q-Hypergeometric representations of the multiple Hurwitz Zeta function

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Abstract

The basic hypergeometric series started essentially by Euler back in (1748) that emphasis on generating functions of partitions. Later, Gauss (1813) and Cauchy (1852) found several transformations and summations formulas related to basic hypergeometric series.

In this paper, the main goal is to introduce some new representations for the q-analogue of the multiple Hurwitz Zeta function are derived.

Key words: multiple Hurwitz Zeta function, q-Hypergeometric series, q-shifted factorial.

Introduction, definitions and notations

The Hurwitz or generalized Zeta function at integer points [12]

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \quad , \ 0<a\le 1, \quad (1.1)$$

has a q-analogue defined by

$$\zeta_q(s,a)=\sum_{n=0}^{\infty}\frac{q^{(n+a)(s-1)}}{[n+a]^s} \quad , \ 0<q<1,\ 0<a\le 1. \qquad (1.2)$$

In [11], theq-Hurwitz Zeta function is defined as

$$\zeta(s,z;q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)s}}{[n+z]_q^s} \quad , \qquad (1.3)$$

Also, he defined the multiple Zeta and q-Zeta functions by

$$\zeta(s,z;w)=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{1}{(k_1w_1+k_2w_2+\ldots+k_nw_n+z)^s} \quad , \qquad (1.4)$$

and

$$\zeta(s,z;w,a,b;q)=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{q^{s(k_1a_1+k_2a_2+\ldots+k_na_n+b)}}{[k_1w_1+k_2w_2+\ldots+k_nw_n+z]_q^s} \qquad (1.5)$$

respectively.

Barnes $[4]$ (see also $[1,2,3]$) introduced and studied the generalized multiple Hurwitz Zeta function $\zeta_n(s,a/w_1,\ldots,w_n)$ defined, for $R(s)>n$, by the following series:

$$\zeta_n(s,a/w_1,\ldots,w_n)=\sum_{k_1,\ldots,k_n=0}^{\infty}\frac{1}{(a+\Omega)^s} \qquad (R(s))>n \ ;n\in N \ , \qquad (1.6)$$

where N denotes the set of positive integers, $\Omega=k_1w_1+\ldots+k_nw_n$.

Barnes-Changhee multiple q-Zeta functions are defined by (see $[8]$, $[9]$).

$$\zeta_{q,r}(s,w/a_1,a_2,\ldots,a_r)=\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{q^{w+n_1+n_2+\ldots+n_r}}{[w+n_1a_1+n_2a_2+\ldots+n_ra_r]^s} \qquad (1.7)$$

$\mathrm{Re}(w)>0,\ q\in C$ with $|q|<1$, which , for $a_1=a_2=\ldots=a_r=1$, yields

$$\zeta_{q,r}(s,w/1,\ldots,1)=\sum_{n_1,\ldots,n_r=0}^{\infty}\frac{q^{w+n_1+n_2+\ldots+n_r}}{[w+n_1+n_2+\ldots+n_r]^s}\ .$$

Moreover, if $w=r$ and $s=1-n \quad (n\in Z^+)$, we have

$$\zeta_{q,r}(n-1,r/1,\ldots,1)=(-1)^r\frac{(n-1)!}{(n+r-1)!}B_{n+r-1}^{(r)}(r;q)$$

where $B_{n+r-1}^{(r)}(r;q)$ is called q-Bernoulli numbers.

We also note that

$$\lim_{q\to 1}\zeta_{q,n}(n-1,r/1,\ldots,1)=\zeta_n(n-1,r/1,\ldots,1)=(-1)^r\frac{(n-1)!}{(n+r-1)!}B_{n+r-1}^{(r)}(r)$$

(see [8], [9]).

