Decomposition of Curvatur tensor filed R^i_{jkh} recurrent spaces of first and second order

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Abstract

Finsler geometry has many uses in relative physics and many of mathematicians contributed in this study and improved it. Takano [26] has studied the decomposition of curvature tensor in a recurrent space. Sinha and Singh [25] have studied and defined the decomposition of recurrent curvature tensor field in a Finsler space. Negi and Rawat [11] and [12] have studied decomposition of recurrent curvature tensor fields in K¨aehlerian space. Rawat and Silswal [19] studied and defined the decomposition of recurrent curvature tensor fields in a Tachibana space. Rawat and Singh [21] studied the decomposition of curvature tensor field in K¨aehlerian recurrent space of first order. Further, Rawat and others [20],[22] and [23] studied the decomposition of curvature tensor field in Einstein- K¨aehlerian recurrent space of first order. Al-Qashbari [1], [2], [3] and [4] and Qasem and others [14], [15], [16], [17] and [18] studied the recurrent for different curvature tensors. In the present paper, we have studied the decomposition of curvature tensor fields R^i_{jkh} in recurrent space of First order and second order, and several theorems have been established and proved.

Keywords: Finsler space, Decomposition of curvature, Cartan's fourth curvature tensor R^i_{jkh}, Cartan's third curvature tensor K^i_{jkh}, recurrent and birecurrent curvature tensors.

1. introduction

We consider an n dimensional Finsler space F_n in which the Riemannian curvature tensor field denoted by R^i_{jkh} is given by

(1.1) $R^i_{jkh} = \partial_j \Gamma^i_{kh} - \partial_k \Gamma^i_{jh} + \Gamma^i_{jm}\Gamma^m_{kh} - \Gamma^i_{km}\Gamma^m_{jh}$, where $\partial_j = \partial/\partial x^j$.

Cartan's third curvature tensor R^i_{jkh} , Cartan's fourth curvature tensor K^i_{jkh} , its associate curvature tensor K_{ijkh} and the R-Ricci tensor R_{jk} in the sense of Cartan, respectively, given by [24]

(1.2) a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + \left(\Gamma^{*i}_{ljk}\right) G^l_h + C^i_{jm}\left(G^m_{kh} - G^m_{kl}\, G^l_h \right) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h$,

b) $R^i_{jkh}\, \dot{x}^j = K^i_{jkh}\, \dot{x}^j = H^i_{kh}$, c) $R_{jk}\, \dot{x}^j = H_k$, d) $R_{jk}\, \dot{x}^j = R_k$,

e) $R^i_{jki} = R_{jk}$, f) $R^i_{jkh} = -R^i_{jhk}$, g) $R_{jk}\, g^{jk} = R$ and h) $H^i_{jk}\, \dot{x}^k = -H^i_j$

The Bianchi identity for the Riemannian curvature tensor R^i_{jkh} is given by

(1.3) a) $R^i_{jkh} + R^i_{khj} + R^i_{hjk} = 0$ and b) $R^i_{jkh(l)} + R^i_{jhl(k)} + R^i_{jlk(h)} = 0$.

The vectors $\dot{x}_i$ and $\dot{x}^j$ satisfy the following relations [24]

(1.4) a) $g_{ij}\, \dot{x}^j = y_i$ and b) $\dot{x}_j\, \dot{x}^j = F^2$.

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by [24]

(1.5) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases}$$

The tensor C_{ijk} defined by

(1.6) $C_{ijk} = \frac{1}{2}\, \dot{\partial}_i\, g_{jk} = \frac{1}{4}\, \dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2$

is known as (h) hv - torsion tensor.

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

(1.7) a) $C^i_{jk}\,\dot{x}^j = C^i_{kj}\,\dot{x}^j = 0$, b) $\dot{x}_i\, C^i_{jk} = 0$, c) $C_{ijk}\,\dot{x}^j = 0$ and

d) $G^i_{jkh}\,\dot{x}^j = G^i_{hjk}\,\dot{x}^j = G^i_{khj}\,\dot{x}^j = 0$.

