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Research Article

Some Inequalities Concerning Maximum Modulus of Complex Polynomials with Restricted Zeros

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Abstract

In this paper, certain new results concerning the maximum modulus of polynomials with restricted zeros are obtained. These estimates strengthen some well-known inequalities for polynomial due to Rivlin, Govil, Lal and others.

1. Introduction

Let $P(z)$ be a polynomial of degree n , let us define and denote $m = m(P,k) = \min_{|z|=k}|P(z)|$ and $M(P,r) = \max_{|z|=r}|P(z)|$.

For $P(z)$ be a polynomial of degree, it is known that

$$M(P,r) \geq r^n\, M(P,1) \quad \text{for} \quad 0 < r \leq 1 \tag{1.1}$$

Inequality (1.1) is due to Varga [1] who attributed it to Zarantonello.

It is noted that equality holds in (1.1) if and only if (z) has all its zeros at the origin.

It was shown by Rivlin [2] that $P(z)$ has no zeros in $|z| < 1$, then (1.1) can be replaced

$$M(P,r) \geq \left(\frac{r+1}{2}\right)^n M(P,1) \quad \text{for} \quad 0 < r \leq 1 \tag{1.2}$$

As a generalization of (1.2), Govil [3] proved that if $P(z)$ has no zeros in $|z| < 1$, then for $0 < r \leq R \leq 1$

$$M(P,r) \geq \left(\frac{1+r}{1+R}\right)^n M(P,R) \tag{1.3}$$

2. Lemmas

We need the following Lemmas.

Lemma 2.1. [4] If $P(z) = \sum_{v=0}^{n} a_v\, z^v$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{1+k}\left[\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\right] \tag{2.1}$$

Lemma 2.2. [5] If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v\, z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < 1$, then for $0 < r < R \leq 1$

$$M(P,r) \geq \left(\frac{1+r^\mu}{1+R^\mu}\right)^{n/\mu} M(P,R) \tag{2.2}$$

Lemma 2.3. [6] If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v\, z^v$, $1 \leq \mu \leq n$, be a polynomial of degree n that dose not vanish in $|z| < k$, $k \geq 1$, then for $1 \leq r < R$

$$M(P,r) \geq M(P,R) - \frac{n}{\mu}\left(\frac{R^n + k^\mu}{1+k^\mu}\right)\{\, M(P,1) - m\,\} \tag{2.3}$$

3. Main Results and Proofs

We first present the following generalization and refinement of (1.3).

Theorem 3.1. If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n has no zeros in $|z| < 1$, then for $0 < r < R \leq 1$

$$M(P,r) \geq \frac{(1+r^\mu)^{n/\mu}}{(1+r^\mu)^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}}\left[M(P,R) + n\, \min_{|z|=1}|P(z)|\, \ln\left(\frac{1+R}{1+r}\right)\right] \tag{3.1}$$

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Proof. Let $0 < r < R \leq 1$ and $0 \leq \theta < 2\pi$. Then we have

$$\left|P\left(Re^{i\theta}\right) - P\left(re^{i\theta}\right)\right| \leq \int_r^R \left| P'\left(te^{i\theta}\right) \right| dt \qquad (3.2)$$

If $P(z) \neq 0$, has no zeros in $|z| < 1$, then $P(tz) \neq 0$ has no zeros in $|z| < \frac{1}{t}$.

If $0 < t \leq 1$, then $\frac{1}{t} \geq 1$ and using inequality (2.1), we obtain

$$t\left|P'(tz)\right| \leq \frac{n\,t}{1+t}\left[\max_{|z|=t}|P(z)| - \min_{|z|=1}|P(z)|\right] \quad (3.3)$$

Combining (3.2) and (3.3), Let $m(P,1) = \min_{|z|=1}|P(z)|$

$$\left|P\left(Re^{i\theta}\right)\right| \leq \left|P\left(re^{i\theta}\right)\right| + \int_r^R \frac{n}{1+t} M(P,t)dt - n\,m(P,1)\int_r^R \frac{1}{1+t}dt$$

Which implies

$$M(P,R) \leq M(P,r) + \int_r^R \frac{n}{1+t} M(P,t)dt - n\,m(P,1)\int_r^R \frac{1}{1+t}dt$$

Now, using inequality (2.2) in above, we get

$$\begin{aligned} M(P,R) &\leq M(P,r) + \int_r^R \frac{n}{1+t}\left(\frac{1+t^{\mu}}{1+r^{\mu}}\right)^{n/\mu} M(P,r)\,dt - n\,m\int_r^R \frac{1}{1+t}dt \\ &\leq M(P,r) + \frac{n\,M(P,r)}{(1+r^{\mu})^{n/\mu}}\int_r^R \frac{(1+t)^{n/\mu}}{(1+t)}dt - n\,m\int_r^R \frac{1}{1+t}dt \\ &= M(P,r) + \frac{\mu\,M(P,r)}{(1+r^{\mu})^{n/\mu}}\left[(1+R)^{n/\mu} - (1+r)^{n/\mu}\right] - n\,m\ln\left(\frac{1+R}{1+r}\right) \end{aligned}$$

Thus, we get

$$M(P,r)\left[\frac{(1+r^{\mu})^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}}{(1+r^{\mu})^{n/\mu}}\right] \geq M(P,R) + n\,m\ln\left(\frac{1+R}{1+r}\right)$$

Which is equivalent to

$$M(P,r) \geq \frac{(1+r^{\mu})^{n/\mu}}{(1+r^{\mu})^{n/\mu} + \mu(1+R)^{n/\mu} - \mu(1+r)^{n/\mu}}\left[M(P,R) + n\min_{|z|=1}|P(z)|\ln\left(\frac{1+R}{1+r}\right)\right]$$

Remark 3.1. For $R = 1$ in inequality (3.1), we obtain

$$M(P,r) \geq \frac{(1+r^{\mu})^{n/\mu}}{(1+r^{\mu})^{n/\mu} + \mu\,2^{n/\mu} - \mu(1+r)^{n/\mu}}\left[M(P,1) + n\min_{|z|=1}|P(z)|\ln\left(\frac{2}{1+r}\right)\right]$$

This inequality is due to Govil el at. [7, Theo.2.1, P.3].

