A study of recurrent Finsler spaces of higher order with Cartan’s Curvature Tensor

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Abstract

In the present communication, we have derived Bianchi and Veblen identities along with a few more related results in a recurrent and generalized $n^{th}$-recurrent Finsler space with Cartan’s curvature tensor field. A Finsler space $F_n$ whose Cartan’s third curvature tensor $R^i_{jkh}$ satisfies the condition $R^i_{jkh}|m_1|m_2|...|m_n = \lambda_{m_1m_2...m_n} R^i_{jkh} + \mu_{m_1m_2...m_n} \left( \delta_{ih} g_{jk} - \delta_{ik} g_{jh} \right)$, where $R^i_{jkh} \neq 0$ and $|m_1|m_2|...|m_n$ are h-covariant differentiation (Cartan’s second kind covariant differential operator) with respect to $x^m$ to nth order, $\lambda_{m_1m_2...m_n}$ and $\mu_{m_1m_2...m_n}$ are recurrence tensors fields.

Keywords: Finsler space, Generalized $R^{-n^{th}}$-recurrent space, Cartan’s covariant derivative of higher order, Cartan’s third curvature tensor $R^i_{jkh}$ and Cartan’s second curvature tensor $P^i_{jkh}$.

1. Introduction

The generalized curvature tensors in recurrent Finsler space used the sense of Berwald curvature tensor discussed by Al-Qashbari [5] and ([7], [9], [14], [15], [16], [17], [18], [23], and [25]). Some properties of Weyl's projective curvature tensor studied by Abu-Donia [1]. Complete Finsler space of constant negative Ricci curvature were studied by Bidabad and Sepasi [8]. Decomposability of projective curvature tensor in recurrent Finsler space has been studied by Al-Qashbari [4] and Al-Qufail [6]. Semiconformal symmetry- A new symmetry of the spacetime manifold of the general relative discussed by Ali, Pundeer and Ahsan [2]. The generalized birecurrent and trirecurrent Finsler space are studied in ([3], [12], [19], [20], [24]). Also, Dwivedi [10] introduced the P*-Reducible Finsler space and Application. On Lie-recurrent in Finsler space studied by Saxsena and Pandey [22] and Pandey and Pandey [13]. The differential geometry of Finsler space was studied by Rund [21]. Ricci coefficients of Rotation of generalized Finsler space studied by Mincic, Stankovic and Zlatanovic [11]. Curvature tensors and pseudotensors in generalized Finsler space were studied by Zlatanovic, Mincic, and Petrovic [26] and others.

Cartan in his second postulate, represented the variation of an arbitrary vector field $X^i$ under the infinitesimal change of its line element $(x,y)$ to $(x + dx, y + dy)$ by means of covariant (absolute) differential given by

$$DX^i = dx^i + X^i \left( C^i_{jk} dy^k + \Gamma^i_{jk} dx^k \right),$$

where

$$\Gamma^i_{jk} = y^k_{,ij} - C^m_{jk} G^m_j + g^{th} C^m_{jkm} G^m_h,$$

$$G^i = \frac{1}{2} y^i_{jk} y^j y^k$$

and

$$\Gamma^i_{jk} = \hat{\partial}^i_j G^i.$$

The function $G^i$ is positively homogeneous of degree two in the direction argument.

Eliminating $dy^k$ from (1.1) and in terms of the absolute differential of $l^i$, Cartan deduced

$$DX^i = F X^i_{lk} D^l k + X^i_{lk} dx^k + y^k \left( \hat{\partial}^i_k X^i \right) \frac{df}{f},$$

where

$$\Gamma^i_{rk} = C^i_{mr} \Gamma^m_{sk} y^s.$$

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The function \( \Gamma^i_{jk} \) defined by (1.4c) is the connection parameter of Cartan, this is symmetric in the lower indices \( r \) and \( k \) and positively homogeneous of degree zero in the directional argument and satisfies :

\[
(1.5) \quad g_{ih} \Gamma^i_{jk} = \Gamma^r_{rk} .
\]

The equations (1.4a) and (1.4b) give two processes of covariant differentiation called \( v \)-covariant differentiation (Cartan's first kind covariant differentiation) and \( h \)-covariant differentiation (Cartan's second kind covariant differentiation), respectively. So \( X^i|_k \) and \( X^i|_k \) are respectively \( v \)-covariant derivative and \( h \)-covariant derivative of the vector field \( X^i \). We note that this notation for covariant differentiation was used by Cartan and followed by Rund and Matsumoto calls these derivatives as "\( v \)-covariant derivative " and " \( h \)-covariant derivative ", respectively and his symbols for covariant differentiations are similar to that of Cartan with the only difference that \( \frac{1}{F} X^i|_k \) of Cartan coincides with \( X^i|_k \) of Matsumoto due to this change we have an extra \( F \) in the first term of the right hand side of the equation (1.5). K. Yano denoted \( \frac{1}{F} X^i|_k \) and \( X^i|_k \) by \( \tilde{\nabla}_k X^i \) and \( \nabla_j X^i \), respectively.