Berwald's covariant derivative $T^i_{j(k)}$ of an arbitrary tensor filed T^i_j with respect to $\dot{x}^k$ is given by

(1.8) $T^i_{j(k)} := \partial_k T^i_j - (\dot{\partial}_r T^i_j)\, G^r_k + T^r_j G^i_{rk} - T^i_r\, G^r_{jk}$.

Berwald's covariant derivative of the metric function , the vectors $\dot{x}^i$, $\dot{x}_i$ and the unit vector l^i vanish identically [24], i.e.

(1.9) a) $\dot{x}^i_{(k)} = 0$, b) $F_{(k)} = 0$, c) $\dot{x}_{i(k)} = 0$ and d) $l^i_{(k)} = 0$.

But Berwald's covariant derivative of the metric tensor g_{ij} doesn't vanish, and given by

(1.10) $g_{ij(k)} = -2\, C_{ijk|h}\,\dot{x}^h = -2\, C_{ijk(h)}\,\dot{x}^h$.

Where $|h$ is h-covariant derivative with respect to $\dot{x}^i$ (Cartan's second kind covariant differentiation).

Berwald's covariant differential operator with respect to $\dot{x}^h$ commutes with partial differential operator with respect to $\dot{x}^k$, according to [24]

(1.11) $\dot{\partial}_k\, T^i_{j(h)} = (\dot{\partial}_k\, T^i_j)_{(h)} + T^r_j G^i_{khr} - T^i_r\, G^r_{khj}$,

where T^i_j is any arbitrary tensor field.

The commutative formulae for the curvature tensor field are given as follows

(1.12) $T^i_{(j)(k)} - T^i_{(k)(j)} = T^r\, R^i_{rjk}$.

(1.13) $T^i_{j(k)(m)} - T^i_{j(m)(k)} = T^r_j\, R^i_{rkm} - T^i_r\, R^r_{jmk}$.

The second covariant derivative of an arbitrary tensor field T^i_j with respect to x^k and x^h in the sense of Berwald may written as

(1.14) $T^i_{j(k)(h)} = \dot{\partial}_k T^i_{j(h)} - (\dot{\partial}_s\, T^i_{j(h)})\, G^s_k + (T^r_{j(h)})\, G^i_{rk} - (T^i_{r(h)})\, G^r_{ik} - (T^i_{j(r)})\, G^r_{hk}$.

The commutation formula for Berwald's curvature differentiation as follows

(1.15) $T^i_{j(k)(h)} - T^i_{j(h)(k)} = T^r_j\, H^i_{hkr} - T^i_r\, H^r_{hkj} - (\dot{\partial}_r\, T^i_j)\, H^r_{hk}$,

where H^i_{jkh} defined by

(1.16) $H^i_{jkh} = 2\,\{\, \partial_{[j}\, G^i_{k]h} + G^i_{rh[j}\, G^r_{k]} + G^i_{r[j}\, G^r_{k]h} \,\}$,

are components of Berwald curvature tensor and

(1.17) a) $H^i_{kh} = H^i_{jkh}\,\dot{x}^j$, b) $H^i_h = H^i_{kh}\,\dot{x}^k$, c) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$ and d) $H^i_{kh} = \dot{\partial}_k H^i_h$.

It is clear from the definition that Berwald curvature tensor H^i_{jkh} is skew-symmetric in its first two lower indices and positively homogeneous of degree zero in the directional arguments $\dot{x}^i$.

2. R-Curvature Tensor in Recurrent and Bi-Recurrent Finsler Space

Definition (2.1)

In a non-flat Finsler space F_n if there exists anon zero covariant vector A_l such that the R-curvature tenser field R^i_{jki} satisfies

(2.1) $R^i_{jkh(l)} = A_l\, R^i_{jkh}$.

where (l) is h-covariant derivative of t first order (Cartan's second kind covariant differential operator) with respect to x^l, the quantities A_l is a non-null covariant vector field.

Then the space is called a recurrent Finsler space (Sinha and Singh 1971) .

Transvecting the equation (2.1) by $\dot{x}^j$, using (1.9a) and (1.2b), we get

(2.2) $H^i_{kh(l)} = A_l\, H^i_{kh}$.