The q-number $[z]_q$ is defined through

$$[z]_q=\frac{1-q^z}{1-q}\quad ,z\in C\ ,q\neq 1\,. \tag{1.8}$$

A special case of (1.8) when $z\in N$ is $[n]_q=\frac{1-q^n}{1-q}=\sum_{0\le i\le n-1}q^i\quad ,n\in N$,

which is called the q-analogue of $n\in N$, since

$$\lim_{q\to 1^-}[n]_q=\lim_{q\to 1^-}\sum_{0\le i\le n-1}q^i=n\,.$$

The Pochhammersymbol $(.)_k$, also called the shifted factorial, is defined by

$$(z)_k=\prod_{j=0}^{k-1}(z+j)\ ,k\ge 1\ ,(z)_0=1\ ,(-z)_k=0,if\ \ z<k\,,$$

which in terms of the Gamma function is given by

$$(z)_k=\frac{\Gamma(z+k)}{\Gamma(z)}\ ,\ k=0,1,2,3,\ldots,z\neq 0,-1,-2,\ldots$$

and ${}_rF_s$ denoted the ordinary hypergeometric series ([4],[10]) with variable z is defined by

$${}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\Big/ z\right)=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k}\frac{z^k}{k!}\ , \tag{1.9}$$

being $(a_1,\ldots,a_r)_k=\prod_{i=1}^{r}(a_i)_k$, with $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ complex numbers subject to the condition that $b_j\neq -n$ with $n\in N/\{0\}$ for $j=1,2,3,\ldots,s$.

Here we will give some usual definitions and notations used in q-calculus, i.e. the q-analogues of the usual calculus .

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [7,10]

$$(a;q)_n=\begin{cases}1 & ,\ n=0\\ \prod\limits_{0\le j\le n-1}(1-aq^j) & ,\ n=1,2,3,\ldots\end{cases} \tag{1.10}$$

where $(q^{-n};q)=0$, whenever $n<k$,(1.11)

$(0;q)_n=1$,

also $(a;q)_{n+k}=(a;q)_n(aq^n;q)_k$ (1.12) and $\lim_{q\to1^-}\frac{(q^z;q)_k}{(1-q)^k}=(z)_k$.

The formula (2.1) is known as the Watson notation $[5,6]$.

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \ , k,n\in N \tag{1.13}$$

and for complex z is defined by

$$\begin{bmatrix} z \\ k \end{bmatrix}_q=\frac{(q^{-z};q)_k}{(q;q)_k}(-1)^k q^{zk-\binom{k}{2}} ;\ k\in N \tag{1.14}$$

In addition, using the above definitions, we have that the binomial theorem

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \ , \ n=0,1,2,3,\dots, \tag{1.15}$$

has a q-analogue of the form

$$(xy;q)_n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}y^k(x;q)_k(y;q)_{n-k}$$
$$=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^{n-k}(x;q)_k(y;q)_{n-k} \ , \tag{1.16}$$

In particular, when $y=0$ we have that

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}x^{n-k}(x;q)_k=(0;q)=1 \ . \tag{1.17}$$

Let $\{a_i\}_{i=1}^{r}$ and $\{b_j\}_{j=1}^{s}$ complex numbers subject to the condition that $b_j\neq q^{-n}$ with $n\in N\setminus\{0\}$ for $j=1,2,3,\dots,s$.

Then the basic hypergeometric or q-hypergeometric ${}_r\phi_s$ series with variable z is defined by

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\Big/ q;z\right)=\sum_{k\geq0}\frac{(a_1,\dots,a_r;q)_k}{(b_1,\dots,b_s;q)_k}(-1)^{(1+s-r)k}q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k}, \tag{1.18}$$

where $(a_1,\dots,a_r;q)_k=\prod_{1\leq j\leq r}(a_j;q)_k$.

In addition, for brevity, let us denote by

$$\left[{}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\Big/ q;z\right)\right]^n={}_r\phi_s^{\,n}\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\Big/ q;z\right), \ n=1,2,3,\dots \ . \tag{1.19}$$

Analogously to the ordinary hypergeometric ${}_{s+1}F_s$ series, the q-hypergemoetric ${}_{s+1}\phi_s$ series is called k-balanced if $b_1b_2\dots b_s=q^k a_1a_2\dots a_{s+1}$.

