Integral transforms and Laguerre-Gould Hopper polynomials

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Abstract

In this article, new families of generalized special polynomials by combining the properties of exponential operators with suitable integral transforms have been introduced. Certain properties of these special polynomials are established.

Key words: Laguerre-Gould Hopper polynomials, Special polynomials, Integral representation.
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Introduction

The combined use of integral transforms and special polynomials provides a powerful tool to deal with fractional derivatives and integrals. Dattoli et al. [2,5] have shown that by combining the properties of exponential operators and suitable integral representations one can find an efficient way of treating fractional operators. They introduced new families of special polynomials starting from a suitable operational definition. Very recently, Al-Gonah [1] used the same method to introduce and study a new form of special polynomials associated with Laguerre-Gould Hopper polynomials. In order to take further advantage from the various forms of operational representations of the Laguerre-Gould Hopper polynomials, we extend the procedure outlined in [2,5] to introduce a new generalized forms of the special polynomials associated with the generalized Laguerre polynomials and Laguerre-Gould Hopper polynomials.

For this purpose, we recall that the 2-variable generalized Laguerre polynomials (2VgLP) ${}_mL_n\,(x,y)$ are specified by the generating function [3; p. 214]

$$exp(yt)C_0(-xt^m) = \sum_{n=o}^{\infty} {}_mL_n\,(x,y)\ \frac{t^n}{n!}, \tag{1.1}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as [11]:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!\,(n+r)!}. \tag{1.2}$$

The 2VgLP ${}_mL_n\,(x,y)$ are defined by the series definition [3; p. 213]

$${}_mL_n\,(x,y) = n! \sum_{r=0}^{\left[\frac{n}{m}\right]} \frac{x^r y^{n-mr}}{(r!)^2 (n-mr)!} \tag{1.3}$$

and by the following operational definition:

$${}_mL_n\,(x,y) = \exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\{y^n\}, \tag{1.4}$$

where D_x^{-1} denotes the inverse of the derivative operator $D_x := \frac{\partial}{\partial x}$ and is defined in such a way that

$$D_x^{-n}\{f(x)\} = \frac{1}{(n-1)!}\int_0^x (x-\xi)^{n-1} f(\xi)d\xi\,, \tag{1.5}$$

so that for $f(x) = 1$, we have

$$D_x^{-n}\{1\} = \frac{x^n}{n!}. \tag{1.6}$$

Also, the 2VgLP ${}_mL_n\,(x,y)$ satisfy the differential equation

$$\left(m\frac{\partial^m}{\partial y^m}+y\frac{\partial^2}{\partial x\,\partial y}-n\frac{\partial}{\partial x}\right){}_mL_n\,(x,y)=0\,. \tag{1.7}$$

Next, the higher-order Hermite polynomials or the Gould-Hopper polynomials (GHP) $H_n^{(s)}(x,y)$, are specified by the generating function [8; p.58]

$$exp(xt+yt^s)=\sum_{n=o}^{\infty}H_n^{(s)}(x,y)\;\frac{t^n}{n!}\;\;. \tag{1.8}$$

The GHP $H_n^{(s)}(x,y)$ are defined by the series definition [8; p.58] (see also [4])

$$H_n^{(s)}(x,y)=n!\sum_{k=0}^{\left[\frac{n}{s}\right]}\frac{y^k x^{n-sk}}{k!\,(n-sk)!} \tag{1.9}$$

and by the following operational definition:

$$H_n^{(s)}(x,y)=exp\left(y\frac{\partial^s}{\partial x^s}\right)\{x^n\}\,. \tag{1.10}$$

We note the following links between 2VgLP ${}_mL_n\,(x,y\,)$ and GHP $H_n^{(m)}(x,y)$:

$${}_mL_n\,(x,y\,)=H_n^{(m)}\,(y,D_x^{-1}\,) \tag{1.11a}$$

and

$$H_n^{(m)}(x,y)={}_mL_n\,\big(yD_y y,x\,\big). \tag{1.11b}$$

Further, the Laguerre-Gould Hopper polynomials (LGHP) ${}_LH\,{}_n^{(m,s)}(x,y,z)$ are specified by the generating function [9; p.9933]

$$exp(y\mathrm{t}+\mathrm{z}\mathrm{t}^s)C_0(-xt^m)=\sum_{n=o}^{\infty}{}_LH\,{}_n^{(m,s)}(x,y,z)\;\frac{t^n}{n!}, \tag{1.12}$$