Transvecting the equation (2.2) by $\dot{x}^k$, using (1.9a) and (1.17b), we get

(2.3) $H^i_{h(l)} = A_l H^i_h$.

The vector tensor λ_l behave like the recurrent vector [14]

(2.4) $\lambda_{l(m)} = \mu_m \lambda_l$.

The Ricci tensor and the vector tensor λ_l are satisfies

(2.5) $\lambda_{l(m)} \left(\lambda_h R_{jk} - \lambda_k R_{jh} \right) = \lambda_l \left(\lambda_{h(m)} R_{jk} - \lambda_{k(m)} R_{jh} \right)$.

Definition (2.2)

In a non-flat Finsler space F_n if there exists anon zero tensor A_{lm} such that the curvature tensor field satisfies the following

(2.6) $R^i_{jkh(l)(m)} = A_{lm} R^i_{jkh}$, where $A_{lm} = A_{l(m)} + A_l A_m$.

Then the Finsler space is called a bi-recurrent Finsler space (Sinha and Singh 1971) and the tensor field A_{lm} is called a bi-recurrent tensor field.

Transvecting the equation (2.6) by $\dot{x}^j$, using (1.9a) and (1.2b), we get

(2.7) $H^i_{kh(l)(m)} = A_{lm} H^i_{kh}$.

Transvecting the equation (2.7) by $\dot{x}^k$, using (1.9a) and (1.17b), we get

(2.8) $H^i_{h(l)(m)} = A_{lm} H^i_h$.

Differentiating (2.7) and (2.8) partially with respect to $\dot{x}^j$ and $\dot{x}^k$, respectively, using (1.17c), (1.17d) and (1.11), we get

(2.9) $\dot{\partial}_j (H^i_{kh(l)(m)}) = (\dot{\partial}_j A_{lm}) H^i_{kh} + A_{lm} H^i_{jkh}$.

(2.10) $\dot{\partial}_k (H^i_{h(l)(m)}) = (\dot{\partial}_k A_{lm}) H^i_h + A_{lm} H^i_{kh}$, respectively.

Using the commutation formula (1.11) for $\left(\dot{\partial}_j H^i_{kh(l)(m)}\right)$ in (2.9), we get

(2.11) $\left(\dot{\partial}_j H^i_{kh}\right)_{(l)(m)} + H^r_{kh(m)} G^i_{jlr} - \left(H^i_{rh(m)}\right) G^r_{jlk} - \left(H^i_{kr(m)}\right) G^r_{jlh} + H^r_{kh(l)} G^i_{jmr}$
$- \left(H^i_{rh(l)}\right) G^r_{jmk} - \left(H^i_{kr(l)}\right) G^r_{jmh} = \left(\dot{\partial}_j A_{lm}\right) H^i_{kh} + A_{lm} H^i_{jkh}$.

Using (1.17c) and (2.7) in equation (2.11), we get

(2.12) $H^r_{kh(m)} G^i_{jlr} - \left(H^i_{rh(m)}\right) G^r_{jlk} - \left(H^i_{kr(m)}\right) G^r_{jlh} + H^r_{kh(l)} G^i_{jmr} - \left(H^i_{rh(l)}\right) G^r_{jmk}$
$- \left(H^i_{kr(l)}\right) G^r_{jmh} - \left(\dot{\partial}_j A_{lm}\right) H^i_{kh} = 0$.

Transvecting (2.12) by $\dot{x}^k$, using (1.9a), (1.17b) and (1.7d), we get

(2.13) $H^r_{h(m)} G^i_{jlr} - \left(H^i_{r(m)}\right) G^r_{jlh} + H^r_{h(l)} G^i_{jmr} - \left(H^i_{r(l)}\right) G^r_{jmh} - \left(\dot{\partial}_j A_{lm}\right) H^i_h = 0$.

Transvecting (2.13) by $\dot{x}^j$, using (1.9a) and (1.7d), we get

(2.14) $\left(\dot{\partial}_j A_{lm}\right) H^i_h \dot{x}^j = 0$.

Since the condition $\left(\dot{\partial}_j A_{lm}\right) H^i_h \dot{x}^j = 0$, implies $\dot{\partial}_j A_{lm} = 0$, i.e. the covariant tensor field A_{lm} is independent of the directional argument.

