_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{p}{t(1-t)}\right)dt,$$

(1.9)

$$\left(\text{Re}(p)>0,\text{Re}(x)>0,\text{Re}(y)>0,\text{Re}(\alpha)>0,\text{Re}(\beta)>0\right),$$

$$F_p^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^\infty (a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

$$\left(p\geq 0;|z|<1;\text{Re}(c)>0,\text{Re}(b)>0,\text{Re}(\alpha)>0,\text{Re}(\beta)>0\right),$$

(1.10)

and

$$\phi_p^{(\alpha,\beta)}(b;c;z)=\sum_{n=0}^\infty \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!};$$

(1.11)

$$\left(p\geq 0;\text{Re}(c)>\text{Re}(b)>0,\text{Re}(\alpha)>0,\text{Re}(\beta)>0\right),$$

respectively.

Parmar [12] gave a new generalized gamma and Beta functions

$$\Gamma_p^{(\alpha,\beta;m)}(x)=\int_0^\infty t^{x-1}{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t^m}\right)dt,$$

(1.12)

$$\left(\mathrm{Re}(m)>0,\mathrm{Re}(p)>0,\mathrm{Re}(x)>0,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0\right),$$

$$B_p^{(\alpha,\beta;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t^m(1-t)^m}\right)dt,$$

(1.13) $\left(\mathrm{Re}(m)>0,\mathrm{Re}(p)>0,\mathrm{Re}(x)>0,\mathrm{Re}(y)>0,,\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0\right).$

The following relations

$$\Gamma_p^{(\alpha,\beta;1)}(x)=\Gamma_p^{(\alpha,\beta)}(x),\qquad \Gamma_p^{(\alpha,\alpha;1)}(x)=\Gamma_p(x),\qquad \Gamma_0^{(\alpha,\alpha)}(x)=\Gamma(x)$$

$$B_p^{(\alpha,\beta;1)}(x,y)=B_p^{(\alpha,\beta)}(x,y),\quad B_p^{(\alpha,\alpha;\mathrm{m})}(x,y)=B_p^m(x,y),\qquad B_0^{(\alpha,\alpha;1)}(x,y)=B(x,y).$$

Generalized Gamma and Beta functions

In this section generalized Gamma and Beta functions are defined as follow.

$$\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(x)=\int_0^\infty t^{x-1}\,{}_rF_r\left(\alpha_1,\alpha_2,\ldots,\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-t-\frac{p}{t^m}\right)dt,$$

(2.1)

$$\left(\mathrm{Re}(\alpha_i)>0,\mathrm{Re}(\beta_i)>0,\quad i=1,2,\ldots,r,\mathrm{Re}(p)>0,\mathrm{Re}(x)>0,\mathrm{Re}(m)>0\right),$$

$$B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,\ldots,\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;\frac{-p}{t^m(1-t)^m}\right)dt,$$

(2.2)

$$\left(\mathrm{Re}(\alpha_i)>0,\mathrm{Re}(\beta_i)>0,\quad i=1,2,\ldots,r,\mathrm{Re}(p)>0,\mathrm{Re}(x)>0,\mathrm{Re(y)}>0,\mathrm{Re}(m)>0\right).$$

$${}_rF_r\left(\alpha_1,\alpha_2,\ldots,\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-u-\frac{p}{u}\right)==\prod_{i=1}^r\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^1\ldots\int_0^1\exp\left[\left(-u-\frac{p}{u}\right)t_1\ldots t_r\right]t_1^{\alpha_1-1}(1-t_1)^{\beta_1-\alpha_1-1}\ldots t_r^{\alpha_i-1}(1-t_r)^{\beta_r-\alpha_r-1}dt_1\ldots dt_r,$$

(2.3)

$$\left(p\ge 0;\mathrm{Re}(\beta_i)>\mathrm{Re}(\alpha_i)>0,i=1,2,\ldots,r\right),$$

It is obvious, putting $r=1,\alpha=\beta$ and $m=1$ in (2.1) and (2.2), we get the results given by (1.1) and (1.2).