And are defined by the following series definition:

$${}_LH_n^{(m,s)}\,(x,y,z)=n!\sum_{k=0}^{\left[\frac{n}{s}\right]}\frac{z^k\,{}_mL_{n-sk}\,(x,y)}{k!\,(n-sk)!}\;\;. \tag{1.13}$$

The LGHP ${}_LH_n^{(m,s)}\,(\mathrm{x},y,z)$ satisfy the differential equation

$$\left(m\frac{\partial^m}{\partial y^m}+sz\frac{\partial^{\,s+1}}{\partial x\,\partial y^s}+y\frac{\partial^2}{\partial x\,\partial y}-n\frac{\partial}{\partial x}\right){}_LH_n^{(m,s)}\,(x,y,z)=0 \tag{1.14}$$

and the monomiality recurrence

$$\left(mD_x^{-1}\frac{\partial^{m-1}}{\partial y^{m-1}}+sz\frac{\partial^{\,s-1}}{\partial y^{s-1}}+y\right)\;\;{}_LH_n^{(m,s)}\,(x,y,z)={}_LH_{n+1}^{(m,s)}\,(x,y,,z). \tag{1.15}$$

Also, the polynomials ${}_LH_n^{(m,s)}\,(\mathrm{x},y,z)$ are defined by means of the following operational representations:

$$exp\left(z\frac{\partial^s}{\partial y^s}\right)\{{}_mL_n\,(x,y)\}={}_LH_n^{(m,s)}\,(x,y,z), \tag{1.16}$$

$$\exp\!\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\left\{H_n^{(s)}\,(y,z)\right\}={}_LH_n^{(m,s)}\,(x,y,z) \tag{1.17}$$

and

$$\exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}+z\frac{\partial^s}{\partial y^s}\right)\{\mathrm{y}^n\}={}_LH_n^{(m,s)}\,(x,y,z). \tag{1.18}$$

Integral transforms and special polynomials

It is well known that one of the starting point of the theory of fractional operators, i.e. operators raised to a fractional power, is the Euler's integral [11; p.218]:

$$\alpha^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-\alpha t} t^{v-1} dt. \tag{2.1}$$

Multiplying both sides of equation (2.1) by $f(y)$ and replacing α by $\alpha - D_x^{-1} \frac{\partial^m}{\partial y^m}$ in the resultant equation, we find

$$\left(\alpha - D_x^{-1} \frac{\partial^m}{\partial y^m}\right)^{-v} f(y) = \frac{1}{\Gamma(v)} \int_0^\infty e^{-\alpha t} t^{v-1} exp\left(D_x^{-1} t \frac{\partial^m}{\partial y^m}\right) f(y) dt. \tag{2.2}$$

Now, let us consider the simpler case $f(y) = y^n$ in equation (2.2), we get

$$\left(\alpha - D_x^{-1} \frac{\partial^m}{\partial y^m}\right)^{-v} \{y^n\} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-\alpha t} t^{v-1} exp\left(D_x^{-1} t \frac{\partial^m}{\partial y^m}\right) \{y^n\} dt. \tag{2.3}$$

Using operational formula (1.4) in the r.h.s. of the above equation, we obtain

$$\left(\alpha - D_x^{-1} \frac{\partial^m}{\partial y^m}\right)^{-v} \{y^n\} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-\alpha t} t^{v-1} {}_mL_n (xt, y) dt. \tag{2.4}$$

The transform on the r.h.s. of equation (2.4) defines a new special polynomials, denoted by ${}_mL_{n,v} (x, y; \alpha)$, *i.e.*

$${}_mL_{n,v} (x, y; \alpha) = \frac{1}{\Gamma(v)} \int_0^\infty e^{-\alpha t} t^{v-1} {}_mL_n (xt, y) dt. \tag{2.5}$$

From equations (2.4) and (2.5), we get the following operational definition:

$$\left(\alpha - D_x^{-1} \frac{\partial^m}{\partial y^m}\right)^{-v} \{y^n\} = {}_mL_{n,v} (x, y; \alpha). \tag{2.6}$$