The q-hypergeomwtric ${}_r\phi_s$ series is a q-analogue of the ordinary hypergeometric ${}_rF_s$ series defined by[7]

$$\lim_{q\to1^-}{}_r\phi_s\left(\begin{matrix} q^{a_1},\dots,q^{a_r} \\ q^{b_1},\dots,q^{b_s} \end{matrix}\Big/ q;z(q-1)^{1+s-r}\right)={}_rF_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\Big/ z\right). \tag{1.20}$$

The q-analogue of the Chu-Vandermonde convolution is given by[10]

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ b \end{matrix} \middle/ q; \frac{bq^n}{a}\right) = \frac{(a^{-1}b;q)_n}{(b;q)_n},\ n = 0,1,2,3,\dots \ , \quad (1.21)$$

$${}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ b \end{matrix} \middle/ q; q\right) = \frac{(a^{-1}b;q)_n}{(b;q)_n} a^n,\ n = 0,1,2,3,\dots \ . \quad (1.22)$$

Soria-Lorente et.al.[12] gave the following relation

$${}_2\phi_0\left(\begin{matrix} q^{-n}, a \\ - \end{matrix} \middle/ q; q^n z^{-1}\right) = z^{-n} \ ,\ n = 0,1,2,3,\dots . \quad (1.23)$$

Main Results

In this section, some representations for the q-analogue of the multiple Hurwitz Zeta function are establish:

$$\zeta_n(s,w;q) = \sum_{r_1,\dots,r_n=0}^{\infty} \frac{q^{(r_1+r_2+\dots+r_n+w)(s-1)}}{[w+r_1+r_2+\dots+r_n]_q^s} \quad (2.1)$$

Theorem1. Let s be an integer number, with $s>1$, $|q|<1$ and $0<w\le 1$. Then, the q-analogue of the multiple Hurwitz zeta function(2.1) admits the following representations

i. $$\zeta_n(s,w;q) = q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s {}_{s+1}\phi_s\left(\begin{matrix} q, q^w, \dots, q^w \\ q^{w+1}, \dots, q^{w+1} \end{matrix} \middle/ q; q^{s-1}\right)$$

$$\times {}_{s+1}\phi_s\left(\begin{matrix} q, q^{w+r_1}, \dots, q^{w+r_1} \\ q^{w+1+r_1}, \dots, q^{w+1+r_1} \end{matrix} \middle/ q; q^{s-1}\right) \cdots {}_{s+1}\phi_s\left(\begin{matrix} q, q^{w+r_1+\dots+r_{n-1}}, \dots, q^{w+r_1+\dots+r_{n-1}} \\ q^{w+1+r_1+\dots+r_{n-1}}, \dots, q^{w+1+r_1+\dots+r_{n-1}} \end{matrix} \middle/ q; q^{s-1}\right) \quad (2.2)$$

ii. $$\zeta_n(s,w;q) = q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s \sum_{r_1,\dots,r_n=0}^{\infty} {}_2\phi_0\left(q^{-(r_1+r_2+\dots+r_n)}; q/q; q^{(r_1+r_2+\dots+r_n)-1}\right)$$

$$\times {}_2\phi_1^s\left(\begin{matrix} q^{-(r_1)}; q \\ q^{w+1} \end{matrix} \middle/ q; q\right) {}_2\phi_1^s\left(\begin{matrix} q^{-(r_2)}; q \\ q^{w+1+r_1} \end{matrix} \middle/ q; q\right) \cdots {}_2\phi_1^s\left(\begin{matrix} q^{-(r_n)}; q \\ q^{w+1+r_1+r_2+\dots+r_{n-1}} \end{matrix} \middle/ q; q\right) \quad (2.3)$$

iii. $$\zeta_n(s,w;q) = q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s {}_s\phi_{s-1}\left(\begin{matrix} q; q^w, \dots, q^w \\ q^{w+1}, \dots, q^{w+1} \end{matrix} \middle/ q; q^{s-1}\right)$$

$$\times {}_s\phi_{s-1}\left(\begin{matrix} q; q^{w+r_1+\dots+r_{n-1}}, \dots, q^{w+r_1+\dots+r_{n-1}} \\ q^{w+1+r_1+\dots+r_{n-1}}, \dots, q^{w+1+r_1+\dots+r_{n-1}} \end{matrix} \middle/ q; q^{s-1}\right)$$

$$\times \sum_{k_1=0}^{\infty} (-1)^{k_1} q^{\binom{k_1}{2}+(wk_1)} {}_2\phi_1\left(\begin{matrix} q^{-k_1}, q^{r_1-w} \\ q^{w+1} \end{matrix} \middle/ q; q^{2w+1}\right)$$

$$\times {}_2\phi_1^{s-1}\left(\begin{matrix} q^{-k_1}, q^{r_1-w} \\ q^{w+1+r_1} \end{matrix} \middle/ q; q\right) \cdots \sum_{k_n=0}^{\infty} (-1)^{k_n} q^{\binom{k_n}{2}+(w+r_1+\dots+r_n)k_n}$$