Putting $r=1,m=1$ in (2.1) and (2.2), we get the results (3) and (4) given by [11].

Also ,in (2.1) and (2.2) if we put $r=1,$,we get the results (1.18) and (1.19) given [12].

Main results

Theorem 1. For the generalized Gamma function $\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(s)$,we have

$$\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(s)$$

$$=\prod_{i=1}^r\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\int_0^1\ldots\int_0^1\Gamma_{p(\mu_1,\mu_2,\ldots,\mu_r)^{2m}}(s)\mu_1^{\alpha_1-s-1}(1-\mu_1)^{\beta_1-\alpha_1-1}\ldots\mu_r^{\alpha_r-s-1}(1-\mu_r)^{\beta_r-\alpha_r-1}d\mu_1\ldots d\mu_r.$$

(3.1)

Proof. Using the integral representation of ${}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r,\beta_1,\beta_2,...,\beta_r;-u-\frac{p}{u^m}\right)$ in (2.3) ,we have

$$\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r,\beta_1,\beta_2,...,\beta_r;m)}$$

$$=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^\infty\int_0^1...\int_0^1 u^{s-1}\exp\left[\left(-ut_1...t_r-\frac{p(t_1...t_r)}{u^m}\right)\right]t_1^{\alpha_1-1}(1-t_1)^{\beta_1-\alpha_1-1}...t_r^{\alpha_r-1}(1-t_r)^{\beta_r-\alpha_r-1}dt_1...dt_r du,$$

Now using a one –to-one transformation(except possibly at the boundaries and maps the region onto itself) $\nu=ut_1t_2..t_r,\mu_i=t_i,i=1,2,...,r$, in the above equality and considering that the Jacobian of the transformation is $J=\frac{1}{\mu}$, and

$$\nu=ut_1t_2...t_r$$

$$\mu_1=t_1\Rightarrow d\mu_1=dt_1$$

$$\mu_2=t_2\Rightarrow d\mu_2=dt_2$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

$$\mu_r=t_r\Rightarrow d\mu_r=dt_r$$

$$u=\frac{\nu}{t_1t_2...t_r}=\frac{\nu}{\mu_1\mu_2...\mu_r}\Rightarrow du=(\mu_1\mu_2...\mu_r)^{-1}d\nu,$$

we get

$$\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r,\beta_1,\beta_2,...,\beta_r;m)}(s)$$

$$=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^\infty\int_0^1...\int_0^1\left(\frac{\nu}{\mu_1...\mu_r}\right)^{s-1}\exp\left[\left(-\nu-\frac{p(\mu_1...\mu_r)^{m+1}}{\nu^m}\right)\right]$$

$$\mu_1^{\alpha_1-1}(1-\mu_1)^{\beta_1-\alpha_1-1}...\mu_r^{\alpha_r-1}(1-\mu_r)^{\beta_r-\alpha_r-1}d\mu_1...d\mu_r\frac{d\nu}{\mu_1...\mu_r},$$

$$=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^\infty\int_0^1...\int_0^1(\nu)^{s-1}(\mu_1...\mu_r)^{-s}\exp\left[\left(-\nu-\frac{p(\mu_1...\mu_r)^m}{\nu^m}\right)\right]$$

$$\mu_1^{\alpha_1-1}(1-\mu_1)^{\beta_1-\alpha_1-1}...\mu_r^{\alpha_r-1}(1-\mu_r)^{\beta_r-\alpha_r-1}d\mu_1...d\mu_r d\nu,$$

$$=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^1\cdots\int_0^1 \nu^{s-1}\mu_1^{\alpha_1-s-1}(1-\mu_1)^{\beta_1-\alpha_1-1}\cdots\mu_r^{\alpha_r-s-1}(1-\mu_r)^{\beta_r-\alpha_r-1}d\mu_1\ldots d\mu_r$$

$$\times\int_0^\infty \nu^{s-1}\exp\left[\left(-\nu-\frac{p(\mu_1\ldots\mu_r)^{m+1}}{\nu^m}\right)\right]d\nu,$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(s)=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}$$

$$\times\int_0^1\cdots\int_0^1\Gamma_{p(\mu_1\mu_2\ldots\mu_r)^{m+1}}(s)\,\mu_1^{\alpha_1-s-1}(1-\mu_1)^{\beta_1-\alpha_1-1}\cdots\mu_r^{\alpha_r-s-1}(1-\mu_r)^{\beta_r-\alpha_r-1}d\mu_1\ldots d\mu_r$$

which is the required result.

