Inequalities for the polar derivative and the generalized polar derivative of complex polynomials with restricted zeros

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Abstract

In this paper, certain new results concerning the maximum modulus of the polar derivative and the generalized polar derivative of a polynomial with restricted zeros are obtained. These estimates strengthen some well known inequalities for polynomial due to Turàn, Rather and Dar, Rather, Ali, Shafi and Dar and others.

Keywords: Polynomials, Polar Derivative, Generalized Polar Derivative, Inequalities in the Complex Domain.

1. Introduction

There is always a desire to look for better and improved bounds than those available in this domain. It is this aspiration of obtaining more refined and revamped bounds that has inspired our work in this article .In this paper, we have generalized and refined some well known results concerning the polynomials due to Turàn [21], Rather and Dar [19] , Rather, Ali ,Shafi and Dar [18] and others.

Let $\mathcal{P}_n$ denote the class of all algebraic polynomials of the from $P(z) = \sum_{v=o}^{n} a_v z^v$ of degree $n \geq 1$.

If $P \in \mathcal{P}_n$,then

$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)| . \tag{1.1}$$

The above inequality is the well-known Bernstein inequality [6]. Inequality (1.1) is best possible and equality holding for a polynomial that has all zeros at the origin.

If $P \in \mathcal{P}_n$ has no zeros in $|z| < 1$, then Erdös [10] conjectured and Lax [15] proved

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)| . \tag{1.2}$$

If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then it was proved by Turàn [21] , that

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{2} \max_{|z|=1}|P(z)| . \tag{1.3}$$

The inequalities (1.2) and (1.3) are also best possible and become equality for polynomials which have all its zeros on $|z| = 1$.

As an extension of (1.3) , Govil [11] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \geq \left(\frac{n}{1+k^n}\right) \max_{|z|=1}|P(z)| \ . \tag{1.4}$$

The result is sharp, as shown by the polynomial $P(z) = z^n + k^n$.

Further Govil [12] proved that, if $P \in \mathcal{P}_n$ has no zeros in $|z| < k$, $k \leq 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \leq \left(\frac{n}{1+k^n}\right) \max_{|z|=1}|P(z)|. \tag{1.5}$$

Provided $\left|P^{'}(z)\right|$ and $\left|q^{'}(z)\right|$ attain their maxima at the same point on the circle $|z| = 1$, where $q(z) = z^n \overline{P(1/\bar{z})}$.

By involving the minimum modulus of $P(z)$ on $|z| = 1$, Aziz and Dawood [2], proved under the hypothesis of inequality (1.3) that

$$\max_{|z|=1}\left|P^{'}(z)\right| \geq \frac{n}{2}\left[\max_{|z|=1}|P(z)| + \min_{|z|=1}|P(z)|\right]. \tag{1.6}$$

Equality in (1.6) holds for $P(z) = a\,z^n + b$, $|a| = |b| = 1$.

Dubinin [9] obtained a refinement of (1.3) by involving some of the coefficients of polynomial $P \in \mathcal{P}_n$ in the bound of inequality (1.3). More precisely, proved. then if all the zeros of the polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \geq \frac{1}{2}\left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right)\max_{|z|=1}|P(z)| \ . \tag{1.7}$$

Recently, Rather and Dar [19] generalized this inequality and proved that, if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \geq \frac{1}{1+k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\max_{|z|=1}|P(z)| \ . \tag{1.8}$$

The result is sharp and equality holds for $P(z) = z^n + k^n$.

The polar derivative $D_\alpha P(z)$ of $P \in \mathcal{P}_n$ with respect to a complex number α is defined by $D_\alpha P(z) := n\,P(z) + (\alpha - z)\,P^{'}(z)$ see [16].The polynomial $D_\alpha P(z)$ is of degree at most $(n-1)$,

and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha P(z)}{\alpha}\right] = P^{'}(z)\,,$$

uniformly for $|z| \leq R$, $R > 0$.

Aziz [1] , Aziz and Rather ([3,5]) obtained several sharp estimates for maximum modulus of $D_\alpha P(z)$ on $|z| = 1$ and among other things they extended inequality (1.4) to the polar derivative of a polynomial by showing that, if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)|. \tag{1.9}$$

Rather and Dar [19] extended inequality (1.8) to the polar derivative of a polynomial by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}|D_\alpha P(z)| \ge \frac{|\alpha|-k}{(1+k^n)}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)|. \tag{1.10}$$

denote the collection of all monic polynomials in $\mathcal{P}_n$ and $\mathbb{R}_+^n$ be the set For each positive integer n , $\partial\mathcal{P}_n$ with of all $n-$tuples $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ of non –negative real numbers (not all zeros)

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = \Lambda .$$

Recently, Rather el at. [18] consider the lower bound estimates for the generalized polar derivative of certain polynomials, which include various results due to Aziz and Rather , Turàn and Govil as special cases.

Let $D_\alpha^\gamma [P](z)$ denote the generalized polar derivative of the polynomial $P(z)$ as

$$D_\alpha^\gamma[P](z) = \Lambda\, P(z) + (\alpha - z)\, P^\gamma(z) \ ,$$

where $\Lambda = \sum_{j=1}^{n} \gamma_j$, for all $\gamma \in \mathbb{R}_+^n$, see [18].

Noting that for $\gamma = (1,1,1,\dots,1)$, $D_\alpha^\gamma[P](z) = D_\alpha P(z)$.

Recently, Rather el at. [18] extended inequality (1.9) to the generalized polar derivative of a polynomial by showing that, if $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}\left|D_\alpha^\gamma [P](z)\right| \ge \frac{\Lambda}{(1+k^n)}(|\alpha|-k)\max_{|z|=1}|P(z)| . \tag{1.11}$$

2. Lemmas

We need the following Lemmas for the proof of our theorems. The first lemma is due to Dubinin [9].

Lemma 2.1. If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then

$$\max_{|z|=1}\left|P'(z)\right| \ge \frac{1}{2}\left(n+\frac{|a_n|-|a_0|}{|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)| \tag{2.1}$$

The next lemma is a special case of a result due to Aziz and Rather [4 ,5].

Lemma 2.2. If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for $|z| = 1$

$$\left|Q^{'}(z)\right| \le \left|P^{'}(z)\right| \ , \tag{2.2}$$

where $Q(z) = z^n \overline{P\left(1/\bar{z}\right)}$.

