A study of recurrent Finsler spaces of higher order with Cartan's Curvature Tensor

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Abstract

In the present communication, we have derived Bianchi and Veblen identities along with a few more related results in a recurrent and generalized n[th]-recurrent Finsler space with Cartan's curvature tensor field. A Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies the condition $R^i_{jkh|m_1|m_2|...|m_n} = \lambda_{m_1m_2...m_n} R^i_{jkh} + \mu_{m_1m_2...m_n}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, where $R^i_{jkh} \neq 0$ and $|m_1|m_2| ... |m_n$ are h-covariant differentiation (Cartan's second kind covariant differential operator) with respect to x^m to nth order, $\lambda_{m_1m_2...m_n}$ and $\mu_{m_1m_2...m_n}$ is recurence tensors fields.

Keywords: Finsler space F_n , Generalized R- n[th]-recurrent space, Cartan's covariant derivative of higher order, Cartan's third curvature tensor R^i_{jkh} and Cartan's second curvature tensor P^i_{jkh}.

1 . Introduction

The generalized curvature tensors in recurrent Finsler space used the sense of Berwald curvature tensor discussed by Al-Qashbari [5] and ([7], [9], [14], [15], [16], [17], [18], [23], and [25]). Some properties of Weyl's projective curvature tensor studied by Abu-Donia [1]. Complete Finsler space of constant negative Ricci curvature were studied by Bidabad and Sepasi [8]. Decomposability of projective curvature tensor in recurrent Finsler space has been studied by Al- Qashbari [4] and Al-Qufail [6]. Semiconformal symmetry- A new symmetry of the spacetime manifold of the general relative discussed by Ali, Pundeer and Ahsan [2]. The generalized birecurrent and trirecurrent Finsler space are studied in ([3], [12], [19], [20], [24]). Also, Dwivedi [10] introduced the P*-Reducible Finsler space and Application. On Lie-recurrent in Finsler space studied by Saxsena and Pandey [22] and Pandey and Pandey [13]. The differential geometry of Finsler spacewase studied by Rund [21]. Ricci coefficients of Rotation of generalized Finsler space studied by Mincic, Stankovic and Zlatanovic [11]. Curvature tensors and pseudotensors in generalized Finsler space were studied by Zlatanovic, Mincic, and Petrovic [26] and others.

Cartan in his second postulate, represented the variation of an arbitrary vector field X^i under the infinitesimal change of its line element (x , y) to $(x + dx , y + dy)$ by means of covariant (absolute) differential given by

(1.1) $DX^i = dX^i + X^j \left(C^i_{jk}\, dy^k + \Gamma^i_{jk}\, dx^k \right)$, where

(1.2) a) $\Gamma^i_{jk} = \gamma^i_{jk} - C^i_{mk} G^m_j + g^{ih} C_{jkm}\, G^m_h$,

b) $G^i = \frac{1}{2}\, \gamma^i_{jk}\, y^j\, y^k$ and c) $G^i_j = \dot{\partial}_j\, G^i$.

The function G^i is positively homogeneous of degree two in the directional argument.
Eliminating dy^k from (1.1) and in terms of the absolute differential of l^i, Cartan deduced

(1.3) $DX^i = F\, X^i|_k\, D\, l^k + X^i_{|k}\, dx^k + y^k \left(\dot{\partial}_k\, X^i \right) \frac{dF}{F}$, where

(1.4) a) $X^i|_k = \dot{\partial}_k\, X^i + X^r\, C^i_{rk}$,

b) $X^i_{|k} = \partial_k\, X^i + X^r\, \Gamma^{*i}_{rk} - \left(\dot{\partial}_m\, X^i \right) \Gamma^m_{sk}\, y^s$, and

c) $\Gamma^{*i}_{rk} = \Gamma^i_{rk} - C^i_{mr}\, \Gamma^m_{sk}\, y^s$.

The function Γ_{rk}^{*i} defined by (1.4c) is the connection parameter of Cartan, this is symmetric in the lower indices r and k and positively homogeneous of degree zero in the directional argument and satisfies :

(1.5) $\quad g_{ih}\,\Gamma_{rk}^{*i} = \Gamma_{rhk}^{*}$.