Remark 1. In Theorem 1, choosing $p=0$, we get

$$\Gamma_0^{(\alpha_1,\alpha_2,\ldots\alpha_r,\beta_1,\beta_2,\ldots,\beta_r;m)}(s)=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)\Gamma(s)}{\Gamma(\alpha_i)\Gamma(\beta_i-\alpha_i)}B(\alpha_i-s,\beta_i-\alpha_i)$$

Also choosing $r=1$,we get

$$\Gamma_p^{(\alpha,\beta;m)}(s)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\Gamma_{p(\mu)^{m+1}}(s)\,\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.$$

which is the result Theorem 2.1 in [12].
Also in Theorem 1, puting $\mathrm{m}=1=r$, we get

$$\Gamma_p^{(\alpha,\beta;1)}(s)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\Gamma_{p\mu^2}(s)\,\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$

which is the result Theorem 2.1 in [11].
Also in Theorem 1, if we choose $m=r=1,\ p=0$, we get (see [12,p.38], [11,p.4603]).

$$\Gamma_0^{(\alpha,\beta;1)}(s)=\frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1\Gamma(s)\,\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$

$$=\frac{\Gamma(\beta)\Gamma(s)\Gamma(\alpha-s)}{\Gamma(\alpha)\Gamma(\beta-s)}$$

(3.2)

Integral representation of generalized Beta function:

Theorem 1. For the generalized Beta function, we have the following integral representations:

$$B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x,y)=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_rF_r\left(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-\sec^{2m}\theta\csc^{2m}\theta\right)d\theta,$$

(4.1)

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y)=\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,{}_rF_r\left(\alpha_1,\alpha_2,...,\alpha_r;\beta_1,\beta_2,...,\beta_r;-p\left(2+u+\frac{1}{u}\right)^m\right)du,$$

(4.2)

Proof. letting $t=\cos^2\theta$ in (2.2), we get

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...,\alpha_r;\beta_1,\beta_2,...,\beta_r;\frac{-p}{t^m(1-t)^m}\right)dt$$

$$=2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_rF_r\left(\alpha_1,\alpha_2,...,\alpha_r;\beta_1,\beta_2,...,\beta_r;-p\sec^{2m}\theta\csc^{2m}\theta\right)d\theta$$

On the other hand, letting $t=\dfrac{u}{1+u}$ in (2.2) , we get

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$=\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-p\left(2+u+\frac{1}{u}\right)^m\right)du,$$

which ends the proof.

Remark 1. When $r=1$, in Theorem 1, we get the result Theorem 2.11 in [12].

Theorem 2. For the generalized Beta function, we have the following functional relation:

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y+1)+B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x+1,y)=B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y)$$

(4.3)

Proof. Direct calculations yield

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y+1)+B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x+1,y)$$

$$=\int_0^1 t^x(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$+\int_0^1 t^{x-1}(1-t)^y\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$=B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,y).$$

which completes the proof.

Remark 2. In Theorem 2, putting $r=1$, it will be reduced to the result Theorem 2.3 in [12].

Theorem 3. For $\mathrm{Re}(p)>0, \mathrm{Re}(m)>0, \mathrm{Re}(\alpha_i)>0, \mathrm{Re}(\beta_i)>0$

$$B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x,y)=\sum_{n=0}^{\infty}B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x+n,y+1)$$

(4.4)

Proof. Replacing $(1-t)^{y-1}$ in (2.2) by its series representation

$$(1-t)^{y-1}=(1-t)^{y}\sum_{n=0}^{\infty}t^{n}$$

we obtain

$$B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x,y)=\int_0^1(1-t)^{y}\sum_{n=0}^{\infty}t^{x+n-1}{}_rF_r\left(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

Interchanging the order of integration and summation, we have

$$B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x,y)=\sum_{n=0}^{\infty}\int_0^1 t^{x+n-1}(1-t)^{y}{}_rF_r\left(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$=\sum_{n=0}^{\infty}B_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x+n,y+1)$$

which complete the proof.