Lemma 2.3. If all the zeros of an nth degree polynomial $P(z)$ lie in a circular region c and w is any zero of $D_\alpha P(z)$, then at most one of the points w and α may lie outside c .

The above Lemma is due to Laguerre (see [16]). The following Lemma is due to Dewan el at. [8]

Lemma 2.4. If $P(z)$ is a polynomial of degree n , then for $R \ge 1$

$$\max_{|z|=R}|P(z)| \le R^n \max_{|z|=1}|P(z)| - \frac{2(R^n-1)}{n+2}|P(0)| - \left[\frac{(R^n-1)}{n} - \frac{R^{n-2}-1}{n-2}\right]\left|P^{'}(0)\right|; \text{ provided } \ n > 2\,. \tag{2.3}$$

And

$$\max_{|z|=R}|P(z)| \le R^n \max_{|z|=1}|P(z)| - \frac{(R-1)}{2}\left[(R+1)|P(0)| + (R-1)\left|P^{'}(0)\right|\right]; \text{ provided } n = 2\,. \tag{2.4}$$

The next lemma is the famous result of Lax [15].

Lemma 2.5. If $P \in \mathcal{P}_n$ dose not vanish in $|z| < 1$, then

$$\max_{|z|=1}\left|P^{'}(z)\right| \le \frac{n}{2}\max_{|z|=1}|P(z)| \ . \tag{2.5}$$

We also need the following Lemma.

Lemma 2.6. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \ge 3$ having no zeros in $|z| < 1$, then for $R \ge 1$

$$\max_{|z|=R}|P(z)| \le \frac{(R^n+1)}{2}\max_{|z|=1}|P(z)| - \frac{2\left|P^{'}(0)\right|}{(n+1)}\left[\frac{(R^n-1)}{n} - (R-1)\right] - \left|P^{''}(0)\right|\left[\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right]; \text{ provided } \ n > 3\,. \tag{2.6}$$

And

$$\max_{|z|=R}|P(z)| \le \frac{(R^n+1)}{2}\max_{|z|=1}|P(z)| - \frac{2\left|P^{'}(0)\right|}{(n+1)}\left[\frac{(R^n-1)}{n} - (R-1)\right] - \frac{(R-1)^n}{n(n-1)}\left|P^{''}(0)\right|; \text{provided } \ n = 3\,. \tag{2.7}$$

Proof of Lemma 2.6. For each θ , $0 \le \theta < 2\pi$ and for $R \ge 1$, we have

$$\left|P\left(Re^{i\theta}\right) - P\left(e^{i\theta}\right)\right| \le \int_1^R \left| P'\left(te^{i\theta}\right) \right| dt \tag{2.8}$$

Since $P(z)$ is a polynomial of degree $n \ge 3$ so that $P'(z)$ is a polynomial of degree $n \ge 2$, so applying inequality (2.3) for $n > 2$ of Lemma 2.4 to $P'(z)$ in (2.8) , we get

$$\left|P\left(Re^{i\theta}\right) - P\left(e^{i\theta}\right)\right| \le \int_1^R t^{n-1} dt \max_{|z|=1}\left|P'(z)\right| - \frac{2\,|P'(0)|}{(n+1)}\int_1^R (t^{n-1}-1)\, dt$$

$$-|P''(0)|\int_1^R \left[\frac{(t^{n-1}-1)}{(n-1)} - \frac{(t^{n-3}-1)}{(n-3)}\right] dt$$

$$= \left(\frac{R^n-1}{n}\right)\max_{|z|=1}\left|P'(z)\right| - \frac{2\,|P'(0)|}{(n+1)}\left(\frac{(R^n-1)}{n} - (R-1)\right)$$

$$-\left|P''(0)\right|\left[\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right].$$

Using inequality (2.5) of Lemma 2.5 above, we get inequality (2.6). The inequality (2.7) follows on the same lines as that of inequality (2.6) but instead of using inequality (2.3) of Lemma 2.4, we use the inequality (2.4) of the same Lemma.

Lemma 2.7. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \ge 3$ having all its zeros in $|z| \le k$, $k \ge 1$, then for $0 \le l < 1$

$$\max_{|z|=k}|P(z)| \ge \frac{2k^n}{(1+k^n)}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)|$$

$$+\frac{4\,k^{n-1}|a_{n-1}|}{(n+1)(1+k^n)}\left[\frac{(k^n-1)}{n} - (k-1)\right] + \frac{2\,k^{n-2}|a_{n-2}|}{1+k^n}$$

$$\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right];\ \text{provied } n > 3. \tag{2.9}$$

And

$$\max_{|z|=k}|P(z)| \ge \frac{2k^n}{(1+k^n)}\max_{|z|=1}|P(z)| + l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)|$$

$$+\frac{4\,k^{n-1}|a_{n-1}|}{(n+1)(1+k^n)}\left[\frac{(k^n-1)}{n} - (k-1)\right] + \frac{2\,k^{n-2}|a_{n-2}|}{1+k^n}\left[\frac{(k-1)^n}{n(n-1)}\right];\ \text{provided } n = 3\,. \tag{2.10}$$

Proof of Lemma 2.7. Since $P(z)$ is a polynomial of degree n has all its zeros in $|z| \leq k, k \geq$ 1,therefore,

all the zeros of $g(z) = P(kz)$ lie in $|z| \leq 1$ and hence all the $f(z) = z^n \overline{g(1/\bar{z})} = z^n \overline{P(k/\bar{z})}$ lie in

$|z| \geq 1$ .Moreover, $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|f(z)|$, so that

$$m\,|z^n| \leq |f(z)| \quad , \qquad \text{for} \quad |z| = 1.$$

We show that for $\lambda \in \mathbb{C}$ with $|\lambda| < 1, f(z) + \lambda\, m\, z^n \neq 0$ in $|z| < 1$ .This is trivially true if $m = 0$. Henceforth, we suppose that $m \neq 0$, so that all the zeros of $f(z)$ lie in $|z| > 1$. By the maximum modulus theorem

$$m\,|z^n| \leq |f(z)| \quad , \qquad \text{for} \quad |z| < 1. \tag{2.11}$$

Now, if there is point $z = z_0$ with $|z_0| < 1$, such that $f(z_0) + \lambda\, m\, {z_0}^n = 0$, then

$$|f(z_0)| = |\lambda||{z_0}^n|m < |{z_0}^n|m \ ,$$

a contradiction to inequality (2.11). Hence, it follows that the polynomial $T(z) = f(z) + \lambda\, m\, z^n$ dose not vanish in $|z| < 1$. Applying inequality (2.6) of Lemma 2.6 to the polynomial $T(z)$, with $R = k \geq 1$ and $n > 3$, we get for $|z| = 1$,

$$|f(kz) + \lambda\, m\, k^n\, z^n| \leq \frac{(k^n+1)}{2}|f(z) + \lambda\, m\, z^n| - \frac{2\,|f'(0)|}{(n+1)}\left[\frac{(k^n-1)}{n} - (k-1)\right]$$
$$-|f''(0)|\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right].$$

