



Research Article

Some Inequalities Concerning Maximum Modulus of Complex Polynomials with Restricted Zeros

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Abstract

In this paper, certain new results concerning the maximum modulus of polynomials with restricted zeros are obtained. These estimates strengthen some well-known inequalities for polynomial due to Rivlin, Govil, Lal and others.

1. Introduction

Let $P(z)$ be a polynomial of degree n , let us define and denote $m = m(P, k) = \min_{|z|=k} |P(z)|$ and $M(P, r) = \max_{|z|=r} |P(z)|$.

For $P(z)$ be a polynomial of degree, it is known that

$$M(P, r) \geq r^n M(P, 1) \quad \text{for} \quad 0 < r \leq 1 \quad (1.1)$$

Inequality (1.1) is due to Varga [1] who attributed it to Zarantonello.

It is noted that equality holds in (1.1) if and only if (z) has all its zeros at the origin.

It was shown by Rivlin [2] that $P(z)$ has no zeros in $|z| < 1$, then (1.1) can be replaced

$$M(P, r) \geq \left(\frac{r+1}{2}\right)^n M(P, 1) \quad \text{for} \quad 0 < r \leq 1 \quad (1.2)$$

As a generalization of (1.2), Govil [3] proved that if $P(z)$ has no zeros in $|z| < 1$, then for $0 < r \leq R \leq 1$

$$M(P, r) \geq \left(\frac{1+r}{1+R}\right)^n M(P, R) \quad (1.3)$$

2. Lemmas

We need the following Lemmas.

Lemma 2.1. [4] If $P(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree n having no zeros in $|z| < k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1+k} \left[\max_{|z|=1} |P(z)| - \min_{|z|=k} |P(z)| \right] \quad (2.1)$$

Lemma 2.2. [5] If $P(z) = a_0 + \sum_{v=\mu}^n a_v z^v, 1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 < r < R \leq 1$

$$M(P, r) \geq \left(\frac{1+r^\mu}{1+R^\mu}\right)^{n/\mu} M(P, R) \quad (2.2)$$

Lemma 2.3. [6] If $P(z) = a_0 + \sum_{v=\mu}^n a_v z^v, 1 \leq \mu \leq n$, be a polynomial of degree n that dose not vanish in $|z| < k, k \geq 1$, then for $1 \leq r < R$

$$M(P, r) \geq M(P, R) - \frac{n}{\mu} \left(\frac{R^n+k^\mu}{1+k^\mu}\right) \{ M(P, 1) - m \} \quad (2.3)$$

3. Main Results and Proofs

We first present the following generalization and refinement of (1.3).

Theorem 3.1. If $P(z) = a_0 + \sum_{v=\mu}^n a_v z^v, 1 \leq \mu \leq n$, is a polynomial of degree n has no zeros in $|z| < 1$, then for $0 < r < R \leq 1$

$$M(P, r) \geq \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}} \left[M(P, R) + n \min_{|z|=1} |P(z)| \ln \left(\frac{1+R}{1+r} \right) \right] \quad (3.1)$$

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Proof. Let $0 < r < R \leq 1$ and $0 \leq \theta < 2\pi$. Then we have

$$|P(Re^{i\theta}) - P(re^{i\theta})| \leq \int_r^R |P'(te^{i\theta})| dt \tag{3.2}$$

If $P(z) \neq 0$, has no zeros in $|z| < 1$, then $P(tz) \neq 0$ has no zeros in $|z| < \frac{1}{t}$.

If $0 < t \leq 1$, then $\frac{1}{t} \geq 1$ and using inequality (2.1), we obtain

$$t |P'(tz)| \leq \frac{nt}{1+t} [\max_{|z|=t} |P(z)| - \min_{|z|=1} |P(z)|] \tag{3.3}$$

Combining (3.2) and (3.3), Let $m(P, 1) = \min_{|z|=1} |P(z)|$

$$|P(Re^{i\theta})| \leq |P(re^{i\theta})| + \int_r^R \frac{n}{1+t} M(P, t) dt - n m(P, 1) \int_r^R \frac{1}{1+t} dt$$

Which implies

$$M(P, R) \leq M(P, r) + \int_r^R \frac{n}{1+t} M(P, t) dt - n m(P, 1) \int_r^R \frac{1}{1+t} dt$$

Now, using inequality (2.2) in above, we get

$$\begin{aligned} M(P, R) &\leq M(P, r) + \int_r^R \frac{n}{1+t} \left(\frac{1+t^\mu}{1+r^\mu}\right)^{n/\mu} M(P, r) dt \\ &\quad - n m \int_r^R \frac{1}{1+t} dt \\ &\leq M(P, r) \\ &\quad + \frac{n M(P, r)}{(1+r^\mu)^{n/\mu}} \int_r^R \frac{(1+t)^{n/\mu}}{(1+t)} dt \\ &\quad - n m \int_r^R \frac{1}{1+t} dt \\ &= M(P, r) \\ &\quad + \frac{\mu M(P, r)}{(1+r^\mu)^{n/\mu}} [(1+R)^{n/\mu} - (1+r)^{n/\mu}] - n m \ln \left(\frac{1+R}{1+r}\right) \end{aligned}$$

Thus, we get

$$M(P, r) \left[\frac{(1+r^\mu)^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}}{(1+r^\mu)^{n/\mu}} \right] \geq M(P, R) + n m \ln \left(\frac{1+R}{1+r}\right)$$

Which is equivalent to

$$\begin{aligned} M(P, r) &\geq \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}} [M(P, R) \\ &\quad + n \min_{|z|=1} |P(z)| \ln \left(\frac{1+R}{1+r}\right)] \end{aligned}$$

Remark 3.1. For $R = 1$ in inequality (3.1), we obtain

$$\begin{aligned} M(P, r) &\geq \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu 2^{n/\mu} - \mu(1+r)^{n/\mu}} [M(P, 1) \\ &\quad + n \min_{|z|=1} |P(z)| \ln \left(\frac{2}{1+r}\right)] \end{aligned}$$

This inequality is due to Govil et al. [7, Theo.2.1, P.3].