Remark 3. In Theorem 3, putting $r=1$,we get the result Theorem 2.5 in [12].

Theorem 4. For the product of two generalized Gamma functions, we have the following integral representation:

$$\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(x)\Gamma_p^{(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;m)}(y)=4\int_0^{\pi/2}\int_0^{\infty}r_1^{2(x-y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta$$

$${}_rF_r\left(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-r_1^{2m}\cos^{2m}\theta-\frac{p}{r_1^{2m}\cos^{2m}\theta}\right)$$

$$\times{}_rF_r\left(\alpha_1,\alpha_2,\ldots\alpha_r;\beta_1,\beta_2,\ldots,\beta_r;-r_1^{2m}\sin^{2m}\theta-\frac{p}{r_1^{2m}\sin^{2m}\theta}\right)dr_1d\theta$$

(4.5)

Proof. Substituting $t=\eta^2$ in (2.1) ,we get

$$\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x)=2\int_0^\infty \eta^{2x-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\eta^2-\frac{p}{\eta^{2m}}\right)d\eta.$$

Therefore

$$\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x)\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(y)$$

$$=4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\eta^2-\frac{p}{\eta^{2m}}\right){}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\xi^2-\frac{p}{\xi^{2m}}\right)d\eta\, d\xi.$$

Letting $\eta=r_1\cos\theta$ and $\xi=r_1\sin\theta$ in the above equality,

$$\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x)\Gamma_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(y)$$

$$=4\int_0^{\pi/2}\int_0^\infty r_1^{2(x-y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-r_1^{2m}\cos^{2m}\theta-\frac{p}{r_1^{2m}\cos^{2m}\theta}\right)$$

$$\times{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-r_1^{2m}\sin^{2m}\theta-\frac{p}{r_1^{2m}\sin^{2m}\theta}\right)dr_1d\theta$$

which complete the proof of the Theorem.

Remark 4. Putting $r=1$, in Theorem 4, we get Theorem 2.9 in [12] and putting $r=m=1$, we get the result theorem 2.6 in [11].

Remark 5. Putting $p=0$ and $r=m=1$ in (3.6), we get the classical relation between the Gamma and Beta functions :

$$\Gamma(x)\Gamma(y)=\Gamma(x+y)B(x,y)$$

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 5. For the new generalized beta function, we have the following summation relation:

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,1-y)=\sum_{n=0}^\infty\frac{(y)_n}{n!}B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x+n,1)$$

(4.5)

$$\operatorname{Re}(p)>0,\operatorname{Re}(m)>0$$

Proof. From the definition of the generalized Beta function , we get

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,1-y)=\int_0^1 t^{x-1}(1-t)^{-y}\,{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

Using the following binomial series expansion

$$(1-t)^{-y}=\sum_{n=0}^\infty (y)_n\frac{t^n}{n!},\quad |t|<1.$$

we obtain

$$B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x,1-y)=\int_0^1\sum_{n=0}^{\infty}\frac{(y)_n}{n!}t^{x+n-1}{}_rF_r\left(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;-\frac{p}{t^m(1-t)^m}\right)dt$$

$$=\sum_{n=0}^{\infty}\frac{(y)_n}{n!}B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(x+n,1)$$

which is the proof.

Remark 6. In Theorem 5, the case $r=1$,we get the result Theorem2.4 [12], and choosing $r=m=1$, we get the result Theorem 2.7 in [11].