Which implies, for $n > 3$

$$|f(kz) + \lambda\, m\, k^n\, z^n| \leq \frac{(k^n+1)}{2}(|f(z)| + |\lambda|m) - \frac{2\,|f'(0)|}{(n+1)}\left[\frac{(k^n-1)}{n} - (k-1)\right]$$
$$-|f''(0)|\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right]. \tag{2.12}$$

Choosing argument of λ suitably in the left hand side of inequality (2.12), we get for $n > 3$

$$|f(kz)| + |\lambda|mk^n \leq \frac{(k^n+1)}{2}(|f(z)| + |\lambda|m) - \frac{2k^{n-1}|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n} - (k-1)\right]$$
$$- k^{n-2}\,|a_{n-2}|\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right].$$

Replacing $f(z)$ by $z^n\,\overline{P(k/\bar{z})}$, we obtain for $n > 3$ and $|z| = 1$

$$k^n \max_{|z|=1}|P(z)| + |\lambda|m\, k^n$$
$$\leq \frac{(k^n+1)}{2}\left(\max_{|z|=k}|P(z)| + |\lambda|m\right) - \frac{2\,k^{n-1}|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n} - (k-1)\right]$$
$$-k^{n-2}|a_{n-2}|\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right].$$

Which on simplification yields inequality (2.9). In a similar manner, we can prove inequality (2.10) by applying inequality (2.7) of Lemma 2.6 instead of inequality (2.6) to the polynomial $T(z)$. This proves Lemma 2.7.

Lemma 2.8. If $P(z)$ is a polynomial of degree n and α is any real or complex number, then on $|z| = 1$,

$$\max_{|z|=1}|D_\alpha q(z)| \le n\,(|\alpha|+1)\max_{|z|=1}|P(z)| - \max_{|z|=1}|D_\alpha P(z)|. \tag{2.13}$$

where $q(z) = z^n\,\overline{P(1/\bar{z})}$.

The above Lemma due to Chanam el at. [7]. The following Lemma is due to Kumar [13].

Lemma 2.9. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for any complex α with $1+k+k^n \ge |\alpha| \ge k$, satisfying $\max_{|z|=1}|P(z)| \ge \left(\frac{1+k^n}{|\alpha|-k}\right)\min_{|z|=k}|P(z)|$, we have

$$\max_{|z|=1}|D_\alpha P(z)| \ge n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|P(z)| + n\left(\frac{(1+k+k^n)-|\alpha|}{1+k^n}\right)\min_{|z|=k}|P(z)|. \tag{2.14}$$

The next lemma is due to [17].

Lemma 2.10. If $P(z)$ is a polynomial of degree n, then for $R \ge 1$

$$\max_{|z|=R}|P(z)| \le R^n \max_{|z|=1}|P(z)|. \tag{2.15}$$

Lemma 2.11. If $P \in \mathcal{P}_n$ has all its zeros in $|z| \le k$, $k \le 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right| \ge \frac{\Lambda}{(1+k)}\,(|\alpha|-k)\max_{|z|=1}|P(z)|. \tag{2.16}$$

The above Lemma due Rather el at. [18].

3. Main Results and Proofs

In this paper, we obtain certain refinements and generalizations of inequalities (1.3), (1.5), (1.6), (1.10) and (1.11).

We first prove the following result which is a generalization of Rather el at.[20] to polar derivative of
$P(z)$.

Theorem 3.1. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \ge 3$ having all its zeros in $|z| \le k, k \ge 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \ge k$

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{(|\alpha|-k)}{(1+k^n)}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)$$

$$\left[\max_{|z|=1}|P(z)|+\frac{2\,|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\}+\frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right.\right.$$

$$\left.\left.-\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]+\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}\,|na_0+\alpha\, a_1|$$

$$+\frac{1}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)}-\frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1+2\alpha\, a_2|\;;\text{ provided } n>3\,. \quad (3.1)$$

And

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{(|\alpha|-k)}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)$$

$$\left[\max_{|z|=1}|P(z)|+\frac{2|a_{n-1}|}{k(n+1)}+\frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\}+\frac{|a_{n-2}|}{k^2}\frac{(k-1)^n}{n(n-1)}\right]$$

$$+\frac{k-1}{2\,k^{n-1}}\,[\,(k+1)|na_0+\alpha\, a_1|+(k-1)|(n-1)a_1+2\alpha\, a_2|\,]\;;\text{ provided } n=3. \quad (3.2)$$

Proof. Let $f(z)=P(kz)$. Since $P(z)$ is a polynomial of degree n has all its zeros in $|z|\leq k, k\geq 1$, therefore, $f(z)$ is a polynomial of degree n has all its zeros in $|z|\leq 1$. If $Q(z)= z^n\,\overline{f(1/\bar{z})}$, then for $|z|=1$

$$|Q'(z)|=|n\,f(z)-z\,f'(z)| \quad (3.3)$$

Combining (3.3) with Lemma 2.2 , we get for $|z|=1$

$$|f'(z)|\geq|n\,f(z)-z\,f'(z)| \quad (3.4)$$

Now, every $\alpha\in\mathbb{C}$ with $|\alpha|\geq k$, we have for $|z|=1$

$$\left|D_{\alpha/k}f(z)\right|=\left|n\,f(z)-(\frac{\alpha}{k}-z)\,f'(z)\right|\geq\left|\frac{\alpha}{k}\right|\,|f'(z)|-|n\,f(z)-z\,f'(z)|$$