The equations (1.4a) and (1.4b) give two processes of covariant differentiation called v-covariant differentiation (Cartan's first kind covariant differentiation) and h-covariant differentiation (Cartan's second kind covariant differentiation), respectively. So $X^i|_k$ and $X^i_{|k}$ are respectively v-covariant derivative and h-covariant derivative of the vector field X^i. We note that this notation for covariant differentiation was used by Cartan and followed by Rund and Matsumoto calls these derivatives as " v-covariant derivative " and " h-covariant derivative ", respectively and his symbols for covariant differentiations are similar to that of Cartan with the only difference that $\frac{1}{F}\, X^i|_k$ of Cartan coincides with $X^i|_k$ of Matsumoto due to this change we have an extra F in the first term of the right hand side of the equation (1.5). K. Yano denoted $\frac{1}{F}\, X^i|_k$ and $X^i|_k$ by $\dot\nabla_k X^i$ and $\nabla_j X^i$, respectively.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are related by

(1.6) $\quad g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 & \text{if} \quad i = k \ , \\ 0 & \text{if} \quad i \neq k \ . \end{cases}$

The quantities g_{ij} , g^{ij} and δ_j^i are satisfies

(1.7) a) $g_{ij}\, g^{ij} = n$ and b) $\delta_j^i\, g_{ik} = g_{jk}$.

The vector y_i satisfies relation

(1.8) $\quad y_i\, y^i = F^2$

The vectors y_i and δ_k^i also satisfy the following relations

(1.9) a) $\delta_k^i\, y^k = y^i$, b) $\delta_j^i\, g^{jk} = g^{ik}$ and c) $g_{ij}\, y^j = y_i$.

By using Euler's theorem, the C_{ijk} and C_{jk}^i tensors satisfy, the following identities

(1.10) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C_{jk}^i\, y^j = C_{kj}^i\, y^j = 0$.

The metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constants with respect to both processes

(1.11) a) $g_{ij|m} = 0$ and b) $g^{ij}_{|m} = 0$.

The vectors y^i , y_i are vanish under h-covariant differentiation

(1.12) a) $y_{i|m} = 0$ and b) $y^i_{|m} = 0$.

The h-curvature tensor R_{jkh}^i (Cartan's third curvature tensor), is defined by

(1.13)
$$R_{jkh}^i = \dot\partial_h\, \Gamma_{jk}^{*i} + \left(\dot\partial_l\, \Gamma_{jk}^{*i}\right) G_h^l + C_{jm}^i\left(\dot\partial_k\, G_h^m - G_{kl}^m\, G_h^l\right) + \Gamma_{mk}^{*i}\, \Gamma_{jh}^{*m}$$
$$-[\dot\partial_k\, \Gamma_{jh}^{*i} + \left(\dot\partial_l\, \Gamma_{jh}^{*i}\right) G_k^l + C_{jm}^i\left(\dot\partial_h\, G_k^m - G_{hl}^m\, G_k^l\right) + \Gamma_{mh}^{*i}\, \Gamma_{jk}^{*m}] \ .$$

The h-curvature tensor R_{jkh}^i is positively homogeneous of degree -1 in the directional argument and skew-symmetric in the last two lower indices h and k , i.e.

(1.14) $\quad R_{jkh}^i = -\, R_{jhk}^i$

and this tensor satisfies the following relation too

(1.15) $\quad R_{jkh}^i = K_{jkh}^i + C_{js}^i\, K_{rkh}^s\, y^r$.

The associate curvature tensor R_{ijkh} of the curvature tensor R_{jkh}^i is given by

(1.16) a) $R_{ijkh} = g_{rj}\, R_{ikh}^r$ and b) $R_{jrkh}\, g^{ir} = R_{jkh}^i$.

The R-Ricci tensor R_{jk} , the curvature scalar R and the deviation tensor R_j^i related by

(1.17) a) $R_{jki}^i = R_{jk}$, b) $R_{jk}\, y^k = R_j$, c) $R_{jk}\, y^j = H_k$ and d) $R_{jk}\, g^{jk} = R$.