$${}_2\phi_1\left(\begin{matrix} q^{-k_n}, q^{-(w+r_1+\dots+r_{n-1}-r_n)} \\ q^{w+1+r_1+\dots+r_{n-1}} \end{matrix} \middle/ q; q^{2(w+r_1+\dots+r_{n-1})+1}\right) {}_2\phi_1^{s-1}\left(\begin{matrix} q^{-k_n}, q^{r_1-w} \\ q^{w+1+r_1+\dots+r_{n-1}} \end{matrix} \middle/ q; q\right) \quad (2.4)$$

Proof .

(i) From

$$\zeta_n(s,w;q)=q^{w(s-1)}\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{q^{(r_1+r_2+\ldots+r_n)(s-1)}}{[w+r_1+r_2+\ldots+r_n]_q^s}\ .$$

By using (1.8), we get

$$\zeta_n(s,w;q)$$

$$=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{(1-q^w)^s(1-q^{w+1})^s\ldots(1-q^{w+r_1+r_2+\ldots+r_n-1})^s q^{(r_1+r_2+\ldots+r_n)(s-1)}}{(1-q^{w+1})^s\ldots(1-q^{w+1+r_1+r_2+\ldots+r_n-2})^s(1-q^{w+1+r_1+r_2+\ldots+r_n-1})^s}$$

$$=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}\frac{(q^w;q)^s_{r_1+r_2+\ldots+r_n}q^{(r_1+r_2+\ldots+r_n)(s-1)}}{(q^{w+1};q)^s_{r_1+r_2+\ldots+r_n}}$$

$$=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s{}_{s+1}\phi_s\left(\begin{matrix}q,q^w,\ldots,q^w\\ q^{w+1},\ldots,q^{w+1}\end{matrix}\Big/q;q^{s-1}\right){}_{s+1}\phi_s\left(\begin{matrix}q,q^{w+r_1},\ldots,q^{w+r_1}\\ q^{w+1+r_1},\ldots,q^{w+1+r_1}\end{matrix}\Big/q;q^{s-1}\right)$$

$$\ldots\ldots{}_{s+1}\phi_s\left(\begin{matrix}q,q^{w+r_1+\cdots+r_{n-1}},\ldots,q^{w+r_1+\cdots+r_{n-1}}\\ q^{w+1+r_1+\cdots+r_{n-1}},\ldots,q^{w+1+r_1+\cdots+r_{n-1}}\end{matrix}\Big/q;q^{s-1}\right)$$

Now, let's prove (ii):

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}q^{-(r_1+r_2+\cdots+r_n)}\left[\frac{(q^w;q)_{r_1+r_2+\cdots+r_n}q^{(r_1+r_2+\cdots+r_n)}}{(q^{w+1};q)_{r_1+r_2+\cdots+r_n}}\right]^s.$$

Putting $n=r_1+r_2+\cdots+r_n$ in (1.23), we obtain

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}{}_2\phi_0\left(q^{-(r_1+r_2+\cdots+r_n)};q/q;q^{(r_1+r_2+\cdots+r_n)-1}\right)$$

$$\left[\frac{(q^{-1}q^{w+1};q)_{r_1+r_2+\cdots+r_n}q^{(r_1+r_2+\cdots+r_n)}}{(q^{w+1};q)_{r_1+r_2+\cdots+r_n}}\right]^s$$

$$=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}{}_2\phi_0\left(q^{-(r_1+r_2+\cdots+r_n)};q/q;q^{(r_1+r_2+\cdots+r_n)-1}\right)$$

$${}_2\phi_1^s\left(\begin{matrix}q^{-(r_1)};q\\ q^{w+1}\end{matrix}\Big/q;q\right){}_2\phi_1^s\left(\begin{matrix}q^{-(r_2)};q\\ q^{w+1+r_1}\end{matrix}\Big/q;q\right)\cdots{}_2\phi_1^s\left(\begin{matrix}q^{-(r_n)};q\\ q^{w+1+r_1+r_2+\cdots+r_{n-1}}\end{matrix}\Big/q;q\right)$$

which is the required result.