Which gives with the help of (3.4)

$$\left|D_{\alpha/k}f(z)\right|\geq\frac{|\alpha|-k}{k}|f'(z)| \quad (3.5)$$

Consequently,

$$\max_{|z|=k}|D_\alpha P(z)|\geq(|\alpha|-k)\max_{|z|=k}|P'(z)| \quad (3.6)$$

Again, since all the zeros of $f(z)=P(kz)$ lie in $|z|\leq 1$, therefore, using Lemma 2.1 , we have for $|z|=1$

$$|f'(z)|\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|f(z)|$$

Replacing $f(z)$ by $P(kz)$, we obtain

$$\max_{|z|=k}|P'(z)|\geq\frac{1}{2k}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)| \quad (3.7)$$

Combining inequality (3.6) and inequality (3.7), we have

$$\max_{|z|=k}|D_\alpha P(z)|\geq\frac{(|\alpha|-k)}{2k}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)| \quad (3.8)$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $(n-1)$, using inequality (2.3) of Lemma 2.4, we have for $n > 3$

$$\max_{|z|=R}|D_\alpha P(z)| \le R^{n-1}\max_{|z|=1}|D_\alpha P(z)| - \frac{2(R^{n-1}-1)}{(n+1)}|n\,a_0+\alpha\,a_1| - \left[\frac{(R^{n-1}-1)}{(n-1)} - \frac{(R^{n-3}-1)}{(n-3)}\right]|(n-1)a_1+2\alpha a_2|$$

Using this inequality and inequality (2.9) of Lemma 2.7 with $l = 0$ and $R = k \ge 1$ in (3.8) , we have for
$n > 3$

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - \frac{2(k^{n-1}-1)}{(n+1)}|na_0+\alpha\,a_1| - \left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1+2\alpha\,a_2|$$
$$\ge \frac{(|\alpha|-k)}{2\,k}\left(n+\frac{k^n|a_n|-|\,a_0|}{k^n|a_n|+|\,a_0|}\right)\Big[\frac{2k^n}{(k^n+1)}\max_{|z|=1}|P(z)| + \frac{4\,k^{n-1}\,|a_{n-1}|}{(n+1)(k^n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\} + \frac{2\,k^{n-2}\,|a_{n-2}|}{(k^n+1)}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\Big],$$

Which is equivalent to

$$\max_{|z|=1}|D_\alpha P(z)| \ge \frac{(|\alpha|-k)}{(1+k^n)}\left(n+\frac{k^n|a_n|-|\,a_0|}{k^n|a_n|+|\,a_0|}\right)\left[\max_{|z|=1}|P(z)| + \frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\} + \frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right] + \frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0+\alpha\,a_1| + \frac{1}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1+2\alpha\,a_2|$$

The above inequality is equivalent to the inequality (3.1) for $n > 3$. For $n = 3$, the result follows on similar lines by using inequality (2.4) of Lemma 2.4 and inequality (2.10) of Lemma 2.7 in the inequality (3.8) . This completes the proof of Theorem 3.1.

If we divide both sides of inequalities (3.1) and (3.2) by $|\alpha|$ and let $|\alpha| \to \infty$, we get

Corollary 3.1. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \ge 3$ having all its zeros in $|z| \le k, k \ge 1$, then

$$\max_{|z|=1}|P'(z)| \ge \frac{1}{(1+k^n)}\left(n+\frac{k^n|a_n|-|\,a_0|}{k^n|a_n|+|\,a_0|}\right)$$

$$\left[\max_{|z|=1}|P(z)| + \frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n} - (k-1)\right\} + \frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right] + \frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|a_1| + \frac{2}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|a_2|\text{; provided } n > 3. \quad (3.9)$$

And

$$\max_{|z|=1}|P^{\prime}(z)| \geq \frac{1}{(1+k^n)}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right)\left[\max_{|z|=1}|P(z)| + \frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n} - (k-1)\right\} + \frac{|a_{n-2}|}{k^2}\frac{(k-1)^2}{n(n-1)}\right] + \frac{(k-1)}{2\,k^{n-1}}\{(k+1)|a_1| + 2(k-1)|a_2|\}\text{; provided } n = 3. \quad (3.10)$$

Remark 3.1. For $k = 1$ in inequalities (3.9) and (3.10), we get inequality (1.7).

Next , we prove the following result which is a generalization of Theorem 3.1.

Theorem 3.2. If $P(z) = \sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k, k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$, $0 \leq l < 1$ and $m = \min_{|z|=k}|P(z)|$

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n}{(1+k^n)}\left[(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha| + 1/_{k^{n-1}})\, l\, m\right] + \frac{(|\alpha|-k)}{k^n(1+k^n)}\left(\frac{k^n|a_n| - l\,m - |a_0|}{k^n|a_n| - l\,m + |a_0|}\right)\left[k^n\max_{|z|=1}|P(z)| - l\,m\right] + \frac{(|\alpha|-k)}{(1+k^n)}\left(n + \frac{k^n|a_n| - l\,m - |a_0|}{k^n|a_n| - l\,m + |a_0|}\right)\left[\frac{2\,|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n} - (k-1)\right\} + \frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right] + \frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0 + \alpha\, a_1| + \frac{1}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1 + 2\alpha a_2|\text{; provided } n > 3\,. \quad (3.11)$$

And

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n}{(1+k^n)}\left[(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha| + 1/_{k^{n-1}})\, l\, m\right] + \frac{(|\alpha|-k)}{k^n(1+k^n)}\left(\frac{k^n|a_n| - l\,m - |a_0|}{k^n|a_n| - l\,m + |a_0|}\right)\left[k^n\max_{|z|=1}|P(z)| - l\,m\right] + \frac{(|\alpha|-k)}{(1+k^n)}\left(n + \frac{k^n|a_n| - l\,m - |a_0|}{k^n|a_n| - l\,m + |a_0|}\right)\left[\frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n} - (k-1)\right\} + \frac{|a_{n-2}|(k-1)^n}{k^2\,n(n-1)}\right] + \frac{(k-1)}{(2\,k^{n-1})}\{(\mathrm{k}+1)|na_0 + \alpha\, a_1| + (k-1)|(n-1)a_1 + 2\alpha\, a_2|\}\text{; provided } n = 3. \quad (3.12)$$

Proof. Since $P(z)$ is a polynomial of degree n has all its zeros in $|z| \le k, k \ge 1$. If $P(z)$ has a zero on

$|z| = k$, then $m = \min_{|z|=k} |P(z)| = 0$ and result follows Theorem 3.1. Henceforth, we suppose that $P(z)$ has all its zeros in $|z| \le k$, $k \ge 1$, so that $m > 0$. Now, if $f(z) = P(kz)$, then $f(z)$ is a polynomial, of degree n has all its zeros in $|z| \le 1$ and $m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |f(z)|$.