1. Generalized Gauss and confluent hypergeometric function

In this section the result (2.2) is used in order to introduce the following generalized hypergeometric and confluent hypergeometric functions

$$F_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},$$

(5.1)

and

$$\phi_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(b;c;z)=\sum_{n=0}^{\infty}\frac{B_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}.$$

(5.2)

We call the $F_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(a,b;c;z)$ by the generalized Gauss hypergeometric function and $\phi_p^{(\alpha_1,\alpha_2,...\alpha_r;\beta_1,\beta_2,...,\beta_r;m)}(b;c;z)$ by the generalized confluent hypergeometric function

Observe if we put $r=1;\ r=1,\alpha=\beta;$ and $r=1,m=1,p=0$ in (5.1) ,we get the generalized hypergeometric functions $F_p^{(\alpha,\beta;m)}(a,b;c;z), F_p^m(a,b;c;z)$ and ${}_2F_1(a,b;c;z)$, respectively (see. [6]).

Also by the same procedure above we get the generalized confluent hypergeometric functions $\phi_p^{(\alpha,\beta;m)}(b;c;z),\phi_p^m(b;c;z)$ and $\phi(b;c;z)$.

Reference

1. Agarwal, P., 2014. Certain properties of the generalized Gauss hypergeometric function, Appl.Math.Inf.sci.8.no.5.2315-2320.
2. Agarwal, P., choi, J., and paris, R. B., 2015. Extended Riemann- Liouville fractional derivative operator and its applications , J. Nonlinear Sci. Appl. 8, 451-466.
3. Andrews, G.E., Askey, R., Roy, R., 1999. Special Functions, Cambridge university press, Cambridge.
4. Chand**,** M., Agarwal, P., Choi, J., 2016. Note on generating relations associated with the generalized Gauss hypergeometric function, Appl.Math.Sci.10.no.35.1747-1754.
5. Chaudhry, M. A., Zubair, S. M., 1994. Generalized incomplete gamma functions with applications, J. comput, Appl. 55, 99-124.
6. Chaudhry, M. A., Qadir, A., Rafique, M., Zubair, S. M., 1997. Extension of Euler's beta function, J. comput, Appl, Math, 78,19-32.

7. Chaudhry, M. A., Qadir, A., Srivastava, H. M., Paris, R. B., 2004. Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159, 589-602.
8. Lee, D. M., Rathic, A. K., Parmar, R. K., and Kim, Y. S., 2011. Generalization of extended Beta function, hypergeometric and confluent hypergeometric functions, Honam Math.J.33.no. 2. 187-206.
9. Luo, M. J., Milovanovic, G. V., and Agawal, P., 2014. Some results on the extended beta and extended hypergeometric functions, Appl. Math. Comput. 248, 631-651.
10. Ozergin, E., 2011. Some properties of hypergeometric functions.Ph. D. thesis, Eastern Mediterranean University, North Cyprus. February
11. Ozergin, E., Ozarslan, M. A., and Altin, A., 2011. Extension of gamma, beta and hypergeometric functions, J. comput. Appl. Math. 235, no. 16, 4601-4610.
12. Parmar, R. K., 2013. Anew generalization of Gamma, Beta, Hypergeometric and Confluent hypergeometric functions, Le Matematiche, Vol. LXVIII-Fasc. II, pp. 33-52.
13. Srivastava, H.M., Monocha, 1984. H.L., A Treaties on Generating functions, Halsted press (Ellis Horwood Limited, Chichester), John wiley and Sons, New York, Chichester, Brisbane and Toronto..

بعض الخواص لتعميم دالة جاما وبيتا

فضل بن فضل وميسون أحمد كليب
قسم الرياضيات، كلية التربية، جامعة عدن

الملخص

في هذه الورقة البحثية تم استحداث تعميم جديد لدالة جاما و دالة بيتا أيضا بالنسبة لدالة بيتا المعممة أوجدنا لتمثيل التكاملي لها والصيغة التنفيذية والعلاقة الجمعية بالنسبة لدالة جاما المعممة أوجدنا التمثيل التكاملي كحاصل ضرب دالتي.

أيضا قمنا بعمل تعميم جديد للدالة فوق الهندسية والدالة فوق الهندسية المندمجة.

الكلمات المفتاحية: دالة جاما، دالة بيتا، الدالة فوق الهندسية والدالة فوق الهندسية المندمجة.