(iii) According to the q-Chu-Vandermonde formula (1.21), we have that

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^w}\right)^s\sum_{r_1,\ldots,r_n=0}^{\infty}q^{-(r_1+r_2+\cdots+r_n)}$$

$$\left[\frac{(q^w;q)^{(s-1)}_{r_1+r_2+\cdots+r_n}q^{(r_1+r_2+\cdots+r_n)(s-1)}}{(q^{w+1};q)^{(s-1)}_{r_1+r_2+\cdots+r_n}}\frac{(q^w;q)_{r_1+r_2+\cdots+r_n}q^{(r_1+r_2+\cdots+r_n)}}{(q^{w+1};q)_{r_1+r_2+\cdots+r_n}}\right]$$

$$= q^{w(s-1)}\left(\frac{1-q}{1-q^{w}}\right)^{s}\sum_{r_1=0}^{\infty}\frac{\left(q^{w};q\right)_{r_1}^{(s-1)}q^{(r_1)(s-1)}}{\left(q^{w+1};q\right)_{r_1}^{(s-1)}}\cdots\sum_{r_n=0}^{\infty}\frac{\left(q^{w+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_1}^{(s-1)}q^{(r_n)(s-1)}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n}^{(s-1)}}$$

$$\sum_{k_1=0}^{\infty}\frac{\left(q^{-r_1};q\right)_{k_1}\left(q;q\right)_{k_1}}{\left(q^{w+1};q\right)_{k_1}}\frac{q^{(w+r_1)k_1}}{\left(q;q\right)_{k_1}}\cdots\sum_{k_n=0}^{\infty}\frac{\left(q^{-r_n};q\right)_{k_n}\left(q;q\right)_{k_n}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{k_n}}\frac{q^{(w+r_1+r_2+\cdots+r_n)k_n}}{\left(q;q\right)_{k_n}}$$

Since $\left(q^{-n};q\right)_k=\dfrac{(q;q)_n}{(q;q)_{n-k}}(-1)^k q^{\binom{k}{2}-nk}$, $k\le n$.

Then, the result obtainedis

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^{w}}\right)^{s}\sum_{r_1=0}^{\infty}\frac{\left(q^{w};q\right)_{r_1}^{(s-1)}q^{(r_1)(s-1)}}{\left(q^{w+1};q\right)_{r_1}^{(s-1)}}\cdots\sum_{r_n=0}^{\infty}\frac{\left(q^{w+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_1}^{(s-1)}q^{(r_n)(s-1)}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n}^{(s-1)}}$$

$$\sum_{k_1=0}^{[r_1]}\frac{(q;q)_{r_1}(-1)^{k_1}q^{\binom{k_1}{2}-r_1k_1}}{\left(q^{w+1};q\right)_{k_1}}\frac{q^{(w+r_1)k_1}}{(q;q)_{r_1-k_1}}\cdots\sum_{k_n=0}^{[r_n]}\frac{(q;q)_{r_n}(-1)^{k_n}q^{\binom{k_n}{2}-r_nk_n}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{k_n}}\frac{q^{(w+r_1+r_2+\cdots+r_n)k_n}}{(q;q)_{r_n-k_n}}$$

$$=q^{w(s-1)}\left(\frac{1-q}{1-q^{w}}\right)^{s}\sum_{r_1,k_1=0}^{\infty}\frac{(q;q)_{r_1+k_1}\left(q^{w};q\right)_{r_1+k_1}^{(s-1)}\left(-q^{w}\right)^{k_1}q^{\binom{k_1}{2}+(r_1+k_1)(s-1)}}{\left(q^{w+1};q\right)_{r_1+k_1}^{(s-1)}\left(q^{w+1};q\right)_{k_1}(q;q)_{r_1}}\cdots$$

$$\cdots\sum_{r_n,k_n=0}^{\infty}\frac{(q;q)_{r_n+k_n}\left(q^{w+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n+k_n}^{(s-1)}\left(-q^{s-1+w}\right)^{k_n}q^{\binom{k_1}{2}+k_n(r_1+r_2+\cdots+r_{n-1})+(r_n)(s-1)}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n+k_n}^{(s-1)}\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{k_n}(q;q)_{r_n}}$$