The metric tensor \( g_{ij} \) and the associate metric tensor \( g^{ij} \) are related by

\[
(1.6) \quad g_{ij} g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}
\]

The quantities \( g_{ij} \), \( g^{ij} \) and \( \delta^j_i \) are satisfies

\[
(1.7) \quad \text{a) } g_{ij} g^{ij} = n \quad \text{and} \quad \text{b) } \delta^j_i g_{ik} = g_{jk} .
\]

The vector \( y_i \) satisfies relation

\[
(1.8) \quad y_i y^i = F^2.
\]

The vectors \( y_i \) and \( \delta^j_i \) also satisfy the following relations

\[
(1.9) \quad \text{a) } \delta^k_i y^k = y^i , \quad \text{b) } \delta^j_i g^{jk} = g^{ik} \quad \text{and} \quad \text{c) } g_{ij} y^j = y_i .
\]

By using Euler's theorem, the \( C_{ijk} \) and \( C^i_{jk} \) tensors satisfy, the following identities

\[
(1.10) \quad \text{a) } C_{ijk} y^l = C_{kij} y^l = C_{jki} y^l = 0 \quad \text{and} \quad \text{b) } C^i_{jk} y^j = C^j_{ki} y^j = 0 .
\]

The metric tensor \( g_{ij} \) and the associate metric tensor \( g^{ij} \) are covariant constants with respect to both processes

\[
(1.11) \quad \text{a) } g_{ij|m} = 0 \quad \text{and} \quad \text{b) } g^{ij|m} = 0 .
\]

The vectors \( y^i \), \( y^i \) are vanish under \( h \)-covariant differentiation

\[
(1.12) \quad \text{a) } y^i_{|lm} = 0 \quad \text{and} \quad \text{b) } y^i_{|lm} = 0 .
\]

The \( h \)-curvature tensor \( R^i_{jkh} \) (Cartan’s third curvature tensor), is defined by

\[
(1.13) \quad R^i_{jkh} = \tilde{\delta}_h \Gamma^i_{jk} + \left( \tilde{\delta}_i \Gamma^l_{jk} \right) G^l_h + C^i_{jm} \left( \tilde{\delta}_k G^m_h - G^m_k \tilde{\delta}_h \right) + \Gamma^i_{mk} \Gamma^m_j
\]

\[
- \left[ \tilde{\delta}_k \Gamma^i_{jh} + \left( \tilde{\delta}_i \Gamma^l_{jh} \right) G^l_h + C^i_{jm} \left( \tilde{\delta}_h G^m_k - G^m_h \tilde{\delta}_k \right) + \Gamma^i_{mh} \Gamma^m_k \right] .
\]

The \( h \)-curvature tensor \( R^i_{jkh} \) is positively homogeneous of degree \(-1\) in the directional argument and skew-symmetric in the last two lower indices \( h \) and \( k \), i.e.

\[
(1.14) \quad R^i_{jkh} = - R^i_{jkh}
\]

and this tensor satisfies the following relation too

\[
(1.15) \quad R^i_{jkh} = R^i_{jkh} + C^i_{js} K^s_{jkh} y^r .
\]

The associate curvature tensor \( R_{ijklm} \) of the curvature tensor \( R^i_{jkh} \) is given by

\[
(1.16) \quad \text{a) } R_{ijklm} = g_{rjk} R^r_{ikh} \quad \text{and} \quad \text{b) } R_{jrkhi} g^{ir} = R^i_{jkh} .
\]

The R-Ricci tensor \( R_{jk} \), the curvature scalar \( R \) and the deviation tensor \( R^i_j \) related by

\[
(1.17) \quad \text{a) } R^i_{jki} = R_{jki} , \quad \text{b) } R_{jk} y^k = R_j , \quad \text{c) } R_{jki} y^j = H_k \quad \text{and} \quad \text{d) } R_{jki} g^{jk} = R .
\]

The curvature tensor \( R^i_{jkh} \) and the associate tensor \( R^i_j \) satisfy the relations

\[
(1.18) \quad \text{a) } R^i_{jkh} y^j = K^i_{jkh} y^j = H^i_{kh}
\]

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and b) \( R_k^i = R_{ik}^{jl} g^{jk} \).