Thus, we conclude

Theorem 2.1. Under the tenser field R^i_{jki} and bi-recurrent Finsler space F_n , the covariant tensor field A_{lm} is independent of the directional argument provided $\left(\dot{\partial}_j A_{lm}\right) H^i_h \dot{x}^j = 0$.

Again applying the commutation formula (1.11) for $\left(\dot{\partial}_j H^i_{h(l)(m)}\right)$ in (2.10), we get

(2.15) $\left(\dot{\partial}_j H^i_h\right)_{(l)(m)} + H^r_{h(m)} G^i_{jlr} - \left(H^i_{r(m)}\right) G^r_{jlh} + H^r_{h(l)} G^i_{jmr} - \left(H^i_{r(l)}\right) G^r_{jmh}$
$= \left(\dot{\partial}_k A_{lm}\right) H^i_h + A_{lm} H^i_{kh}$.

Using (1.17d) and (2.8) in equation (2.11), we get

(2.16) $H^r_{h(m)} G^i_{jlr} - \left(H^i_{r(m)}\right) G^r_{jlh} + H^r_{h(l)} G^i_{jmr} - \left(H^i_{r(l)}\right) G^r_{jmh} - \left(\dot{\partial}_k A_{lm}\right) H^i_h = 0.$

Transvecting (2.16) by $\dot{x}^j$, using (1.9a) and (1.7d), we get

(2.17) $\left(\dot{\partial}_k A_{lm}\right) H^i_h = 0$, where $\dot{x}^j \neq 0$.

Since the condition $\left(\dot{\partial}_j A_{lm}\right) H^i_h \, \dot{x}^j = 0$, implies $\dot{\partial}_j A_{lm} = 0$, i.e. the covariant tensor field A_{lm} is independent of the directional argument.
Thus, we conclude

Theorem 2.2. Under the tenser field R^i_{jki} and bi-recurrent Finsler space F_n , the covariant tensor field A_{lm} is independent of the directional argument provided $\left(\dot{\partial}_k A_{lm}\right) H^i_h = 0$.

In view the equation (1.13), we get

(2.18) $\quad R^i_{jkh(l)(m)} - R^i_{jkh(m)(l)} = R^r_{jkh} R^i_{rlm} - R^i_{rkh} R^r_{jml} - R^i_{jrh} R^r_{kml} - R^i_{jkr} R^r_{hml}$.

Using equation (2.6) in (2.18), we get

$$(A_{lm} - A_{ml}) R^i_{jkh} = R^r_{jkh} R^i_{rlm} - R^i_{rkh} R^r_{jml} - R^i_{jrh} R^r_{kml} - R^i_{jkr} R^r_{hml} .$$

If A_{lm} is skew-symmetric then the above equation can be written as

(2.19) $\quad \lambda_{lm} R^i_{jkh} = R^r_{jkh} R^i_{rlm} - R^i_{rkh} R^r_{jml} - R^i_{jrh} R^r_{kml} - R^i_{jkr} R^r_{hml}$, where $\lambda_{lm} = A_{lm} + A_{ml}$.

If A_{lm} is symmetric then the above equation can be written as

(2.20) $\quad R^r_{jkh} R^i_{rlm} - R^i_{rkh} R^r_{jml} - R^i_{jrh} R^r_{kml} - R^i_{jkr} R^r_{hml} = 0$, where $A_{lm} - A_{ml} = 0$.

Theorem 2.3. Under the tenser field R^i_{jki} and bi-recurrent Finsler space F_n, we have the conditions (2.19) and (2.20) holds.

3. Decomposition of Curvature Tensor Field $\mathbf{R}^i_{jkh}$ By Using Contravariant Vector $\mathbf{V}^i$ and Covariant Tensor $\boldsymbol{\Psi}_{jkh}$

Let us consider the decomposition of curvature tensor field R^i_{jkh} as

(3.1) $\quad \mathrm{R}^i_{jkh} = \mathrm{V}^i \, \Psi_{jkh}$.