Using definition (1.3) in the r.h.s. of equation (2.5), we find

$${}_mL_{n,v} (x, y; \alpha) = \frac{n!}{\Gamma(v)} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{x^k y^{n-mk}}{(k!)^2 (n - mk)!} \int_0^\infty e^{-\alpha t} t^{v-1+k} dt, \tag{2.7}$$

which on using equation (2.1) in the r.h.s., gives the following series definition of ${}_mL_{n,v} (x, y; \alpha)$:

$${}_mL_{n,v} (x, y; \alpha) = \frac{n!}{\alpha^v} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(v)_k x^k y^{n-mk}}{\alpha^k (k!)^2 (n - mk)!}. \tag{2.8}$$

In particular, we note that

$${}_mL_{n,1} (x, y; 1) = H_n^{(m)} (y, x) \tag{2.9}$$

and

$${}_mL_{n,1} (D_x^{-1}, y; 1) = {}_mL_n (x, y). \tag{2.10}$$

Also, we note the following relation link:

$${}_mL_{n,v} (x, y; \alpha) = {}_vH_n^{(m)} (y, D_x^{-1}; \alpha). \tag{2.11}$$

where ${}_vH_n^{(m)} (x, y; \alpha)$, denotes the special polynomials defined by Al - Gonah [1; p.329]

$${}_vH_n^{(m)} (x, y; \alpha) = \frac{n!}{\alpha^v} \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(v)_k\, y^k x^{n-mk}}{\alpha^k k! (n - mk)!}. \tag{2.12}$$

Next, from definition (2.8) and using formula [10]

$$D_x^n x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \qquad (m \geq 0)\,, \tag{2.13}$$

it follows that the special polynomials ${}_mL_{n,v}\,(x,y;\alpha)$ satisfy the following differential relations:

$$\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^s {}_mL_{n,v}\,(x,y;\alpha) = \frac{(v)_s\, n!}{(n-ms)!}\, {}_mL_{n-ms,v+s}(x,y;\alpha), \tag{2.14}$$

$$\frac{\partial^s}{\partial y^s}\, {}_mL_{n,v}\,(x,y;\alpha) = \frac{n!}{(n-s)!}\, {}_mL_{n-s,v}\,(x,y;\alpha) \tag{2.15}$$

and

$$\frac{\partial^s}{\partial \alpha^s}\, {}_mL_{n,v}\,(x,y;\alpha) = (-1)^s (v)_s\, {}_mL_{n,v+s}\,(x,y;\alpha). \tag{2.16}$$

From equations (2.14) and (2.15), we have

$$\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^s {}_mL_{n,v}\,(x,y;\alpha) = (v)_s \frac{\partial^{ms}}{\partial y^{ms}}\, {}_mL_{n,v+s}\,(x,y;\alpha). \tag{2.17}$$

Consequently from equations (2.16) and (2.17), we get

$$\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^s {}_mL_{n,v}\,(x,y;\alpha) = (-1)^s \frac{\partial^{ms+s}}{\partial y^{ms} \partial \alpha^s}\, {}_mL_{n,v}\,(x,y;\alpha). \tag{2.18}$$

Further, The integral representation (2.5) can be used to establish some properties for the special polynomials ${}_mL_{n,v}\,(x,y;\alpha)$ with the help of the corresponding properties of the ${}_mL_n\,(x,y)$. For instance, multiplying both sides of equation (2.5) by $\frac{\xi^n}{n!}$, summing up over *n* and using generating function (1.1) in the r.h.s. of the resultant equation, we get

$$\sum_{n=0}^{\infty} {}_mL_{n,v}\,(x,y;\alpha) \frac{\xi^n}{n!} = \frac{1}{\Gamma(v)} \int_0^{\infty} e^{-\alpha t} t^{v-1} exp(y\xi) C_0(-xt\,\xi^m) dt. \tag{2.19}$$

Now, using equations (1.2) and (2.1), respectively in the r.h.s. of the above equation, we get the following generating function for ${}_mL_{n,v}\,(x,y;\alpha)$:

$$\sum_{n=0}^{\infty} {}_mL_{n,v}\,(x,y;\alpha) \frac{\xi^n}{n!} = \frac{exp(y\xi)}{\alpha^v}\, {}_1F_1\left(v; 1; \frac{x\xi^m}{\alpha}\right), \tag{2.20}$$

where ${}_1F_1(a,b;x)$ denotes the confluent hypergeometric function defined by Srivastava & Man [11]

$${}_1F_1(a,b;x) = \sum_{r=0}^{\infty} \frac{(a)_r\, x^r}{(b)_r\, r!}. \tag{2.21}$$