Cartan's connection parameter \( \Gamma_{jk}^l \) and Berwald's connection parameter \( G_{jm}^l \) given by

\[
(1.19) \quad a) \quad \hat{\partial}_k G_{lj}^i = G_{lj}^i \quad \text{and} \quad b) \quad G_{kj}^i = \Gamma_{sk}^i y^{s}.
\]

The tensor \( P_{kh} \) is called \( (hv) \)-torsion tensor and its associate tensor \( P_{jkh} \) is given by

\[
(1.21) \quad a) \quad y_i \Gamma_{jk}^i = P_{jkh} \quad \text{and} \quad b) \quad g_{rk} P_{jkh} = P_{kjh}.
\]

The tensors \( H_{jkh} \) and \( H_{kh}^i \) form the components of tensors and defined by

\[
(1.22) \quad H_{jkh}^i = \hat{\partial}_h G_{ij}^k + G_{jk}^i G_{rh}^l + G_{rj}^h G_{kh}^i - \hat{\partial}_k G_{ijh} - G_{jkh}^i G_{rk}^l - G_{rjk}^l G_{kh}^i.
\]

The formula (1.23) is called the generalized Ricci identity or Ricci commutation formula.

\[
(1.24) \quad a) \quad H_k = H_{ki}^i \quad \text{and} \quad b) \quad H_k y^k = (n - 1) H.
\]

The curvature tensor \( K_{rjk}^l \) satisfies the following relation too

\[
(1.25) \quad K_{rjk}^l = \hat{\partial}_r \Gamma_{jk}^l + \left( \hat{\partial}_k \Gamma_{jr}^l \right) G_{ij}^k + \Gamma_{mj}^i \Gamma_{kr}^m - \hat{\partial}_k G_{jrh}^i - G_{rh}^i G_{jkr}^l - G_{rk}^i G_{rh}^l.
\]

The tensor \( K_{rjk}^l \) as defined (1.25) above is called Cartan's fourth curvature tensor, this tensor is positively homogeneous of degree zero.

Ricci tensor and the curvature vector of the curvature tensor are given by

\[
(1.28) \quad a) \quad K_{jkl} = K_{jlk} \quad \text{and} \quad b) \quad K_{jkl} y^k = K_j.
\]

2. On Generalized \( R^h \)-Recurrent Finsler Space of \( N \)th order

Let us consider a Finsler space \( F_n \) whose Cartan's third curvature tensor \( R_{jkh}^l \) satisfies the following condition

\[
(2.1) \quad R_{jkh} = \sum_{m=1}^{n} \lambda_{m1,2,...,m} R_{jkh} + \mu_{m1,2,...,m} \left( \delta_{l}^i g_{jk} - \delta_{l}^i g_{jk} \right),
\]

where \( R_{jkh} \neq 0 \) and \( \lambda_{m1,2,...,m} \) are non-null covariant tensors fields.

Definition 2.1. A Finsler space \( F_n \) whose Cartan's third curvature tensor \( R_{jkh}^l \) satisfies the condition

\[
(2.2) \quad R_{jph} = \sum_{m=1}^{n} \lambda_{m1,2,...,m} R_{jph} + \mu_{m1,2,...,m} \left( g_{hp} g_{jk} - g_{hp} g_{jk} \right).
\]

Conversely, the transvection of the condition (2.2) by \( g^{lp} \), yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore a generalized \( R^h \)-nth order space may characterized by the condition (2.2). Therefore, we conclude.
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Theorem 2.1. The generalized $R^n$-th order space may characterized by condition (2.2).

Let us consider an $G^{R^n}$-th order $RF_n$, which is characterized by the condition (2.1). Contracting the indices $i$ and $h$ in (2.1), using (1.17a), (1.6) and (1.7b), we get

(2.3) \[ R_{jk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} R_{jk} + (n-1) \mu_{m_1m_2...m_n} g_{jk} \]

Transvecting the equation (2.3) by $y^k$, using (1.12b), (1.7b) and (1.9c), we get

(2.4) \[ R_{jk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} R_{jk} + \mu_{m_1m_2...m_n} (n-1) y_j \]

Further, transvecting (2.1) by $g^{jk}$, using (1.11b), (1.18b) and in view of (1.6), we get

(2.5) \[ R_{jk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} R_{jk} + (n-1) \mu_{m_1m_2...m_n} \delta^i_h \]

Contracting the indices $i$ and $h$ in the condition (2.5) and using (1.6), we get

(2.6) \[ R_{jk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} R + n (n-1) \mu_{m_1m_2...m_n} \delta^i_h \]

Also, by transvecting the equation (2.3) by $g^{jk}$, using (1.11b), (1.17d) and in view of (1.6), we get the condition (2.6).