Where V^i is contravariant vector and Ψ_{jkh} is covariant tensor .
The decomposition vector V^i should satisfy the relation

(3.2) $\quad A_i V^i = 1$.

Theorem(3.1): Under the decomposition (3.1) the tensor field Ψ_{jkh} satisfies the identity

(3.3) (a) $\quad \Psi_{jkh} = -\Psi_{jhk}$;

(b) $\quad \Psi_{jkh} + \Psi_{khj} + \Psi_{hjk} = 0$;

(c) $\quad A_l \Psi_{jkh} + A_k \Psi_{khj} + A_h \Psi_{hj} = 0$.

Proof

(a) From Equations (1.2f) and (3.1), we have

(3.4) $\quad V^i \, \Psi_{jkh} = - V^i \, \Psi_{jhk}$.

Multiplying the equation (3.4) , by A_i and using (3.2), we get the relation (3.3a) .

(b) Using (3.1) in (1.3a), we get

(3.5) $\quad V^i \left(\Psi_{jkh} + \Psi_{khj} + \Psi_{hjk} \right) = 0$, where $V^i \neq 0$.

Multiplying equation (3.5) by A_i and using (3.2), we get the relation (3.3b).

(c) From Equations (1.3b) , (2.1) and (3.1), we have

$$A_l \Psi_{jkh} + A_k \Psi_{jhl} + A_h \Psi_{jlk} = 0 .$$

Which can be written as

(3.6) $\quad V^i \left(A_l \Psi_{jkh} + A_k \Psi_{jhl} + A_h \Psi_{jlk} \right) = 0$.

Multiplying (3.6) by A_i and using (3.2), we get the relation (3.3c) .

Theorem(3.2): Under the decomposition (3.1) , the tensor field R^i_{jkh} satisfies

(3.7) $\quad \Psi_{jkh} = A_h R_{jk} - A_k R_{jh} = A_l R^l_{jkh}$.

Proof

From Equations (3.3a) and (3.6) , we have

(3.8) $V^i \left(A_l \Psi_{jkh} + A_k \Psi_{jhl} - A_h \Psi_{jkl} \right) = 0$

Multiplying (3.8) by A_i and using (3.2) , we get

(3.9) $A_l \Psi_{jkh} = A_h \Psi_{jkl} - A_k \Psi_{jhl}$.

Multiplying (3.9) by V^l and using (3.2) and (3.1), we get

$$\Psi_{jkh} = A_h R^l_{jkl} - A_k R^l_{jhl}$$

In view (2.1e) , the above equation yields to

$$\Psi_{jkh} = A_h R_{jk} - A_k R_{jh}$$

using (3.1) and (3.2), in the above equation , we get the relation (3.7).

Theorem(3. 3): Under the decomposition (3.1), the vector field V^i is behaving like the recurrent vector $V^i_{(m)} = -\mu_m V^i$ **.**

Proof

Taking covariant derivative for (3.2) with respect to x^m and using (2.4), we get

$$\lambda_i \left(\mu_m V^i + V^i_{(m)} \right) = 0 \quad , \quad \text{where} \quad \lambda_i \neq 0 \text{ , we get}$$

(3.10) $V^i_{(m)} = -\mu_m V^i$.

Theorem(3. 4): Under the decomposition (3.1), the tensor field Ψ_{jkh} is behaving like the recurrent tensor field $\Psi_{jkh(m)} = \lambda_m \Psi_{jkh}$ **.**

Proof

Taking covariant derivative for (3.1), with respect to x^m , we get

$$R^i_{jkh(m)} = V^i_{(m)} \Psi_{jkh} + V^i \Psi_{jkh(m)} \quad .$$

Using (2.1) , (3.1) and (3.10) in above equation, we have

$$A_m V^i \Psi_{jkh} = -\mu_m V^i \Psi_{jkh} + V^i \Psi_{jkh(m)} \quad .$$

Which can be written as

$$V^i \Psi_{jkh(m)} = V^i (A_m + \mu_m) \Psi_{jkh} \quad .$$

Multiplying above equation by A_i and using (3.2), we get

(3.11) $\Psi_{jkh(m)} = (A_m + \mu_m) \Psi_{jkh}$,

which can be written as

$$\Psi_{jkh(m)} = \lambda_m \Psi_{jkh} \quad , \quad \text{where} \quad \lambda_m = (A_m + \mu_m) \ .$$

Theorem (3. 5): Under the decomposition (3.1), the vector field V^i , behaves like bi-recurrent tensor field $V^i_{(m)(n)} = \text{Ʉ}_{mn} V^i$.