The curvature tensor R_{jkh}^i and the associate tensor R_h^r are satisfy the relations

(1.18) a) $R_{jkh}^i\, y^j = K_{jkh}^i\, y^j = H_{kh}^i$

and b) $R_h^r = R_{ikh}^r \, g^{ik}$.

Cartan's connection parameter Γ_{jk}^{*i} and Berwald's connection parameter G_{jm}^i given by

(1.19) a) $\dot{\partial}_k \, G_h^i = G_{kh}^i$ and b) $G_k^i = \Gamma_{sk}^{*i} \, y^s$.

(1.20) $\left(\dot{\partial}_h \, \Gamma_{jk}^{*i}\right) y^h = \Gamma_{jkh}^{*i} \, y^h = G_{jkh}^i \, y^h = 0$.

The tensor P_{kh}^i is called v(hv)-torsion tensor and its associate tensor P_{kjh} is given by

(1.21) a) $\Gamma_{jkh}^{*i} \, y^j = P_{kh}^i$, b) $y_i \, \Gamma_{kjh}^{*i} = -P_{kjh}$ and c) $g_{rj} \, P_{kh}^r = P_{kjh}$.

the tensors H_{jkh}^i and H_{kh}^i form the components of tensors and defined by

(1.22) $H_{jkh}^i = \dot{\partial}_h \, G_{jk}^i + G_{jk}^r \, G_{rh}^i + G_{rjh}^i \, G_k^r - \dot{\partial}_k \, G_{jh}^i - G_{jh}^r \, G_{rk}^i - G_{rjk}^i \, G_h^r$

and

(1.23) $H_{kh}^i = \dot{\partial}_h \, G_k^i + G_k^r \, C_{rh}^i - \dot{\partial}_k \, G_h^i - G_h^r \, C_{rk}^i$.

The formula (1.23) is called the generalized Ricci identity or Ricci commutation formula.

(1.24) a) $H_k = H_{ki}^i$ and b) $H_k \, y^k = (n-1)H$,

where H_{hk}^i and H_k^i are called H-Ricci tensor and the curvature scalar, respectively and defined by

(1.24) $H_{hk}^i \, y^h = H_k^i$.

(1.25) $K_{rkj}^i = \partial_j \, \Gamma_{kr}^{*i} + \left(\dot{\partial}_l \, \Gamma_{rj}^{*i}\right) G_k^l + \Gamma_{mj}^{*i} \, \Gamma_{kr}^{*m} - \partial_k \, \Gamma_{jr}^{*i} - \left(\dot{\partial}_l \, \Gamma_{rk}^{*i}\right) G_j^l - \Gamma_{mk}^{*i} \, \Gamma_{jr}^{*m}$.

The tensor K_{rkj}^i as defined (1.25) above is called Cartan’s fourth curvature tensor, this tensor is positively homogeneous of degree zero .

The curvature tensor K_{jkh}^i satisfies the following relation too

(1.26) a) $g_{rj} \, K_{ikh}^r = K_{ijkh}$ and b) $K_{jkh}^i \, y^j = H_{kh}^i$.

he associate curvature tensor K_{ijkh} satisfies the condition

(1.27) $K_{jikh} + K_{ijkh} = -2 \, C_{ijs} \, K_{rkh}^s \, y^r$.

Ricci tensor K_{jk} and the curvature vector K_j of the curvature tensor K_{jkh}^i are given by

(1.28) a) $K_{jki}^i = K_{jk}$

and b) $K_{jk} \, y^k = K_j$.

2. On Generalized R^h-Recurrent Finsler Space of N^th order

Let us consider a Finsler space F_n whose Cartan's third curvature tensor R_{jkh}^i satisfies the following condition

(2.1) $R_{jkh|m_1|m_2|\dots|m_n}^i = \lambda_{m_1 m_2 \dots m_n} \, R_{jkh}^i + \mu_{m_1 m_2 \dots m_n}\left(\delta_h^i \, g_{jk} - \delta_k^i \, g_{jh}\right)$,

where $R_{jkh}^i \neq 0$ and $|m_1|m_2| \dots |m_n$ are h-covariant differentiation (Cartan's second kind covariant differential operator) with respect to x^{m_n} to nth order, $\lambda_{m_1 m_2 \dots m_n}$ and $\mu_{m_1 m_2 \dots m_n}$ are recurrence tensors fields.