Next, we prove the following result which is a generalization of Theorem 3.1.

Theorem 3.2. If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leq \mu \leq n$, is a polynomial of degree n has no zeros in $|z| < k$, $k \geq 1$, then for $0 < r < R \leq k$

$$M(P,r) \geq \frac{k^{-n}(k^{\mu}+r^{\mu})^{n/\mu}}{k^{-n}(k^{\mu}+r^{\mu})^{n/\mu} + \mu k^{-n/\mu}(1+R)^{n/\mu} - \mu k^{-n/\mu}(k+r)^{n/\mu}}\left[M(P,R) + n\min_{|z|=k}|P(z)|\ln\left(\frac{k+R}{k+r}\right)\right] \qquad (3.4)$$

Proof. If $P(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then the polynomial $u(z) = P(kz) \neq 0$ for $|z| < 1$. Further, if $0 < r < k$, then $0 < r/k < R/k \leq 1$ and applying (3.1) to $u(z)$, we get

$$M\left(u, r/k\right) \geq \frac{\left(1+(r/k)^{\mu}\right)^{n/\mu}}{\left(1+(r/k)^{\mu}\right)^{n/\mu} + \mu(1+R/k)^{n/\mu} - \mu(1+r/k)^{n/\mu}}\left[M\left(u, R/k\right) + n\min_{|z|=1}|u(z)|\ln\left(\frac{1+R/k}{1+r/k}\right)\right] \qquad (3.5)$$

Which yields

$$M(P,r) \geq \frac{k^{-n}(k^{\mu}+r^{\mu})^{n/\mu}}{k^{-n}(k^{\mu}+r^{\mu})^{n/\mu} + \mu k^{-n/\mu}(1+R)^{n/\mu} - \mu k^{-n/\mu}(k+r)^{n/\mu}}\left[M(P,R) + n\,m\ln\left(\frac{k+R}{k+r}\right)\right]$$

Now, we prove a result recently proved by Lal [6].

Theorem 3.3. If $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$, $k > 1$, then

$$M(P,k) \ge \frac{1}{B_\mu}\left\{k^n\, M(P,1) + \frac{n}{\mu}\left(\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right)\min_{|z|=k}|P(z)|\right\} \tag{3.6}$$

where

$$B_\mu = \left[1 + \frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right]$$

Proof. Since $P(z)$ has all its zeros in $|z| \le k$, therefore, the polynomial $Q(z) = z^n P\left(1/z\right)$ has all its zeros in $|z| \ge \frac{1}{k}$ and hence the polynomial $Q\left(\frac{z}{k}\right)$ has all its zeros in $|z| \ge 1$. Applying Lemma 2.3 when $r = 1$ and $k = 1$, we have

$$M(P,1) \ge M(P,R) - \frac{n}{\mu}\left(\frac{R^n+1}{2}\right)\{M(P,1) - m(P,1)\}\ln\left(\frac{R^\mu+1}{2}\right) \tag{3.7}$$

To the polynomial $Q\left(\frac{z}{k}\right)$, for $k > 1$ and replacing R by k in (3.7), we obtain

$$\max_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right| \le \max_{|z|=k}\left|Q\left(\frac{z}{k}\right)\right| - \left(\frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right)\max_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right| + \left(\frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right)\min_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right| \tag{3.8}$$

Since

$$Q\left(\frac{z}{k}\right) = \left(\frac{z}{k}\right)^n P\left(\frac{k}{z}\right)$$

Therefore

$$\max_{|z|=k}\left|Q\left(\frac{z}{k}\right)\right| = \max_{|z|=1}|P(z)|,$$

$$\max_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right| = \frac{1}{k^n}\max_{|z|=k}|P(z)| \text{ and}$$

$$\min_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| \tag{3.9}$$

Using (3.9) in inequality (3.8), we get

$$\frac{1}{k^n}M(P,k) \ge M(P,1) - \left(\frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\frac{1}{k^n}\ln\left(\frac{k^\mu+1}{2}\right)\right)M(P,k) + \left(\frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\frac{1}{k^n}\ln\left(\frac{k^\mu+1}{2}\right)\right)m$$

Which is equivalent to

$$\left[1 + \frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right]M(P,k) \ge k^n\, M(P,1) + \left(\frac{n}{\mu}\left(\frac{k^n+1}{2}\right)\ln\left(\frac{k^\mu+1}{2}\right)\right)m$$

This proves Theorem 3.3.

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بحث علمي

بعض المتباينات المتعلقة بالمقياس الأقصى لمتعددات الحدود المركبة بأصفار مقيدة

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الملخص

في هذا البحث نقدم بعض النتائج الجديدة المتعلقة بالمقياس الأقصى لمتعددات الحدود بأصفار مقيدة تم الحصول عليها. هذه التقديرات تضاف لبعض المتباينات المعروفة لمتعددات الحدود لكل من ريفلين ، جوفيل ، لال وآخرين.

مفاتيح البحث

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