Also, replacing *x* by xt in differential equation (1.7) and multiplying by $\frac{1}{\Gamma(v)}\, e^{-\alpha t} t^v$ and integrating the resultant equation with respect to *t* between the limits 0 to ∞, we have

$$m \frac{\partial^m}{\partial y^m} \frac{1}{\Gamma(v)} \int_0^{\infty} e^{-\alpha t} t^v\, {}_mL_n\,(xt,y) dt$$
$$+ \left(y \frac{\partial^2}{\partial x\, \partial y} - n \frac{\partial}{\partial x}\right) \frac{1}{\Gamma(v)} \int_0^{\infty} e^{-\alpha t} t^{v-1}\, {}_mL_n\,(xt,y) dt = 0, \tag{2.22}$$

with the help of the relation

$$\frac{\partial}{\partial (xt)} = \frac{1}{t} \frac{\partial}{\partial x}. \tag{2.23}$$

Using equations (2.5) in the above equation, we get

$$v\, m \frac{\partial^m}{\partial y^m}\, {}_mL_{n,v+1}\,(x,y;\alpha) + \left(y \frac{\partial^2}{\partial x\, \partial y} - n \frac{\partial}{\partial x}\right) {}_mL_{n,v}\,(x,y;\alpha) = 0, \tag{2.24}$$

which on using relation (2.16) (for *s=1*) gives the following differential equation for ${}_mL_{n,v}\,(x,y;\alpha)$:

$$\left(m\frac{\partial^{m+1}}{\partial y^m\partial\alpha}-y\frac{\partial^{\,2}}{\partial x\,\partial y}+n\frac{\partial}{\partial x}\right){}_mL_{n,v}\,(x,y;\alpha)=0. \tag{2.25}$$

Furthermore, from definitions (2.8), (2.12) and by using relation (2.1) (for $a=1$) and the Hankel formula [7]

$$\frac{1}{\Gamma(z)}=\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{-t}t^{-z}dz, \tag{2.26}$$

we get the following integral relations:

$${}_vH_n^{(m)}\,(x,y;\alpha)=\int_0^{\infty} e^{-t}\,{}_mL_n\,(yt,x;\alpha)dt \tag{2.27}$$

and

$${}_mL_{n,v}\,(x,y;\alpha)=\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{t}t^{-1}\,{}_vH_n^{(m)}\,(y,xt^{-1})d \tag{2.28}$$

respectively.

Finally, using the identity [11]

$$(-n)_{mk}=\frac{(-1)^{mk}\,n!}{(n-mk)!}\qquad 0\le k\le\left[\frac{n}{m}\right], \tag{2.29}$$

in the r.h.s. of definition (2.8), we get the following hypergeometric representation for the special polynomials ${}_mL_{n,v}\,(x,y;\alpha)$:

$${}_mL_{n,v}\,(x,y;\alpha)=\frac{y^n}{\alpha^v}\,{}_{m+1}F_1\left[\Delta(m,-n),v;1;\left(-\frac{m}{y}\right)^m\frac{x}{\alpha}\right], \tag{2.30}$$

where ${}_pF_q[.]$ denotes the generalized hypergeometric function defined byRainville [10] and $\Delta(m,-n)$ denotes the array of m parameters $\frac{-n}{m},\frac{-n+1}{m},\frac{-n+2}{m},\ldots,\frac{-n+m-1}{m},\quad m\ge 1.$

In the next section, the special polynomials ${}_mL_{n,v}\,(x,y;\alpha)$ will be used to introduce another new generalized special polynomials.

New generalized special polynomials

It is now evident that the above procedure can be extended to any family of special polynomials. Here, we explore the possibility of introducing a new generalized special polynomials associated with the LGHP ${}_LH_n^{(m,s)}\,(x,y,z)$. For this purpose, multiplying both sides of equation (2.1) by $f(y,z)$ and replacing *a* by $\alpha-\,D_x^{-1}\frac{\partial^m}{\partial y^m}$ in the resultant equation, we find

$$\left(\alpha-\,D_x^{-1}\frac{\partial^m}{\partial y^m}\right)^{-v}f(y,z)=\frac{1}{\Gamma(v)}\int_0^{\infty} e^{-\alpha t}t^{v-1}exp\left(tD_x^{-1}\frac{\partial^m}{\partial y^m}\right)f(y,z)dt, \tag{3.1}$$