Definition 2.1. A Finsler space F_n whose Cartan's third curvature tensor R_{jkh}^i satisfies the condition (2.1), where $\lambda_{m_1 m_2 \dots m_n}$ and $\mu_{m_1 m_2 \dots m_n}$ are non-null covariant tensors fields, is called a generalized R^h-nth order space and the tensor will be called generalized h-nth tensor. We shall denote this space briefly by GR^h-$n^{th}RF_n$.

Since the metric tensor is a covariant constant, the transvecting of the condition (2.1) by g_{ip} , using (1.11a), (1.16a) and (1.7b), we get

(2.2) $R_{jpkh|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \, R_{jpkh} + \mu_{m_1 m_2 \dots m_n}\left(g_{hp} \, g_{jk} - g_{kp} \, g_{jh}\right)$.

Conversely, the transvection of the condition (2.2) by g^{ip} , yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore a generalized R^h-nth order space may characterized by the condition (2.2).

Therefore, we conclude

Theorem 2.1. The generalized R^h-nth order space may characterized by condition (2.2).

Let us consider an GR^h-$n^{th}RF_n$, which is characterized by the condition (2.1).
Contracting the indices i and h in (2.1), using (1.17a), (1.6) and (1.7b), we get

(2.3) $$R_{jk|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R_{jk} + (n-1)\,\mu_{m_1m_2\dots m_n}\, g_{jk} \quad .$$

Transvecting the equation (2.3) by y^k , using (1.12b), (1.17b) and (1.9c), we get

(2.4) $$R_{j|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R_j + \mu_{m_1m_2\dots m_n}\,(n-1)\, y_j \quad .$$

Further, transvecting (2.1) by g^{jk} , using (1.11b), (1.18b) and in view of (1.6), we get

(2.5) $$R^i_{h|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R^i_h + (n-1)\,\mu_{m_1m_2\dots m_n}\,\delta^i_h \quad .$$

Contracting the indices i and h in the condition (2.5) and using (1.6), we get

(2.6) $$R_{|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R + n\,(n-1)\,\mu_{m_1m_2\dots m_n} \quad , \quad \text{where } R^r_r = R \, .$$

Also, by transvecting the equation (2.3) by g^{jk} , using (1.11b), (1.17d) and in view of (1.6), we get the condition (2.6).

The conditions (2.3), (2.4), (2.5), and (2.6), show that, Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R^i_h and the curvature scalar R (all for Cartan's third curvature tensor R^i_{jkh}) of a generalized R^h-nth order space cannot vanish, because the vanishing of them imply the vanishing of the covariant tensors fields $\mu_{m_1m_2\dots m_n}$, i.e. $\mu_{m_1m_2\dots m_n} = 0$, a contradiction.
Thus, we conclude

Theorem 2.3. In GR^h-$n^{th}RF_n$, Ricci tensor R_{jk} , the curvature vector R_j , the deviation tensor R^i_h and the curvature scalar R (all for Cartan's third curvature tensor R^i_{jkh}) are non-vanishing.

Transvecting the condition (2.3) by y^j , using (1.12b), (1.17c) and (1.9c), we get

(2.7) $$H_{k|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} H_k + (n-1)\,\mu_{m_1m_2\dots m_n}\, y_k \quad .$$

The condition (2.7), shows that, the curvature vector H_k of a generalized R^h- nth order space can not vanish, because the vanishing of it would imply the vanishing of the covariant tensors fields $\mu_{m_1m_2\dots m_n}$, i.e. $\mu_{m_1m_2\dots m_n} = 0$, a contradiction.
Transvecting the condition (2.7) by y^k, using (1.12b), (1.24b) and (1.8), we get

(2.8) $$H_{|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} H + \mu_{m_1m_2\dots m_n} F^2 \quad .$$

Thus, we conclude

Theorem 2.4. In GR^h-$n^{th}RF_n$, the curvature vector H_k and the curvature scalar H are non-vanishing.