The conditions (2.3), (2.4), (2.5), and (2.6), show that, Ricci tensor $R_{jk}$, the curvature vector $R_j$, the deviation tensor $R_{ik}$ and the curvature scalar $R_{jj}$ (all for Cartan's third curvature tensor $R_{jk}$) of a generalized $R^n$-th order space cannot vanish, because the vanishing of them imply the vanishing of the covariant tensors fields $\mu_{m_1m_2...m_n}$, i.e. $\mu_{m_1m_2...m_n} = 0$, a contradiction.

Thus, we conclude

Theorem 2.3. In $G^{R^n}$-th order $RF_n$, Ricci tensor $R_{jk}$, the curvature vector $R_j$, the deviation tensor $R_{ik}$ and the curvature scalar $R$ (all for Cartan's third curvature tensor $R_{jk}$) are non-vanishing.

Transvecting the condition (2.3) by $y^l$, using (1.12b), (1.17c) and (1.9c), we get

(2.7) \[ R_{lk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} h_{lk} + (n-1) \mu_{m_1m_2...m_n} y_k \]

The condition (2.7), shows that, the curvature vector $h_k$ of a generalized $R^n$-th order space cannot vanish, because the vanishing of it would imply the vanishing of the covariant tensors fields $\mu_{m_1m_2...m_n}$, i.e. $\mu_{m_1m_2...m_n} = 0$, a contradiction.

Transvecting the condition (2.7) by $y^k$, using (1.12b), (1.24b) and (1.8), we get

(2.8) \[ R_{lk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} h_k + \mu_{m_1m_2...m_n} R^2 \]

Thus, we conclude

Theorem 2.4. In $G^{R^n}$-th order $RF_n$, the curvature vector $h_k$ and the curvature scalar $H$ are non-vanishing.

Now, we have seen that in a generalized $R^n$-th order space, Ricci tensor $R_{jk}$ (of Cartan's third curvature tensor $R_{jk}$) satisfies the condition (2.3). Conversely, if Ricci tensor $R_{jk}$ of a Finsler space satisfy the condition (2.3), then it need not be the space is a generalized $R^n$-th order space. However, the converse is true if the dimension of a Finsler space is 3 or the space is $R3$-like. The proof of this fact is follows:

We know that the associate curvature tensor $R_{ijkh}$ (of Cartan's third curvature tensor $R_{ijkh}$) for three dimensioned Finsler space is of the form

(2.9) \[ R_{ijkh} = g_{ih} L_{jk} + g_{jh} L_{ih} - g_{ih} L_{jk} - g_{jk} L_{ih} \]

(2.10) \[ L_{ik} = \frac{1}{n-2} R_{ik} - \frac{C}{z} g_{ik} \]

(2.11) \[ r = \frac{1}{n-1} R_{ij} \]

Transvecting the condition (2.3) by $g^{ij}$, using (1.11b), (1.18b) and (1.6), we get

(2.12) \[ R_{jk|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} R_{jk} + \mu_{m_1m_2...m_n} (n-1) \delta^i_k \]

Contracting the indices $p$ and $k$ in the condition (2.12), using (2.11) and (1.6), we get

(2.13) \[ r_{|m_1|,m_2|...|m_n} = \lambda_{m_1m_2...m_n} r + n \mu_{m_1m_2...m_n} \]
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Taking the h-covariant derivative of the condition (2.10) with respect to \(x^m\) to nth order and using (1.6), we get

\[
L_{ik|m_1 m_2|... m_n} = \frac{1}{n-2} \left( R_{ik|m_1 m_2|... m_n} - \frac{1}{2} r_{m_1 m_2|... m_n} g_{ik} \right).
\]

Using the conditions (2.3) and (2.13) in (2.14), we get

\[
L_{ik|m_1 m_2|... m_n} = \lambda_{m_1 m_2|... m_n} \left[ \frac{1}{n-2} \left( R_{ik} - \frac{1}{2} r g_{ik} \right) \right] + \frac{1}{2} \mu_{m_1 m_2|... m_n} g_{ik}.
\]