This implies

$$m \le |f(z)| \ , \quad \text{for} \quad |z| = 1.$$

From Rouche´s Theorem , we conclude that for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, the polynomial $g(z) = f(z) - \lambda\, m\, z^n$ has all its zeros in $|z| \le 1$. Applying (3.5) to the polynomial $g(z)$, it follows for

$|z| = 1$ and $|\alpha| \ge k$

$$\left|D_{\alpha/k} g(z)\right| \ge \frac{|\alpha| - k}{k} \left|g'(z)\right| .$$

Since all the zeros of $g(z)$ lie in $|z| < 1$, using Lemma 2.1, we obtain for $|z| = 1$ and $|\alpha| \ge k$

$$\left|D_{\alpha/k} g(z)\right| \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{|\,k^n a_n - \lambda\, m| - |a_0|}{|\,k^n\, a_n - \lambda\, m| + |a_0|}\right)|g(z)| .$$

Using the fact that the function $t(x) = \frac{x - |a|}{x + |a|}$, $x > 0$ is non-decreasing function of x and $|k^n\, a_n - \lambda\, m\,| \ge k^n\, |a_n| - |\lambda\, m|$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$,

$$\left|D_{\alpha/k} g(z)\right| \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n |\,a_n| - |\lambda| m - |a_0|}{k^n |\,a_n| - |\lambda| m + |a_0|}\right)|g(z)| . \tag{3.13}$$

Replacing $g(z)$ by $f(z) - \lambda\, m\, z^n$ in (3.13), we get for $|z| = 1$ and $|\alpha| \ge k$

$$\left|D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k} z^{n-1}\right| \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\lambda| m - |a_0|}{k^n|a_n| - |\lambda| m + |a_0|}\right) \ (|f(z) - \lambda\, m|\,) \tag{3.14}$$

Since all the zeros of $f(z) - \lambda\, m\, z^n = g(z)$ lie in $|z| < 1$ and $\left|\alpha/k\right| \ge 1$, it follows from Lemma 2.3 that all the zeros of

$$D_{\alpha/k}(f(z) - m\, \lambda\, z^n) = D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k}\, z^{n-1} \quad ,$$

lie in $|z| < 1$. This implies that

$$|D_{\alpha/k} f(z)| \ge \frac{nm|\alpha||\lambda|}{k}\, |z|^{n-1} \quad , \qquad \text{for} \quad |z| \ge 1 \tag{3.15}$$

In view of this inequality, choosing argument of λ in the left hand side of inequality (3.14) such that

$$\left|D_{\alpha/k} f(z) - \frac{nm\alpha\lambda}{k}\, z^{n-1}\right| = |D_{\alpha/k} f(z)| - \frac{nm|\alpha||\lambda|}{k} \quad , \qquad \text{for } |z| = 1$$

we get for $|z| = 1$ and $|\alpha| \ge k$

$$|D_{\alpha/k} f(z)| - \frac{nm|\alpha||\lambda|}{k} \ge \frac{1}{2}\left(\frac{|\alpha| - k}{k}\right)\left(n + \frac{k^n|a_n| - |\lambda| m - |a_0|}{k^n|a_n| - |\lambda| m + |a_0|}\right) (|f(z)| - |\lambda| m)$$

Which on simplification yields

$$|D_{\alpha/k}f(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|f(z)|$$
$$-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|\lambda|m+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m$$

This implies for $|z|=1$ and $|\alpha| \geq k$

$$\max_{|z|=k}|D_\alpha P(z)| \geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)\max_{|z|=k}|P(z)|$$
$$-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)|\lambda|m+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)|\lambda|m \quad (3.16)$$

Since $D_\alpha P(z)$ is a polynomial of degree at most $(n-1)$, applying inequality (2.3) of Lemma 2.4 and inequality (2.9) of Lemma 2.7 with $R=k \geq 1$, we obtain for $|\alpha| \geq k$, $0 \leq l < 1$ and $|z|=1$

$$k^{n-1}\max_{|z|=1}|D_\alpha P(z)| - \frac{2(k^{n-1}-1)}{(n+1)}|na_0+\alpha\, a_1| - \left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1$$
$$+2\alpha\, a_2|$$
$$\geq \frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(n+\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)$$
$$\{\frac{2k^n}{(1+k^n)}\max_{|z|=1}|P(z)| + \left(\frac{k^n-1}{k^n+1}\right)l\,m + \frac{4\,k^{n-1}|a_{n-1}|}{(n+1)(1+k^n)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$+\frac{2k^{n-2}|a_{n-2}|}{(1+k^n)}\left[\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right]\}$$
$$-\frac{1}{2}\left(\frac{|\alpha|-k}{k}\right)\left(\frac{k^n|a_n|-|\lambda|m-|a_0|}{k^n|a_n|-|\lambda|m+|a_0|}\right)l\,m+\frac{n}{2}\left(\frac{|\alpha|+k}{k}\right)l\,m\ , \qquad \text{for} \quad n>3$$

Equivalently, we have for $|\alpha| \geq k$, $0 \leq l < 1$ and $|z|=1$

$$\max_{|z|=1}|D_\alpha P(z)| \geq \frac{n}{(1+k^n)}\left[(|\alpha|-k)\max_{|z|=1}|P(z)| + (|\alpha| + {}^{1}/_{k^{n-1}})\,l\,m\right] +$$
$$\frac{(|\alpha|-k)}{k^n(1+k^n)}\left(\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)[k^n\max_{|z|=1}|P(z)|-l\,m] + \frac{(|\alpha|-k)}{(1+k^n)}$$
$$\left(n+\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)\left[\frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\}\right.$$
$$\left.+\frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)} - \frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]$$
$$+\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)} - \frac{(k^{n-3}-1)}{(n-3)}\right\}|(n-1)a_1+2\alpha a_2|$$

That proves the inequality (3.11) for $n>3$. For the case $n=3$, the result follows on similar lines by using inequality (2.4) of Lemma 2.4 and inequality (2.10) of Lemma 2.7 in the inequality (3.16). This complete the proof of Theorem 3.2.