Let us consider the case $f(y,z)=H_n^{(s)}\,(y,z)$ in equation (3.1), we get

$$\left(\alpha-\,D_x^{-1}\frac{\partial^m}{\partial y^m}\right)^{-v}H_n^{(s)}\,(y,z)=\frac{1}{\Gamma(v)}\int_0^{\infty} e^{-\alpha t}t^{v-1}exp\left(tD_x^{-1}\frac{\partial^m}{\partial y^m}\right)H_n^{(s)}\,(y,z)dt. \tag{3.2}$$

Using operational representation (1.17) in the r.h.s. of the above equation, we find

$$\left(\alpha-\,D_x^{-1}\frac{\partial^m}{\partial y^m}\right)^{-v}H_n^{(s)}\,(y,z)=\frac{1}{\Gamma(v)}\int_0^{\infty} e^{-\alpha t}t^{v-1}\,{}_LH_n^{(m,s)}\,(xt,y,z)dt, \tag{3.3}$$

the transform on the r.h.s. of equation (3.3) defines a new generalized special polynomials, denoted by ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$; *i.e.*

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}\,{}_LH_n^{(m,s)}(xt,y,z)dt. \quad (3.4)$$

From equations (3.3) and (3.4), we get the following operational definition:

$$\left(\alpha - D_x^{-1}\frac{\partial^m}{\partial y^m}\right)^{-v}\left\{H_n^{(s)}(y,z)\right\} = {}_{vL}H_n^{(m,s)}(x,y,z;\alpha). \quad (3.5)$$

Using equation (1.13) in the r.h.s. of equation (3.4), we obtain

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = n!\sum_{k=0}^{\left[\frac{n}{s}\right]}\frac{z^k}{k!\,(n-sk)!}\,\frac{1}{\Gamma(v)}\int_0^\infty e^{-\alpha t}t^{v-1}\,{}_mL_{n-sk}(xt,y)dt, \quad (3.6)$$

which on using definition (2.5) in the r.h.s. gives the definition

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = n!\sum_{k=0}^{\left[\frac{n}{s}\right]}\frac{z^k\,{}_mL_{n-sk,v}(x,y;\alpha)}{k!\,(n-sk)!}. \quad (3.7)$$

Using definition (2.8) in the r.h.s. of the above equation, we get the following series definition for ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{n!}{\alpha^v}\sum_{k=0}^{\left[\frac{n}{s}\right]}\sum_{r=0}^{\left[\frac{n-sk}{m}\right]}\frac{(v)_r x^r z^k y^{n-sk-mr}}{\alpha^r k!\,(r!)^2(n-sk-mr)!}. \quad (3.8)$$

Now, using series definition (1.9) of the GHP $H_n^{(s)}(y,z)$ in equation (3.8), we have a second form of definition for ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ as:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{n!}{\alpha^v}\sum_{r=0}^{\left[\frac{n}{m}\right]}\frac{(v)_r x^r H_{n-mr}^{(s)}(y,z)}{\alpha^r (r!)^2(n-mr)!}, \quad (3.9)$$

which on using definitions (1.10) and (2.8) respectively in the r.h.s. gives the following second form of operational definition for ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = exp\left(z\frac{\partial^s}{\partial y^s}\right){}_mL_{n,v}(x,y;\alpha). \quad (3.10)$$

Also, using operational definition (2.6) in the r.h.s. of equation (3.10), we get the following operational rule:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = exp\left(z\frac{\partial^s}{\partial y^s}\right)\left(\alpha - D_x^{-1}\frac{\partial^m}{\partial y^m}\right)^{-v}\{y^n\}. \quad (3.11)$$

Next, from equations (3.8) and (2.13), the generalized special polynomials satisfy the following differential relations:

$$\left(\frac{\partial}{\partial x}x\frac{\partial}{\partial x}\right)^p {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{(v)_p\,n!}{(n-pm)!}\,{}_{v+pL}H_{n-pm}^{(m,s)} \quad (3.12)$$

$$\frac{\partial^p}{\partial y^p}\,{}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{n!}{(n-p)!}\,{}_{vL}H_{n-p}^{(m,s)}(x,y,z;\alpha), \quad (3.13)$$

$$\frac{\partial^p}{\partial z^p}\,{}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{n!}{(n-ps)!}\,{}_{vL}H_{n-ps}^{(m,s)}(x,y,z;\alpha) \quad (3.14)$$

and

$$\frac{\partial^p}{\partial \alpha^p}\, {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha) = (-1)^p (v)_p\, {}_{v+p}{}_{L}H_n^{(m,s)}(x,y,z;\alpha). \quad (3.15)$$