In view of the condition (2.10), the above equation implies to

\[
L_{ik|m_1 m_2|... m_n} = \lambda_{m_1 m_2|... m_n} L_{ik} + \frac{1}{2} \mu_{m_1 m_2|... m_n} g_{ik}.
\]

This gives the h-covariant derivative of the condition (2.9) with respect to \(x^m\) to nth order and using (1.11a), we get

\[
R_{ijkh|m_1 m_2|... m_n} = g_{ik} L_{jh|m_1 m_2|... m_n} + g_{jh} L_{ik|m_1 m_2|... m_n}.
\]

In view of the condition (2.15), the above equation implies

\[
R_{ijkh|m_1 m_2|... m_n} = \lambda_{m_1 m_2|... m_n} \left( g_{ik} L_{jh} + g_{jh} L_{ik} - g_{ih} L_{jk} - g_{jk} L_{ih} \right) + \mu_{m_1 m_2|... m_n} \left( g_{ik} g_{jh} - g_{ih} g_{jk} \right).
\]

In view of the condition (2.9), the above equation implies

\[
R_{ijkh|m_1 m_2|... m_n} = \lambda_{m_1 m_2|... m_n} R_{ijkh} + \mu_{m_1 m_2|... m_n} \left( g_{ik} g_{jh} - g_{ih} g_{jk} \right)
\]

This shows that, the associate curvature tensor \(R_{ijkh}\) (of Cartan’s third curvature tensor \(R^i_{jk}\)) is a generalized h-recurrent.

In view of (1.16a), the above condition implies (2.1). That is, the h-covariant derivative of the condition (2.9) with respect to \(x^m\) to nth order and in view of (1.8a), gives (2.1). This shows that, a three dimensional Ricci generalized \(R^h\)-nth order space is necessarily generalized \(R^h\)-recurrent space.

Matsumoto [28] introduced a Finsler space \(F_h(n > 3)\) for which the tensor \(R_{ijkh}\) satisfying (2.9) and called it R3-like Finsler space \(F_h\). If we consider a R3-like Ricci generalized \(R^h\)-recurrent space and applied the same process as above we may show that

\[
R_{ijkh|m_1 m_2|... m_n} = \lambda_{m_1 m_2|... m_n} \left( R_{ijkh} + \mu_{m_1 m_2|... m_n} \left( g_{ik} g_{jh} - g_{ih} g_{jk} \right) \right).
\]

This, leads to

**Theorem 2.7.** In \(G^{h^1}_{h^n} R^h F_h\), is Ricci generalized \(R^h\)-recurrent space, but the converse need not be true. However, if the space \(F_h\) is R3-like, then the converse is also true.

Now, taking the h-covariant derivative for (1.13), with respect to \(x^m\) to nth order, we get

\[
\lambda_{m_1 m_2|... m_n} R_{ijkh} + \mu_{m_1 m_2|... m_n} \left( \delta_{ik} g_{jh} - \delta_{jk} g_{ih} \right)
\]

\[
= \left[ \Gamma_{jk}^{li} - \delta_{jk} g_{ih} \right] G^p_k + \Gamma_{jk}^{li} \left( G^m_k - G^m_{ht} G^h_t \right) + \Gamma_{jk}^{li} \Gamma_{jk}^{m} - \delta_{jk} \Gamma_{jk}^{li} G^p_k + \Gamma_{jk}^{li} \left( G^p_k - G^p_{ht} G^h_t \right) - \delta_{jk} \Gamma_{jk}^{li} \Gamma_{jk}^{m} \right]_{m_1 m_2|... m_n}.
\]

By using (1.13), (1.20) and (1.19a), the above equation can be written as

\[
\left[ \Gamma_{jk}^{li} + \Gamma_{jk}^{li} G^p_k + \Gamma_{jk}^{li} \left( G^m_k - G^m_{ht} G^h_t \right) + \Gamma_{jk}^{li} \Gamma_{jk}^{m} - \delta_{jk} \Gamma_{jk}^{li} G^p_k + \Gamma_{jk}^{li} \left( G^p_k - G^p_{ht} G^h_t \right) - \delta_{jk} \Gamma_{jk}^{li} \Gamma_{jk}^{m} \right]_{m_1 m_2|... m_n} - \lambda_{m_1 m_2|... m_n} R_{ijkh} + \mu_{m_1 m_2|... m_n} \left( \delta_{ik} g_{jh} - \delta_{jk} g_{ih} \right).
\]

Thus, we conclude
Theorem 2.8. In $GR^h$- $n$th $RF_n$, the h-covariant derivative of the nth order for the tensor \[ \Gamma_{jkh}^i + \Gamma_{jks}^i G_h^x + C_{ijk} G_{mn}^x - C_{jkm} G_{nh}^x + \Gamma_{mik} \Gamma_{jh}^m - \Gamma_{jkh}^i - \Gamma_{ijs} G_h^x \]
\[- C_{jkm} G_{nh}^x - G_{ik}^x + \Gamma_{mih} \Gamma_{jkh}^m \] is generalized nth- recurrent.