If we divide both sides of inequalities (3.11) and (3.12) by $|\alpha|$ and let $|\alpha| \to \infty$, we get

Corollary 3.2. If $P(z)=\sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{(1+k^n)}\left[\max_{|z|=1}|P(z)|+l\,m\right]+\frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)$$

$$[k^n\max_{|z|=1}|P(z)|-l\,m]+\frac{1}{(1+k^n)}\left(n+\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)$$

$$\left[\frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\}\right.$$

$$\left.+\frac{|a_{n-2}|}{k^2}\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}-\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]+$$

$$\frac{2(k^{n-1}-1)}{k^{n-1}(n+1)}|a_1|+\frac{2}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{(n-1)}-\frac{(k^{n-3}-1)}{(n-3)}\right\}|a_2|;\text{provided } n>3. \quad (3.17)$$

And

$$\max_{|z|=1}|P'(z)| \geq \frac{n}{(1+k^n)}\left[\max_{|z|=1}|P(z)|+l\,\mathrm{m}\right]$$

$$+\frac{1}{k^n(1+k^n)}\left(\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)[k^n\max_{|z|=1}|P(z)|-l\,m]$$

$$+\frac{1}{(1+k^n)}\left(n+\frac{k^n|a_n|-l\,m-|a_0|}{k^n|a_n|-l\,m+|a_0|}\right)\left[\frac{2|a_{n-1}|}{k(n+1)}\left\{\frac{(k^n-1)}{n}-(k-1)\right\}+\frac{|a_{n-2}|(k-1)^n}{k^2\,n(n-1)}\right]$$

$$+\frac{(k-1)}{(2\,k^{n-1})}\{(\mathrm{k}+1)|a_1|+2\,(k-1)|a_2|\};\quad \text{provided} \quad n=3. \quad (3.18)$$

Where $m=\min_{|z|=k}|P(z)|$.

Theorem 3.3. If $P(z)=\sum_{v=o}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k, k \leq 1$, then for every $\alpha \in \mathbb{C}$ with $1+\frac{1}{k}+\frac{1}{k^n} \geq |\alpha| \geq \frac{1}{k}$;satisfying $\max_{|z|=1}|P(z)| \geq \frac{(k^n+1)\,m}{k^{2n-1}\,(k\,|\alpha|-1)}$, we have

$$\max_{|z|=1}|D_\alpha P(z)| \leq n\left(\frac{|\alpha|+k^n+k^{n-1}+1}{(k^n+1)}\right)\max_{|z|=1}|P(z)|-n\left(\frac{(k^n+k^{n-1}+1)-k^n|\alpha|}{k^n\,(k^n+1)}\right)m\,,$$

$$\text{where } m=\min_{|z|=k}|P(z)|. \quad (3.19)$$

Provided $|D_\alpha P(z)|$ and $|D_\alpha q(z)|$ attain their maxima at the same point on the circle $|z|=1$, where $q(z)=z^n\overline{P(1/\bar{z})}$.

Proof. Since $P(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k \leq 1$ and hence all the zeros of $q(z)=z^n\overline{P(1/\bar{z})}$ lie in $|z| \leq 1/k$, $1/k \geq 1$. Applying (2.14) of Lemma 2.9 on $q(z)$, for $1+\frac{1}{k}+\frac{1}{k^n} \geq |\alpha| \geq \frac{1}{k}$, satisfying $\max_{|z|=1}|q(z)| \geq \frac{1+\frac{1}{k^n}}{|\alpha|-\frac{1}{k}}\min_{|z|=\frac{1}{k}}|q(z)|$, we have

$$\max_{|z|=1}|D_\alpha q(z)| \geq n\left(\frac{|\alpha|-\frac{1}{k}}{1+\frac{1}{k^n}}\right)\max_{|z|=1}|q(z)| + n\left(\frac{\left(1+\frac{1}{k}+\frac{1}{k^n}\right)-|\alpha|}{1+\frac{1}{k^n}}\right)\min_{|z|=\frac{1}{k}}|q(z)|$$

Which is equivalent

$$\max_{|z|=1}|D_\alpha q(z)| \geq n\,k^{n-1}\left(\frac{k\,|\alpha|-1}{k^n+1}\right)\max_{|z|=1}|q(z)| + n\left(\frac{(k^n+k^{n-1}+1)-k^n|\alpha|}{k^n+1}\right)\min_{|z|=\frac{1}{k}}|q(z)|$$

Since on $|z|=1, |P(z)|=|q(z)|$ and $\min_{|z|=1/k}|q(z)| = \frac{1}{k^n}\min_{|z|=k}|P(z)|$.

$$\max_{|z|=1}|D_\alpha q(z)| \geq n\,k^{n-1}\left(\frac{k\,|\alpha|-1}{k^n+1}\right)\max_{|z|=1}|P(z)| + n\left(\frac{(k^n+k^{n-1}+1)-k^n|\alpha|}{k^n\,(k^n+1)}\right)m\,,$$

where $m=\min_{|z|=k}|P(z)|$ (3.20)

From Lemma 2.8 , we have on $|z|=1$

$$|D_\alpha P(z)| + |D_\alpha q(z)| \leq n\,(|\alpha|+1)\max_{|z|=1}|P(z)| \tag{3.21}$$

where $q(z)=z^n\,\overline{P(1/\bar{z})}$.

Let z_0 be a point on the unit circle such that $\max_{|z|=1}|D_\alpha q(z)| = |D_\alpha q(z_0)|$. Since $|D_\alpha P(z)|$ and $|D_\alpha q(z)|$
attain their maxima at the same point on $|z|=1$ with $|\alpha|\geq\frac{1}{k}$, we have $\max_{|z|=1}|D_\alpha P(z)| = |D_\alpha P(z_0)|$.