By comparing equations (3.12), (3.13) and (3.14) , we get

$$\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^p {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha) = (v)_p \frac{\partial^{pm}}{\partial y^{pm}}\, {}_{v+p}{}_{L}H_n^{(m,s)} \quad (3.16)$$

$$\frac{\partial^p}{\partial z^p}\, {}_{v}{}_{L}H_n^{(m,s)}(\mathrm{x},y,z;\alpha) = \frac{\partial^{ps}}{\partial y^{ps}}\, {}_{v}{}_{L}H_n^{(m,s)}(\mathrm{x},y,z;\alpha). \quad (3.17)$$

Consequently from equations (3.15) and (3.16) , we get

$$\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^p {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha) = (-1)^p \frac{\partial^{pm+p}}{\partial y^{pm}\partial \alpha^p}\, {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha). \quad (3.18)$$

We note that operational definition (3.10) can be used to establish further properties for the generalize special polynomials ${}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)$ with the help of the corresponding properties of the special polynomials ${}_{m}L_{n,v}(x,y;\alpha)$. For instance, operating $exp\left(z\frac{\partial^s}{\partial y^s}\right)$ on both sides of generating function (2.20) (after replacing ξ by t), we find

$$\sum_{n=0}^{\infty} exp\left(z\frac{\partial^s}{\partial y^s}\right) {}_{m}L_{n,v}(x,y;\alpha)\frac{t^n}{n!} = exp\left(z\frac{\partial^s}{\partial y^s}\right)\left\{\frac{exp(yt)}{\alpha^v}\right\} {}_1F_1\left(v;1;\frac{xt^m}{\alpha}\right), \quad (3.19)$$

which on using operational definition (3.10) and the crofton identity [4]

$$exp\left(\lambda\frac{d^s}{dx^s}\right)\{f(x)\} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}f^{(sk)}(x) = f\left(x+s\lambda\frac{d^{s-1}}{dx^{s-1}}\right) exp\left(\lambda\frac{d^s}{dx^s}\right)\{1\}, \quad (3.20)$$

in the l.h.s. and r.h.s. respectively, gives the following generating function for the generalized special polynomials ${}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)$:

$$\sum_{n=0}^{\infty} {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)\frac{t^n}{n!} = \frac{exp(yt+zt^s)}{\alpha^v}\, {}_1F_1\left(v,1;\frac{x^k t^m}{\alpha}\right). \quad (3.21)$$

It is worthy to mention that generating function (3.21) and further properties for ${}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)$ can also be obtained from generating function of the LGHP ${}_{L}H_n^{(m,s)}(x,y,z)$ and its properties respectively by making use of integral representation (3.4) and following the same procedure leading to results (2.20) and (2.25) in Section 2. For example, proceeding in the same way, we can derive the following differential equation and monomiality recurrence relation for ${}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)$ as:

$$\left(m\frac{\partial^{m+1}}{\partial y^m \partial \alpha} - sz\frac{\partial^{s+1}}{\partial x\,\partial y^s} - y\frac{\partial^2}{\partial x\,\partial y} + n\frac{\partial}{\partial x}\right) {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha) = 0 \quad (3.22)$$

and

$$\left(mD_x^{-1}\frac{\partial^m}{\partial y^{m-1}\partial \alpha} - sz\frac{\partial^{s-1}}{\partial y^{s-1}} - y\right) {}_{v}{}_{L}H_{n+1}^{(m,s)}(x,y,z;\alpha) = {}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha), \quad (3.23)$$

corresponding to differential equation (1.14) and monomiality recurrence (1.15) for the LGHP ${}_{L}H_n^{(m,s)}(x,y,z)$ respectively.