Transvecting (2.18) by $y^i$, using (1.12b), (1.10b), (1.19b) and (1.9c), we get
\[ (2.19) \quad (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x)_{[m_1|m_2|...|m_n]}
\]
\[ = \lambda_{m_1 m_2...m_n} (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x)
\]
\[ + \mu_{m_1 m_2...m_n} (\delta^i_k y_k - \delta^i_k y_h) \]

Transvecting (2.19) by $y^i$, using (1.12a), (1.21b), and (1.7b), we get
\[ (2.20) \quad (\Gamma_{jkh}^i + P_{jks} G_h^x + y_i \Gamma_{mik} \Gamma_{jh}^m + P_{jkh} + P_{jhs} G_h^x - y_i \Gamma_{mih} \Gamma_{jk}^m)_{[m_1|m_2|...|m_n]}
\]
\[ = \lambda_{m_1 m_2...m_n} (-P_{jkh}^i - P_{jks} G_h^x + \Gamma^i_{mik} \Gamma_{jh}^m + P_{jkh} + P_{jhs} G_h^x - y_i \Gamma_{mih} \Gamma_{jk}^m)
\]
\[ + \mu_{m_1 m_2...m_n} (y_h g_{jk} - y_k g_{jh}) \]

Further, transvecting (2.19) by $g_{ij}$, using (1.11a), (1.21c), (1.5), and (1.7b), we get
\[ (2.21) \quad (\Gamma_{jhr} + P_{jrs} G_h^x + \Gamma^i_{mik} G_{rh}^m - P_{hkr} - P_{hrs} G_h^x - \Gamma^i_{mih} G_{k}^x)_{[m_1|m_2|...|m_n]}
\]
\[ = \lambda_{m_1 m_2...m_n} (\Gamma_{jhr} + P_{jrs} G_h^x + \Gamma^i_{mik} G_{rh}^m - P_{hkr} - P_{hrs} G_h^x - \Gamma^i_{mih} G_{k}^x)
\]
\[ + \mu_{m_1 m_2...m_n} (y_h g_{jk} - y_k g_{jh}) \]

The equation (2.21) can be written as
\[ (2.22) \quad (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x)_{[m_1|m_2|...|m_n]}
\]
\[ = \lambda_{m_1 m_2...m_n} (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x)
\]
\[ + \mu_{m_1 m_2...m_n} (\delta^i_k y_k - \delta^i_k y_h) \]

Thus, we conclude

Theorem 2.9. In $GR^h$- $n$th $RF_n$, the h-covariant derivative of the nth order for the tensor \[ (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x),
\]
\[ (P_{jhr} + P_{jrs} G_h^x + \Gamma^i_{mik} G_{rh}^m - P_{hkr} - P_{hrs} G_h^x - \Gamma^i_{mih} G_{k}^x) \]
and
\[ (P_{jkh}^i + P_{jks} G_h^x + \Gamma^i_{mik} G_{nh}^x - P_{jkh}^i - P_{jhs} G_h^x - \Gamma^i_{mih} G_{k}^x),
\]
are given by the conditions (2.19), (2.20), (2.21), and (2.22), respectively.

3. Necessary and Sufficient Condition

We know that Cartan's third curvature tensor $R_{jkh}$ and Cartan's fourth curvature tensor $K_{jkh}$ are connected by the equation (1.15).

Using (1.18a) in (1.15), we get
\[ R_{jkh}^i = C_{jfr} H_{fr}^i \]

Taking the h-covariant derivative for equation (3.1) with respect to $x^m$ to nth order, we get
\[ R_{jkh}^i |m_1|m_2|...|m_n = K_{jkh} |m_1|m_2|...|m_n + \left( C_{jfr} H_{fr}^i \right)_{[m_1|m_2|...|m_n]} \]

Using the condition (2.1) in (3.2), we get
\[ \lambda_{m_1 m_2...m_n} R_{jkh}^i + \mu_{m_1 m_2...m_n} (\delta^i_k g_{jk} - \delta^i_k g_{jh})
\]
\[ = K_{jkh} |m_1|m_2|...|m_n + \left( C_{jfr} H_{fr}^i \right)_{[m_1|m_2|...|m_n]} \]