Proof

Differentiating (3.10) , covariantly with respect to x^n , we ge t

$$V^i_{(m)(n)} = -\mu_{m(n)} V^i - \mu_m V^i_{(n)} \quad .$$

Using (3.10) in the above equation, we get

(3.12) $V^i_{(m)(n)} = \left(\mu_m \mu_n - \mu_{m(n)} \right) V^i$.

We can be written

$$V^i_{(m)(n)} = \text{Ʉ}_{mn} V^i \quad , \quad \text{where} \quad \text{Ʉ}_{mn} = \left(\mu_m \mu_n - \mu_{m(n)} \right) .$$

Theorem(3. 6): Under the decomposition (3.1), the tensor field Ψ_{jkh} is behaving like bi-recurrent tensor field $\Psi_{jkh(m)(n)} = r_{mn} \Psi_{jkh}$.

Proof

Taking covariant derivative for (3.11), with respect to x^n , we get

$$\Psi_{jkh(m)(n)} = (A_m + \mu_m)_{(n)}\, \Psi_{jkh} + (A_m + \mu_m)\, \Psi_{jkh(n)} \ .$$

Using (2.4) and (3.11), in above equation, we get

$$\Psi_{jkh(m)(n)} = \left(\mu_m\, \lambda_m + \mu_{m(n)}\right) \Psi_{jkh} + (\lambda_m + \mu_m)(\lambda_n + \mu_n)\, \Psi_{jkh} \ ,$$

which can be written as

$$\Psi_{jkh(m)(n)} = \left(\, 2\lambda_m\, \mu_n + \mu_{m(n)} + \lambda_m\, \lambda_n + \mu_m\, \lambda_n + \mu_m\, \mu_n\right) \Psi_{jkh} \ .$$

Or $\Psi_{jkh(m)(n)} = r_{mn}\Psi_{jkh}$, where $r_{mn} = \left(2\lambda_m\, \mu_n + \mu_{m(n)} + \lambda_m\, \lambda_n + \mu_m\, \lambda_n + \mu_m\, \mu_n\right)$.

Theorem (3.7): Under the decomposition (3.1), the tensor $\left(\mu_{n(m)} - \mu_{m(n)}\right)$ is behaving like a recurrent tensor field $\left(\mu_{n(m)} - \mu_{m(n)}\right)_{(s)} = \lambda_s \left(\mu_{n(m)} - \mu_{m(n)}\right)$.

Proof

Interchanging the m and n in the equation (3.12) and subtracting the result thus obtained from (3.12), we get

(3.13) $\quad V^i_{(m)(n)} - V^i_{(n)(m)} = \left(\mu_{n(m)} - \mu_{m(n)}\right) V^i$.

Using the commutation formula (1.12), we get

(3.14) $\quad V^h R^i_{hmn} = \left(\mu_{n(m)} - \mu_{m(n)}\right) V^i$

Taking covariant derivative for (3.14), with respect to x^s , we get

(3.15) $\quad V^h_{(s)} R^i_{hmn} + V^h R^i_{imn(s)} = \left(\mu_{n(m)} - \mu_{m(n)}\right)_{(s)} V^i + V^i_{(s)} \left(\mu_{n(m)} - \mu_{m(n)}\right)$.