Next, we prove the following result which is a generalization of Theorem 3.1.

Theorem 3.2. If $P(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n has no zeros in $|z| < k$, $k \geq 1$, then for $0 < r < R \leq k$

$$\begin{aligned} M(P, r) &\geq \frac{k^{-n(k^\mu+r^\mu)^{n/\mu}}}{k^{-n(k^\mu+r^\mu)^{n/\mu} + \mu k^{-n/\mu(1+R)^{n/\mu} - \mu k^{-n/\mu(k+r)^{n/\mu}}} [M(P, R) + \\ &\quad n \min_{|z|=k} |P(z)| \ln \left(\frac{k+R}{k+r}\right)] \end{aligned} \tag{3.4}$$

Proof. If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then the polynomial $u(z) = P(kz) \neq 0$ for $|z| < 1$. Further, if $0 < r < k$, then $0 < r/k < R/k \leq 1$ and applying (3.1) to $u(z)$, we get

$$\begin{aligned} M(u, r/k) &\geq \frac{(1+(r/k)^\mu)^{n/\mu}}{(1+(r/k)^\mu)^{n/\mu} + \mu(1+R/k)^{n/\mu} - \mu(1+r/k)^{n/\mu}} [M(u, R/k) + \\ &\quad n \min_{|z|=1} |u(z)| \ln \left(\frac{1+R/k}{1+r/k}\right)] \end{aligned} \tag{3.5}$$

Which yields

$$\begin{aligned} M(P, r) &\geq \frac{k^{-n(k^\mu+r^\mu)^{n/\mu}}}{k^{-n(k^\mu+r^\mu)^{n/\mu} + \mu k^{-n/\mu(1+R)^{n/\mu} - \mu k^{-n/\mu(k+r)^{n/\mu}}} [M(P, R) + \\ &\quad n m \ln \left(\frac{k+R}{k+r}\right)] \end{aligned}$$

Now, we prove a result recently proved by Lal [6].

Theorem 3.3. If $P(z) = a_0 + \sum_{v=\mu}^n a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having all its zeros in $|z| \leq k$, $k > 1$, then

$$M(P, k) \geq \frac{1}{B_\mu} \left\{ k^n M(P, 1) + \frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \min_{|z|=k} |P(z)| \right\} \tag{3.6}$$

where

$$B_\mu = \left[1 + \frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \right]$$

Proof. Since $P(z)$ has all its zeros in $|z| \leq k$, therefore, the polynomial $Q(z) = z^n P(1/z)$ has all its zeros in $|z| \geq \frac{1}{k}$ and hence the polynomial $Q\left(\frac{z}{k}\right)$ has all its zeros in $|z| \geq 1$. Applying Lemma 2.3 when $r = 1$ and $k = 1$, we have

$$M(P, 1) \geq M(P, R) - \frac{n}{\mu} \left(\frac{R^n + 1}{2} \right) \{ M(P, 1) - m(P, 1) \} \ln \left(\frac{R^\mu + 1}{2} \right) \tag{3.7}$$

To the polynomial $Q\left(\frac{z}{k}\right)$, for $k > 1$ and replacing R by k in (3.7), we obtain

$$\begin{aligned} \max_{|z|=1} \left| Q\left(\frac{z}{k}\right) \right| &\leq \max_{|z|=k} \left| Q\left(\frac{z}{k}\right) \right| - \\ &\left(\frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \right) \max_{|z|=1} \left| Q\left(\frac{z}{k}\right) \right| + \\ &\left(\frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \right) \min_{|z|=1} \left| Q\left(\frac{z}{k}\right) \right| \end{aligned} \tag{3.8}$$

Since

$$Q\left(\frac{z}{k}\right) = \left(\frac{z}{k}\right)^n P\left(\frac{k}{z}\right)$$

Therefore

$$\begin{aligned} \max_{|z|=k} \left| Q\left(\frac{z}{k}\right) \right| &= \max_{|z|=1} |P(z)|, \\ \max_{|z|=1} \left| Q\left(\frac{z}{k}\right) \right| &= \frac{1}{k^n} \max_{|z|=k} |P(z)| \text{ and} \\ \min_{|z|=1} \left| Q\left(\frac{z}{k}\right) \right| &= \frac{1}{k^n} \min_{|z|=k} |P(z)| \end{aligned} \tag{3.9}$$

Using (3.9) in inequality (3.8), we get

$$\begin{aligned} \frac{1}{k^n} M(P, k) &\geq M(P, 1) \\ &- \left(\frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \frac{1}{k^n} \ln \left(\frac{k^\mu + 1}{2} \right) \right) M(P, k) \\ &+ \left(\frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \frac{1}{k^n} \ln \left(\frac{k^\mu + 1}{2} \right) \right) m \end{aligned}$$

Which is equivalent to

$$\begin{aligned} \left[1 + \frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \right] M(P, k) \\ \geq k^n M(P, 1) \\ + \left(\frac{n}{\mu} \left(\frac{k^n + 1}{2} \right) \ln \left(\frac{k^\mu + 1}{2} \right) \right) m \end{aligned}$$

This proves Theorem 3.3.

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بحث علمي

بعض المتباينات المتعلقة بالمقياس الأقصى لمتعددات الحدود المركبة بأصفار مقيدة

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