Now, we have seen that in a generalized R^h-nth order space, Ricci tensor R_{jk} (of Cartan's third curvature tensor R^i_{jkh}) satisfies the condition (2.3). Conversely, if Ricci tensor R_{jk} of a Finsler space satisfy the condition (2.3), then it need not be the space is a generalized R^h-nth order space. However, the converse is true if the dimension of a Finsler space is 3 or the space is $R3$-like. The proof of this fact is follows:

We know that the associate curvature tensor R_{ijkh} (of Cartan's third curvature tensor R^i_{jkh}) for three dimensioned Finsler space is of the form

(2.9) $$R_{ijkh} = g_{ik}\, L_{jh} + g_{jh}\, L_{ik} - g_{ih}\, L_{jk} - g_{jk}\, L_{ih} \quad , \quad \text{where}$$

(2.10) $$L_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{r}{2}\, g_{ik}\right) \quad \text{and}$$

(2.11) $$r = \frac{1}{n-1}\, R^i_j \quad .$$

Transvecting the condition (2.3) by g^{jP} , using (1.11b), (1.18b) and (1.6), we get

(2.12) $$R^p_{k|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R^p_k + \mu_{m_1m_n\dots m_n}\,(n-1)\,\delta^p_k \quad .$$

Contracting the indices p and k in the condition (2.12), using (2.11) and (1.6), we get

(2.13) $$r_{|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\, r + n\, \mu_{m_1m_2\dots m_n} \quad .$$

Taking the h-covariant derivative of the condition (2.10) with respect to x^m to nth order and using (1.6), we get

(2.14) $$L_{ik|m_1|m_2|\dots|m_n} = \frac{1}{n-2}\left(R_{ik|m_1|m_2|\dots|m_n} - \frac{1}{2}\, r_{|m_1|m_2|\dots|m_n}\, g_{ik}\right) .$$

Using the conditions (2.3) and (2.13) in (2.14), we get

$$L_{ik|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\left[\frac{1}{n-2}\left(R_{ik} - \frac{1}{2}\, r\, g_{ik}\right)\right] + \frac{1}{2}\,\mu_{m_1m_2\dots m_n}\, g_{ik} .$$

In view of the condition (2.10), the above equation implies to

(2.15) $$L_{ik|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} L_{ik} + \frac{1}{2}\,\mu_{m_1m_2\dots m_n}\, g_{ik} .$$

This gives the h-covariant derivative of Ricci tensor L_{ik} in generalized R^h-nth order space.

The h-covariant derivative of the condition (2.9) with respect to x^m to nth order and using (1.11a), we get

$$R_{ijkh|m_1|m_2|\dots|m_n} = g_{ik}\, L_{jh|m_1|m_2|\dots|m_n} + g_{jh}\, L_{ik|m_1|m_2|\dots|m_n}$$
$$g_{ih}\, L_{jk|m_1|m_2|\dots|m_n} + g_{jk}\, L_{ih|m_1|m_2|\dots|m_n} .$$

In view of the condition (2.15), the above equation implies

$$R_{ijkh|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\left(g_{ik}\, L_{jh} + g_{jh}\, L_{ik} - g_{ih}\, L_{jk} - g_{jk}\, L_{ih}\right)$$
$$+\mu_{m_1m_2\dots m_n}\left(g_{ik}\, g_{jh} - g_{ih}\, g_{jk}\right) .$$

In view of the condition (2.9), the above equation implies

$$R_{ijkh|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\, R_{ijkh} + \mu_{m_1m_2\dots m_n}\left(g_{ik}\, g_{jh} - g_{ih}\, g_{jk}\right) .$$

This shows that, the associate curvature tensor R_{ijkh} (of Cartan's third curvature tensor R^i_{jkh}) is a generalized h-recurrent.