Furthermore, using identity (2.29) in the r.h.s. of definition (3.8), we get the following hypergeometric representation for ${}_{v}{}_{L}H_n^{(m,s)}(x,y,z;\alpha)$:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{y^n}{\alpha^v} F_{0:0;1}^{1:0;1}\left[\begin{matrix}[-n;s,m]:-;[1:1]; \\ -:-;[1:1];\end{matrix}\left(-\frac{1}{y}\right)^s z, \left(-\frac{1}{y}\right)^m \frac{x}{\alpha}\right], \tag{3.24} $$

where $F_{0:0;1}^{1:0;1}[.]$ denotes the generalized lauricella function of two variables defined by Srivastava and Monacha [11].

Finally, in particular, we note that

$$ {}_{1L}H_n^{(m,s)}(D_x^{-1},y,z;1) = {}_LH_n^{(m,s)}(x,y,z), \tag{3.25} $$

$$ {}_{1L}H_n^{(m,s)}(y,x,z;1) = H_n^{(s,m)}(x,y,z), \tag{3.26} $$

$$ {}_{vL}H_n^{(m,s)}(x,y,0;\alpha) = {}_mL_{n,v}(x,y;\alpha), \tag{3.27} $$

$$ {}_{1L}H_n^{(m,s)}(0,y,z;1) = H_n^{(s)}(y,z), \tag{3.28} $$

where $H_n^{(s,m)}(x,y,z)$ denotes the 3-variable generalized Hermite polynomials defined by [6, p.414]

$$ H_n^{(s,m)}(x,y,z) = n!\sum_{r=0}^{\left[\frac{n}{s}\right]} \frac{z^r H_{n-sk}^{(m)}(x,y)}{k!\,(n-sk)!}. \tag{3.29} $$

Also, by using relation (2.11) (for *m=2*), we note that

$$ {}_{vL}H_n^{(2,2)}(x,y,z;\alpha) = {}_{vH}H_n^{(2)}(y,D_x^{-1},z;\alpha), \tag{3.30} $$

where ${}_{vH}H_n^{(2)}(x,y,z;\alpha)$ denotes the known special polynomials defined by Dattoli et.al[5].

$$ {}_{vH}H_n^{(2)}(x,y,z;\alpha) = n!\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{z^k\, {}_vH_{n-2k}^{(2)}(y,z)}{k!(n-2k)!}. \tag{3.31} $$

Operational and integral representations

Very recently, Al-Gonah [1] introduced the generalized special polynomials ${}_LH_{n,v}^{(m,s)}(x,y,z;\alpha)$ and ${}_vH_n^{(s,m)}(x,y,z;\alpha)$, defined by the series definitions

$$ {}_LH_{n,v}^{(m,s)}(x,y,z;\alpha) = \frac{n!}{\alpha^v}\sum_{k=0}^{\left[\frac{n}{s}\right]} \frac{(v)_k z^k\, {}_mL_{n-sk}(x,y)}{\alpha^k k!\,(n-sk)!} \tag{4.1} $$

and

$$ {}_vH_n^{(m,s)}(x,y,z;\alpha) = n!\sum_{k=0}^{\left[\frac{n}{s}\right]} \frac{z^k\, {}_vH_{n-sk}^{(m)}(x,y;\alpha)}{k!\,(n-sk)!}, \tag{4.2} $$

respectively, these special polynomials also defined by the following operational definitions [1]:

$$ exp\left(D_x^{-1}\frac{\partial^m}{\partial y^m}\right)\left\{{}_vH_n^{(s)}(y,z;\alpha)\right\} = {}_LH_{n,v}^{(m,s)}(x,y,z;\alpha) \tag{4.3} $$

and

$$ exp\left(z\frac{\partial^s}{\partial x^s}\right)\left\{{}_vH_n^{(m)}(x,y;\alpha)\right\} = {}_vH_n^{(s,m)}(x,y,z;\alpha), \tag{4.4} $$

respectively.

Now, from definition(4.1) and using relations (1.11b) and (3.9), we get that the generalized special polynomials ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ defined in Section 3 are also defined by the following operational definition :

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = {}_LH_{n,v}^{(s,m)}(zD_z z,y,D_x^{-1};\alpha). \tag{4.5} $$

Again, from definition (4.2) and using relations (2.11) and (3.7), we get the following operational definition for ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$:

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = {}_vH_n^{(s,m)}(y,D_x^{-1},z;\alpha). \tag{4.6} $$

Also, we get the following operational link between the three generalized polynomials:

$$ {}_{vL}H_n^{(m,s)}(x,y,D_z^{-1};\alpha) = {}_LH_{n,v}^{(s,m)}(z,y,D_x^{-1};\alpha) = {}_vH_n^{(s,m)}(y,D_x^{-1},D_z^{-1};\alpha). \tag{4.7} $$