By using the condition (3.1), the above equation implies to
\[ \lambda_{m_1 m_2...m_n} K_{jkh} + \lambda_{m_1 m_2...m_n} (C_{jfr} H_{fr}^i) + \mu_{m_1 m_2...m_n} (\delta^i_k g_{jk} - \delta^i_k g_{jh})
\]
\[ = K_{jkh} |m_1|m_2|...|m_n + \left( C_{jfr} H_{fr}^i \right)_{[m_1|m_2|...|m_n]} \]

This shows that
\[ K_{jkh} |m_1|m_2|...|m_n = \lambda_{m_1 m_2...m_n} K_{jkh} + \mu_{m_1 m_2...m_n} (\delta^i_k g_{jk} - \delta^i_k g_{jh}) \]
if and only if
\[(C^p_{fr} H^r_{kp})_{m1|m2|...|m_n} = \lambda_{m1 m2 ... m_n} (C^p_{fr} H^r_{kp}) \quad .\]

Thus, we conclude

**Theorem 3.1.** In $GR^{h}_{-nth} RF^n_h$, Cartan’s fourth curvature tensor $K^i_{jk}h$ is generalized nth- recurrent if and only if the tensor $\left( \frac{C^p_{fr} H^r_{kp}}{m1|m2|...|m_n} \right)$ is recurrent of nth order.

Contracting the indices $i$ and $h$ in (3.3), using (1.28a), (1.6) and (1.7b), we get
\[
\lambda_{m1 m2 ... m_n} K^j_k + \lambda_{m1 m2 ... m_n} \left( C^p_{fr} H^r_{kp} \right) = K^j_k \left( m1|m2|...|m_n \right) + \lambda_{m1 m2 ... m_n} (n - 1) g_{jk}.
\]
This shows that
\[
K^j_k \left( m1|m2|...|m_n \right) = \lambda_{m1 m2 ... m_n} K^j_k + \lambda_{m1 m2 ... m_n} (n - 1) g_{jk}
\]
if and only if
\[
\left( C^p_{fr} H^r_{kp} \right)_{m1|m2|...|m_n} = \lambda_{m1 m2 ... m_n} (C^p_{fr} H^r_{kp}) .
\]

Thus, we conclude

**Theorem 3.3.** In $GR^{h}_{-nth} RF^n_h$, Ricci tensor $K^j_k$ (of Cartan’s fourth curvature tensor $K^i_{jk}h$) is non-vanishing if and only if the tensor $\left( \frac{C^p_{fr} H^r_{kp}}{m1|m2|...|m_n} \right)$ is nth- recurrent.

Transvecting (3.6) by $y^K$, using (1.12b), (1.28b), (1.24) and (1.9c), we get
\[
\lambda_{m1 m2 ... m_n} K^j_i + \lambda_{m1 m2 ... m_n} \left( C^p_{fr} H^r_{kp} \right) = K^j_i \left( m1|m2|...|m_n \right) + \lambda_{m1 m2 ... m_n} (n - 1) y^j
\]
This shows that
\[
K^j_i \left( m1|m2|...|m_n \right) = \lambda_{m1 m2 ... m_n} K^j_i + \lambda_{m1 m2 ... m_n} (n - 1) y^j
\]
if and only if
\[
\left( C^p_{fr} H^r_{kp} \right)_{m1|m2|...|m_n} = \lambda_{m1 m2 ... m_n} (C^p_{fr} H^r_{kp}) .
\]

Thus, we conclude

**Theorem 3.4.** In $GR^{h}_{-nth} RF^n_h$, the curvature vector $K^j_i$ (of Cartan’s fourth curvature tensor $K^i_{jk}h$) is non-vanishing if and only if the tensor $\left( \frac{C^p_{fr} H^r_{kp}}{m1|m2|...|m_n} \right)$ is nth-recurrent.