Using the property(1.12), we get

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^{w}}\right)^{s}\sum_{r_1=0}^{\infty}\frac{(q;q)_{r_1}\left(q^{w};q\right)_{r_1}^{(s-1)}q^{(r_1)(s-1)}}{\left(q^{w+1};q\right)_{r_1}^{(s-1)}(q;q)_{r_1}}$$

$$\sum_{k_1=0}^{\infty}\frac{\left(q^{1+r_1};q\right)_{k_1}\left(q^{w+r_1};q\right)_{k_1}^{(s-1)}\left(-q^{w}\right)^{k_1}q^{\binom{k_1}{2}+(k_1)(s-1)}}{\left(q^{w+1+r_1};q\right)_{k_1}^{(s-1)}\left(q^{w+1};q\right)_{k_1}}\cdots\sum_{r_n=0}^{\infty}\frac{(q;q)_{r_n}\left(q^{w+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n}^{(s-1)}q^{(r_n)(s-1)}}{\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{r_n}^{(s-1)}(q;q)_{r_n}}$$

$$\sum_{k_n=0}^{\infty}\frac{\left(q^{1+r_n};q\right)_{k_n}\left(q^{w+r_1+r_2+\cdots+r_n};q\right)_{k_n}^{(s-1)}\left(-q^{s-1+w}\right)^{k_n}q^{\binom{k_1}{2}+k_n(r_1+r_2+\cdots+r_{n-1})}}{\left(q^{w+1+r_1+r_2+\cdots+r_n};q\right)_{k_n}^{(s-1)}\left(q^{w+1+r_1+r_2+\cdots+r_{n-1}};q\right)_{k_n}}$$

$$\zeta_n(s,w;q)=q^{w(s-1)}\left(\frac{1-q}{1-q^{w}}\right)^{s}{}_s\phi_{s-1}\left(\begin{matrix}q;q^{w},\ldots,q^{w}\\ q^{w+1},\ldots,q^{w+1}\end{matrix}\Big/ q;q^{s-1}\right)$$

$$\times{}_s\phi_{s-1}\left(\begin{matrix}q;q^{w+r_1+\cdots+r_{n-1}},\ldots,q^{w+r_1+\cdots+r_{n-1}}\\ q^{w+1+r_1+\cdots+r_{n-1}},\ldots,q^{w+1+r_1+\cdots+r_{n-1}}\end{matrix}\Big/ q;q^{s-1}\right)\sum_{k_1=0}^{\infty}(-1)^{k_1}q^{\binom{k_1}{2}+(wk_1)}$$

$$\times{}_2\phi_1\left(\begin{matrix}q^{-k_1},q^{r_1-w}\\ q^{w+1}\end{matrix}\Big/ q;q^{2w+1}\right){}_2\phi_1^{\,s-1}\left(\begin{matrix}q^{-k_1},q^{r_1-w}\\ q^{w+1+r_1}\end{matrix}\Big/ q;q\right)\cdots\sum_{k_n=0}^{\infty}(-1)^{k_n}q^{\binom{k_n}{2}+(w+r_1+\cdots+r_n)k_n}$$

$$\times{}_2\phi_1\left(\begin{matrix}q^{-k_n},q^{-(w+r_1+\cdots+r_{n-1}-r_n)}\\ q^{w+1+r_1+\cdots+r_{n-1}}\end{matrix}\Big/ q;q^{2(w+r_1+\cdots+r_{n-1})+1}\right){}_2\phi_1^{\,s-1}\left(\begin{matrix}q^{-k_n},q^{r_1-w}\\ q^{w+1+r_1+\cdots+r_{n-1}}\end{matrix}\Big/ q;q\right)$$

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التمثيلات فوق الهندسية لدالة هويرتز زيتا المتعددة من النوع كيو

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الملخص

بدأت المتسلسلات فوق الهندسية الأساسية تظهر جوهرياً بواسطة إيلر في (1748م) للتأكيد علي توليد الدوال المجزئة. لاحقاً جاوس (1813م) و كوشي (1852م) وجدا عدة تحويلات وصيغ المجموع المتعلقة بالمتسلسلات فوق الهندسية الأساسية في هذه الورقة, هدفنا الأساسي هو تقديم بعض التمثيلات فوق الهندسية الأساسية الجديدة لدالة هويرتز زيتا المتعددة واشتقاقها.

الكلمات المفتاحية: دالة هويرتز زيتا المتعددة، المتسلسلة فوق الهندسية من النوع كيو، والانتقال المضروبي من النوع كيو.