Next, we find some integral representations for the generalized special polynomials ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ associated with the special polynomials ${}_{L}H_{n,v}^{(m,s)}(x,y,z;\alpha)$ and ${}_{v}H_n^{(s,m)}(x,y,z;\alpha)$ in the form of the following theorems:

Theorem 4.1. *The following integral representation for the generalized special polynomials* ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ *holds true:*

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \int_0^\infty e^{-t}\,{}_{L}H_{n,v}^{(s,m)}(zt,y,D_x^{-1};\alpha)dt. \tag{4.8}$$

Proof. Denoting the r.h.s. of equation (4.8) by Δ_1 and using definition (4.1) and relation (1.6), we get

$$\Delta_1 = \frac{n!}{\alpha^v}\sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{(v)_k x^k}{\alpha^k (k!)^2 (n-mk)!}\int_0^\infty e^{-t}\,{}_{s}L_{n-mk}(zt,y)dt. \tag{4.9}$$

Using the following integral relation (which obtain by using relations (2.5) (for $v = \alpha = 1$) and (2.9))

$$H_n^{(s)}(y,z) = \int_0^\infty e^{-t}\,{}_{s}L_n(zt,y)\,dt, \tag{4.10}$$

in the r.h.s. of equation(4.9) and then using equation (3.9) in the resultant equation , we get assertion (4.8) of Theorem 4.1.

Theorem 4.2. *The following integral representation for the generalized special polynomials* ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ *holds true:*

$$ {}_{vL}H_n^{(m,s)}(x,y,z;\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{t}t^{-1}\,{}_{v}H_n^{(s,m)}(y,xt^{-1},z;\alpha)dt. \tag{4.11}$$

Proof. Denoting the r.h.s. of equation (4.11) by Δ_2 and using definition (4.2), we get

$$\Delta_2 = \frac{n!}{\alpha^v}\sum_{k=0}^{\left[\frac{n}{s}\right]} \frac{z^k}{k!\,(n-sk)!}\frac{1}{2\pi i}\int_{-\infty}^{(0+)} e^{\,t}t^{-1}\,{}_{v}H_{n-sk}^{m}(y,xt^{-1})dt. \tag{4.12}$$

Using relation (2.28) in the r.h.s. of the above equation and then using equation (3.7) in the resultant equation , we get assertion (4.11) of Theorem 4.2 .

Alternate Proof. Operating $exp\left(z\frac{\partial^s}{\partial y^s}\right)$ on both sides of equation (2.28),we get

$$exp\left(z\frac{\partial^s}{\partial y^s}\right)\,{}_{m}L_{n,v}(x,y;\lambda)$$
$$= \frac{1}{2\pi!}\int_{-\infty}^{(0+)} e^{-t}t^{-1}exp\left(z\frac{\partial^s}{\partial y^s}\right)\,{}_{v}H_n^{(m)}(y,xt^{-1})dt, \tag{4.13}$$

which on using operational definition (3.10) and (4.4) in the l.h.s. and r.h.s. respectively yields assertion (4.11) of Theorem 4.2.

Remark 4.1. Operating $exp\left(z\frac{\partial^s}{\partial y^s}\right)$ on both sides of relation (2.27) and then using operational definitions (4.4) and (3.10) in the l.h.s. and r.h.s. respectively of the resultant equation, we get the following result.

Theorem 4.3. *The following integral involving the generalized special polynomials* ${}_{vL}H_n^{(m,s)}(x,y,z;\alpha)$ *holds true:*

$$ {}_{v}H_n^{(s,m)}(x,y,z;\alpha) = \int_0^\infty e^{-t}\,{}_{vL}H_n^{(m,s)}(yt,x,z;\alpha)dt. \tag{4.14}$$

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تحويلات تكاملية ومتعددة حدود لاجير - جولد هوبر

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الملخص

في هذه المقالة نستعرض عائلات جديدة لمتعددات حدود معممة وخاصة بواسطة دمج خواص المؤثرات الأسية مع تحويلات تكاملية مناسبة. وقد تم إيجاد خواص محددة لتلك متعددات الحدود.

الكلمات المفتاحية: متعددة حدود لاجير- جولد هوبر، متعددة حدود خاصة, تمثيل تكاملي.