4. Some Rules of Tensors in $GR^{h}_{-Nth} RF^n_h$

We know that the associate tensor $R^h_{ijkh}$ of Cartan’s third curvature tensor $R^i_{ijk}$ satisfies the identity
\[
R^h_{ijk} + R^h_{ikj} + R^h_{ikj} + \left( C_{ij}^s K^s_{rhk} + C_{ih}^s K^s_{rkh} + C_{iks} K^s_{rjh} \right) y^r = 0 .
\]
In view of the condition (1.18a), the identity (4.1) becomes
\[
R^h_{ijk} + R^h_{ikj} + R^h_{ikj} + C_{ij}^s H^s_{hk} + C_{ih}^s H^s_{kh} + C_{iks} H^s_{j} = 0 .
\]
The h-covariant differentiation of the identity (4.2), with respect to $x^m$, for nth order gives
\[
R^h_{ijk} \left( m1|m2|...|m_n \right) + \left( C_{ij}^s H^s_{hk} + C_{ih}^s H^s_{kj} + C_{iks} H^s_{j} \right)_{m1|m2|...|m_n} = 0 .
\]
Using the condition (2.2) in (4.3), we get
\[
\lambda_{m1 m2 ... m_n} R^h_{ijk} + \mu_{m1 m2 ... m_n} \left( g_{ih} g_{jk} - g_{ik} g_{jh} \right) + \lambda_{m1 m2 ... m_n} R^h_{ijk} + \mu_{m1 m2 ... m_n} \left( g_{ik} g_{jh} - g_{ij} g_{kh} \right) + \lambda_{m1 m2 ... m_n} R^h_{ijk} + \mu_{m1 m2 ... m_n} \left( g_{ij} g_{kh} - g_{ih} g_{kj} \right) + \left( C_{ij}^s H^s_{hk} + C_{ih}^s H^s_{kj} + C_{iks} H^s_{j} \right)_{m1|m2|...|m_n} = 0 .
\]
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Since the metric tensor $g_{jk}$ is symmetric, then the above equation implies to
\[ \lambda_{m_1m_2...m_n} R_{ijk}^{n} + \lambda_{m_1m_2...m_n} R_{ikj} + \lambda_{m_3m_2...m_n} R_{ikj} = 0. \]
or
\[ \lambda_{m_1m_2...m_n} \left( R_{ijk}^{n} + R_{ikj} + R_{ikj} \right) + \left( C_{ij}^{n} H_{i}^{s} + C_{ik}^{n} H_{j}^{s} \right) \mid_{m_1m_2...m_n} = 0. \]

Using the condition (4.2) in (4.4), we get
\[ \left( C_{ij}^{n} H_{i}^{s} + C_{ik}^{n} H_{j}^{s} \right) \mid_{m_1m_2...m_n} = \lambda_{m_1m_2...m_n} \left( C_{ij}^{n} H_{i}^{s} + C_{ik}^{n} H_{j}^{s} \right). \]

Thus, we conclude

**Theorem 4.1.** In $GR^{n}$-th RF$_{n}$, the tensor $\left( C_{ij}^{n} H_{i}^{s} + C_{ik}^{n} H_{j}^{s} \right)$ behaves as the nth-recurrent.

Transversing (4.5) by $y^{h}$, using (1.12b) and (1.10a), we get
\[ \left( C_{ij}^{n} H_{i}^{s} - C_{ik}^{n} H_{j}^{s} \right) \mid_{m_1m_2...m_n} = \lambda_{m_1m_2...m_n} \left( C_{ij}^{n} H_{i}^{s} - C_{ik}^{n} H_{j}^{s} \right). \]

Thus, we conclude

**Theorem 4.2.** In $GR^{n}$-th RF$_{n}$, the tensor $\left( C_{ij}^{n} H_{i}^{s} - C_{ik}^{n} H_{j}^{s} \right)$ behaves as the nth-recurrent.

**References**

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دراسة الاشتقاق احادي المعودة في فضاء فنسلر من الرتب العليا باستخدام مؤثر الانحناء لكارتاتن

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الملخص

في هذه الورقة قمنا دراسة الاشتقاق أحادي المعودة من الرتب العليا في فضاء فنسلر للمؤثر الرابع لكارتاتن Rjk记得 عن طريق استخدام اشتقاق كارتن من الرتبة الثنائية وعرفنا الفضاء للمشتقة الثنائية للمؤثر Rjk حيث العضود m1m2...mn = λm1m2...mn Rjk + μm1m2...mn (δj k gjk − δj k gjk ) بعض النتائج والعلاقات الهامة لبعض المؤثرات في هذا الفضاء. قمنا بعض النظريات المتعلقة باشتقاق كارتن Rjk من الرتبة الثنائية لبعض المؤثرات للمنشتيات تحت منحنى كارتن Rjk مع إثباتاتها.

الكلمات المفتاحية: فضاء فنسلر، تعميم الاشتقاق التوائي للفضاء، الاشتقاق ذات الرتب العليا لكارتن، النوع الرابع لمؤثر كارتن Rjk والنوع الثاني لمؤثر كارتن Rjk

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