Using (2.1) and (3.10) , in an equation (3.15), we get

$$-\mu_s V^h R^i_{hmn} + \lambda_s V^h R^i_{hmn} = V^i \left(\mu_{n(m)} - \mu_{m(n)}\right)_{(s)} - \mu_s \left(\mu_{n(m)} - \mu_{m(n)}\right) V^i \ .$$

Using (3.14) in the above equation, we get

$$\left(\mu_{n(m)} - \mu_{m(n)}\right)_{(s)} = \lambda_s \left(\mu_{n(m)} - \mu_{m(n)}\right) \ .$$

Theorem (3.8): Under the decomposition (3.1), the tensor field λ_l is behaving like bi-recurrent tensor field $\lambda_{l(m)(n)} = \mu_{mn}\, \lambda_l$

Proof

Taking covariant derivative for (2.5), with respect to x^n , we have

$$\lambda_{l(m)(n)} \left(\lambda_h R_{jk} - \lambda_k R_{jh}\right) + \lambda_{l(m)}\left(\lambda_h R_{jk} - \lambda_k R_{jh}\right)_{(n)}$$
$$= \lambda_{l(n)} \left(\lambda_{h(m)} R_{jk} - \lambda_{k(m)} R_{jh}\right) + \lambda_l \left(\lambda_{h(m)} R_{jk} - \lambda_{k(m)} R_{jh}\right)_{(n)} \ .$$

Which can be written as

(3.16) $\quad \lambda_{l(m)(n)} \left(\lambda_h R_{jk} - \lambda_k R_{jh}\right) + \lambda_{l(m)}\left(\lambda_{h(n)} R_{jk} - \lambda_{k(n)} R_{jh} + \lambda_h R_{jk} - \lambda_k R_{jh}\right) =$
$\lambda_{l(n)}\left(\lambda_{h(m)} R_{jk} - \lambda_{k(m)} R_{jh}\right) + \lambda_l\left(\lambda_{h(m)(n)} R_{jk} - \lambda_{k(m)(n)} R_{jh} + \lambda_{h(m)} R_{jk(n)} - \lambda_{k(m)} R_{jh(n)}\right)$.

Using (2.4) and (2.5) in (3.16), we get

(3.17) $\quad \lambda_{l(m)(n)} \left(\lambda_h R_{jk} - \lambda_k R_{jh}\right) = \lambda_l \left(\lambda_{h(m)(n)} R_{jk} - \lambda_{k(m)(n)} R_{jh}\right)$.

Multiplying (3.17) by λ_s , we get

$$\lambda_{l(m)(n)} \left(\lambda_h R_{jk} - \lambda_k R_{jh}\right) \lambda_s = \lambda_s\, \lambda_l \left(\lambda_{h(m)(n)} R_{jk} - \lambda_{k(m)(n)} R_{jh}\right) \ .$$

Since the expression of the right hand side of the above equation is symmetric l and s . There for

(3.18) $\quad \lambda_{l(m)(n)}\, \lambda_s = \lambda_{s(m)(n)}\, \lambda_l$.

Provided that $\lambda_h R_{jk} - \lambda_k R_{jh} \neq 0$.

The vector field λ_l being non zero, we can have aporortianal vector μ_{mn} such that

(3.19) $\quad \lambda_{l(m)(n)} = \mu_{mn}\, \lambda_l$.

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تحلل مؤثر الانحناء R^{i}_{jkh} في الفضاء الاشتقاقي الأحادي و الثنائي المعاودة

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الملخص

هندسة فنسلر لها العديد من الاستخدامات في الفيزياء النسبية وقد ساهم العديد من علماء الرياضيات في دراستها وتحسينها. وقدم العديد من الباحثين المهتمين بدراسة هندسة فنسلر دراسات مختلفة في تحلل المؤثرات للمنحنيات في الاشتقاق الاحادي في فضاء فنسلر. وفي هذه الورقة قدمنا دراسة تحلل المؤثر الرابع لكارتان R^{i}_{jkh} في فضاء فنسلر عن طريق استخدام اشتقاق كارتان من الرتبة الاولى والرتبة الثانية والذي يحقق التحلل لبعض المؤثرات في هذا الفضاء. وقدمنا بعض النظريات المتعلقة بتحلل المؤثرات للمنحنيات تحت منحنى كارتان R^{i}_{jkh} مع اثباتاتها.

الكلمات المفتاحية: فضاء فنسلر، تحلل المنحنى، النوع الرابع لمؤثر كارتان R^{i}_{jkh}، النوع الثالث لمؤثر كارتان K^{i}_{jkh}, الاشتقاق الاحادي والثنائي لمؤثر المنحنيات.