Thus, in particular (3.21) gives

$$\max_{|z|=1}|D_\alpha q(z)| \leq n\,(|\alpha|+1)\max_{|z|=1}|P(z)| - \max_{|z|=1}|D_\alpha P(z)| \tag{3.22}$$

Combining (3.22) with (3.20), we have

$$n\,(|\alpha|+1)\max_{|z|=1}|P(z)| - \max_{|z|=1}|D_\alpha P(z)| \geq n\,k^{n-1}\left(\frac{k\,|\alpha|-1}{k^n+1}\right)\max_{|z|=1}|P(z)| + n\left(\frac{(k^n+k^{n-1}+1)-k^n|\alpha|}{k^n\,(k^n+1)}\right)m\ .$$

Which on simplification gives

$$\max_{|z|=1}|D_\alpha P(z)| \leq n\left(\frac{|\alpha|+k^n+k^{n-1}+1}{k^n+1}\right)\max_{|z|=1}|P(z)| - n\left(\frac{(k^n+k^{n-1}+1)-k^n|\alpha|}{k^n\,(k^n+1)}\right)m\,.$$

Which is equivalent to Theorem 3.3.

Dividing both sides of Theorem 3.3 by $|\alpha|$ and taking $|\alpha|\to\infty$, we get

Corollary 3.3. If $P(z)=\sum_{v=o}^{n} a_v z^v$ is a polynomial of degree n having no zeros in $|z|<k, k\leq 1$, then satisfying $\max_{|z|=1}|P(z)| \geq \frac{(k^n+1)\, m}{(k^{2n})}$, we have

$$\max_{|z|=1}\left|P'(z)\right| \leq \left(\frac{n}{k^n+1}\right)\left[\max_{|z|=1}|P(z)|-m\right], \text{where} \quad m=\min_{|z|=k}|P(z)|. \tag{3.23}$$

Provided $\left|P'(z)\right|$ and $\left|q'(z)\right|$ attain their maxima at the same point on $|z|=1$, where $q(z)=z^n\, \overline{P\left(1/\bar{z}\right)}$.

Now, we improve a result recently proved by Rather el at.[17].

Theorem 3.4. If $P(z)=\sum_{v=o}^{n} a_v z^v$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z|\leq k, k\geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha|\geq k$

$$\max_{|z|=1}\left|D_\alpha^\gamma[P](z)\right| \geq \Lambda\left(\frac{|\alpha|-k}{1+k^n}\right)\{\max_{|z|=1}|P(z)|+\frac{2\,|a_{n-1}|}{k\,(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$+\frac{|a_{n-2}|}{k^2}\left[\frac{(k^n-1)-n(k-1)}{n(n-1)}-\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right]\}; \text{provided } n>3. \tag{3.24}$$

And

$$\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right|$$
$$\geq \Lambda\left(\frac{|\alpha|-k}{1+k^n}\right)\{\max_{|z|=1}|P(z)|+\frac{2\,|a_{n-1}|}{k\,(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$+\frac{|a_{n-2}|}{k^2}\left[\frac{(k-1)^n}{n(n-1)}\right];$$
$$\text{provided} \quad n=3. \tag{3.25}$$

Proof. Since the polynomial $P(z)$ has all its zeros in $|z|\leq k$, where $k\geq 1$. Hence, the polynomial

$F(z)=P(kz)$ has all its zeros in $|z|\leq 1$. We take $k=1$, in inequality (2.16) of Lemma 2.11, we get

If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha|\geq 1$

$$\max_{|z|=1}\left|D_\alpha^\gamma\,[P](z)\right| \geq \frac{\Lambda}{2}\,(|\alpha|-1)\,\max_{|z|=1}|P(z)| \tag{3.26}$$

Applying inequality (3.26) to the polynomial $F(z)$ and noting that $\left|\frac{\alpha}{k}\right|\geq 1$, we get

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^\gamma\,[F](z)\right| \geq \frac{\Lambda}{2}\left(\frac{|\alpha|}{k}-1\right)\max_{|z|=1}|F(z)|$$

Replacing $F(z)$ by $P(kz)$, we get

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](k\,z)\right| \geq \frac{\Lambda}{2}\left(\frac{|\alpha|}{k}-1\right)\max_{|z|=1}|P(k\,z)|$$

$$=\frac{\Lambda}{2}\left(\frac{|\alpha|-k}{k}\right)\max_{|z|=k}|P(\,z)|\,. \qquad (3.27)$$

With the help inequality (2.9) of Lemma 2.7 where $l=0$, this implies

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](k\,z)\right| \geq \frac{\Lambda}{2}\left(\frac{|\alpha|-k}{k}\right)\max_{|z|=k}|P(\,z)|$$

$$\geq\frac{\Lambda}{2}\left(\frac{|\alpha|-k}{k}\right)\Big\{\frac{2k^n}{(1+k^n)}\max_{|z|=1}|P(z)|+\frac{4\,k^{n-1}|a_{n-1}|}{(n+1)(1+k^n)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$

$$+\frac{2\,k^{n-2}|a_{n-2}|}{1+k^n}\left[\frac{(k^n-1)-n(k-1)}{n(n-1)}-\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right]\Big\}$$

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](k\,z)\right| \geq \Lambda\,k^{n-1}\left(\frac{|\alpha|-k}{1+k^n}\right)$$

$$\left\{\max_{|z|=1}|P(z)|+\frac{2\,|a_{n-1}|}{k\,(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]\right.$$

$$+\frac{|a_{n-2}|}{k^2}\left[\frac{(k^n-1)-n(k-1)}{n(n-1)}\right.$$

$$\left.\left.-\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right]\right\}. \qquad (3.28)$$

Also, we have

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](kz)\right| = \max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha}{k}-z\right)\,P^{\gamma}(k\,z)\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha-k\,z}{k}\right)\,P(k\,z)\sum_{j=1}^{n}\frac{\gamma_j}{z-\frac{z_j}{k}}\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+\left(\frac{\alpha-k\,z}{k}\right)k\,P(k\,z)\sum_{j=1}^{n}\frac{\gamma_j}{k\,z-z_j}\right|$$

$$=\max_{|z|=1}\left|\Lambda\,P(k\,z)+(\alpha-k\,z)\,\,P(k\,z)\sum_{j=1}^{n}\frac{\gamma_j}{k\,z-z_j}\right|$$

$$=\max_{|z|=1}|G(kz)|$$

$$=\max_{|z|=k}|G(z)|\;\;,$$

where $\mathrm{G}(z)=\Lambda\,P(\,z)+(\alpha-\,z)\,\,P(\,z)\,\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}$ is a polynomial of degree at most $(n-1)$. On using
inequality (2.15), this gives

$$\max_{|z|=1}\left|D^{\gamma}_{\frac{\alpha}{k}}[P](kz)\right| = \max_{|z|=k}|G(z)|$$

$$\leq\; k^{n-1}\max_{|z|=1}|G(z)|$$

$$= k^{n-1} \max_{|z|=1}\left|\Lambda P(z) + (\alpha - z) P(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - z_j}\right|$$
$$= k^{n-1} \max_{|z|=1}\left|D_{\alpha}^{\gamma} [P](z)\right| .$$

Hence,

$$\max_{|z|=1}\left|D_{\frac{\alpha}{k}}^{\gamma} [P](kz)\right| \leq k^{n-1} \max_{|z|=1}\left|D_{\alpha}^{\gamma} [P](z)\right| \tag{3.29}$$

$$k^{n-1} \max_{|z|=1}\left|D_{\alpha}^{\gamma} [P](z)\right| \geq \Lambda k^{n-1} \left(\frac{|\alpha| - k}{1 + k^n}\right)$$
$$\left\{\max_{|z|=1}|P(z)| + \frac{2\,|a_{n-1}|}{k\,(n+1)} \left[\frac{k^n - 1}{n} - (k-1)\right]\right.$$
$$\left. + \frac{|a_{n-2}|}{k^2}\left[\frac{(k^n - 1) - n(k-1)}{n(n-1)} - \frac{(k^{n-2} - 1) - (n-2)(k-1)}{(n-2)(n-3)}\right]\right\}.$$

This implies,

$$\max_{|z|=1}\left|D_{\alpha}^{\gamma} [P](z)\right| \geq \Lambda\left(\frac{|\alpha| - k}{1 + k^n}\right)\left\{ \max_{|z|=1}|P(z)| + \frac{2\,|a_{n-1}|}{k\,(n+1)}\left[\frac{k^n - 1}{n} - (k-1)\right]\right.$$
$$\left. + \frac{|a_{n-2}|}{k^2}\left[\frac{(k^n - 1) - n(k-1)}{n(n-1)} - \frac{(k^{n-2} - 1) - (n-2)(k-1)}{(n-2)(n-3)}\right]\right\}.$$

That proves the inequality (3.24) for $n > 3$. For the case $n = 3$, the result follows on similar lines by using inequality (2.10) of Lemma 2.7 in the inequality (3.27). This complete the proof of Theorem 3.4.

Remark 3.2. For $k = 1$ in inequalities (3.24) and (3.25), we get inequality (3.26).

References:

1. [1] Aziz , A. (1983). Inequalities for the derivative of a polynomial, Proc. of the Amer. Math. Soc. 89, 259-266.
2. Aziz , A. and Dawood , Q.M. (1988). Inequalities for a Polynomial and its derivative, J. Approx. Theory, 54, 306-313.
3. Aziz , A. and Rather , N.A. (1998). A refinement of a theorem of Paul Turan concerning Polynomials, J. Math. Ineq. Appl.,1, pp.231-238.
4. Aziz , A. and Rather, N.A. (2004). Some Zygmund type L^q inequalities for polynomials , J. Math. Anal. Appl. 289 ,14-29.
5. Aziz , A. and Rather, N.A. (2003). Inequalities for the polar derivative of a polynomial with restricted zeros, Math. Balkanica 17 , 15-28.

6. Bernstein , S. (1926). Lecons sur les proprie'te's extremales et la meilleure dune functions rella. Paris.
7. Chanam ,B., Devi ,K.B. and Krishnadas ,K. (2021). Some inequalities concerning , Polar derivative of a polynomial, IJPAMS. ISSN 0972-9828 Vol.14, No.1, PP. 9-19.
8. Dewan , K.K. , Kaur , J. and Mir, A. (2002). Inequalities for the derivative of a Polynomial , J. Math. Anal. Appl. 269, pp. 489-499.
9. Dubinin, V.N. (2007). Applications of the Schwarz Lemma to inequalities for entire functions with constraints on zeros, J. Math. Sci. 143 , 3069-3076.
10. [10] Erd*ö*s, P. (1940). On extremal properties of derivatives of polynomials, Ann. Math. 41, 310-313.
11. Govil, N .K. (1973). On the derivative of a polynomial, Proc. Amer. Math. Soc. 41, 543 – 546.
12. Govil, N .K. (1980). On the Theorem of S. Bernstein, Proc. Nat. Acad. Sci., 50, 50-52.
13. Kumar, P. (2019). Some integral inequalities for the polar derivative of polynomials ,Publications
14. De L'institut Mathematique Nouvelle série , tome 106 (120), 85–94. Laguerre, E. Q Euvres. Vol. pp. 48–66. Chelsea, New York, 2.
15. Lax, P.D. (1944). Proof of a conjunction of P. Erd¨os on the derivative of a Polynomial. Bull. Amer.
16. Mathematics Soc. 50, 509–513. Marden , M. (1966). Geometry of Polynomial, Math. Survey No. 3, Amer. Math. Soc., Providence, RI.
17. Polya , G. and Szegö , G. (1945). Aufgaben and Lehrsatze aus der Analysis, Vol. 1, p.137, Problem 269.
18. Rather, N.A., Ali ,L., Shafi, M. and Dar,I. (2022). Inequalities for the generalized polar derivative of a polynomial ,Palestine Journal of Mathematics , Vol. 11 (3), 549-557.
19. Rather ,N.A. and Dar, I.A. (2020). Some Applications of the boundary Schwarz Lemma for polynomials with restricted zeros, Appl. Math. E-Notes, 20 , 422- 431.
20. Rather , N.A. , Dar, I.A. and Iqbal , A. (2020). Inequalities for the derivative of Polynomial with restricted zeros, Korean J. Math. 28 , No. 4, PP.931-924.
21. Turnan , P. (1939) . Uberdie Ableitung Compositio Math. 7 , 89 – 95.

المتباينات للمشتقة القطبية والمشتقة القطبية المعممة لمتعددات الحدود المركبة بأصفار مقيدة

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الملخص

في هذا البحث نقدم بعض النتائج الجديدة المتعلقة بالمقياس الأقصى للمشتقة القطبية و المشتقة القطبية المعممة لمتعددة الحدود بأصفار مقيدة تم الحصول عليها . هذه التقديرات تضاف لبعض المتباينات المعروفة لمتعددات الحدود لكل من توران، راذر و دار، راذر، علي، شافي و دار وآخرين.

الكلمات المفتاحية: متعددة الحدود، مشتقة قطبية، مشتقة قطبية معممة، متباينات في نطاق المركب.