In view of (1.16a), the above condition implies (2.1). That is, the h-covariant derivative of the condition (2.9) with respect to x^m to nth order and in view of (1.8a), gives (2.1). This shows that, a three dimensional Ricci generalized R^h-nth order space is necessarily generalized R^h- recurrent space.

Matsumote [28] introduced a Finsler space $F_n (n > 3)$ for which the tensor R_{ijkh} satisfying (2.9) and called it $R3$-like Finsler space F_n . If we consider a $R3$-like Ricci generalized R^h-recurrent space and applied the same process as above we may show that

(2.16) $$R_{ijkh|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\, R_{ijkh} + \mu_{m_1m_2\dots m_n}\left(g_{ik}\, g_{jh} - g_{ih}\, g_{jk}\right) .$$

This, leads to

Theorem 2.7. In $GR^h\text{-}n^{th}RF_n$, is Ricci generalized R^h-recurrent space, but the converse need not be true. However, if the space F_n is $R3$-like, then the converse is also true.

Now, taking the h-covariant derivative for (1.13), with respect to x^m to nth order, we get

(2.17) $$R^i_{jkh|m_1|m_2|\dots|m_n} = \Big[\dot{\partial}_h \Gamma^{*i}_{jk} + \left(\dot{\partial}_s \Gamma^{*i}_{jk}\right) G^s_h + C^i_{jm}\left(\dot{\partial}_k\, G^m_h - G^m_{kt}\, G^t_h\right) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh}$$
$$- \dot{\partial}_k \Gamma^{*i}_{jh} - \left(\dot{\partial}_s \Gamma^{*i}_{jh}\right) G^s_k - C^i_{jm}\left(\dot{\partial}_h\, G^m_k - G^m_{ht}\, G^t_k\right) - \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\Big]_{|m_1|m_2|\dots|m_n} .$$

Using the conditions (2.1), (1.20) and (1.19a) in (2.17), we get

$$\lambda_{m_1m_2\dots m_n}\, R^i_{jkh} + \mu_{m_1m_2\dots m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$$
$$= \Big[\Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks}\, G^s_h + C^i_{jm}\left(G^m_{kh} - G^m_{kt}\, G^t_h\right) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh}$$
$$- \Gamma^{*i}_{jhk} - \Gamma^{*i}_{jhs}\, G^s_k - C^i_{jm}\left(G^m_{hk} - G^m_{ht}\, G^t_k\right) - \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\Big]_{|m_1|m_2|\dots|m_n}$$

By using (1.13), (1.20) and (1.19a), the above equation can be written as

(2.18) $$\Big[\Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks}\, G^s_h + C^i_{jm}\left(G^m_{kh} - G^m_{kt}\, G^t_h\right) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} - \Gamma^{*i}_{jhk} - \Gamma^{*i}_{jhs}\, G^s_k$$
$$- C^i_{jm}\left(G^m_{hk} - G^m_{ht}\, G^t_k\right) - \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\Big]_{|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n}\Big[\Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks}\, G^s_h$$
$$+ C^i_{jm}\left(G^m_{kh} - G^m_{kt}\, G^t_h\right) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} - \Gamma^{*i}_{jhk} + \Gamma^{*i}_{jhs} G^s_k - C^i_{jm}\left(G^m_{hk} - G^m_{ht}\, G^t_k\right)$$
$$- \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk} + \mu_{m_1m_2\dots m_n}\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right) .$$

Thus, we conclude

Theorem 2.8. In GR^h-$n^{th}RF_n$, the h-covariant derivative of the nth order for the tensor $\big[\Gamma^{*i}_{jkh} + \Gamma^{*i}_{jks}\, G^s_h + C^i_{jm}\,(G^m_{kh} - G^m_{kt}\, G^t_h) + \Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} - \Gamma^{*i}_{jhk} - \Gamma^{*i}_{jhs}\, G^s_k - C^i_{jm}\,(G^m_{hk} - G^m_{ht}\, G^t_k) - \Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\big]_{|m_1|m_2|\ldots|m_n}$ is generalized nth- recurrent.

Transvecting (2.18) by y^j, using (1.12b), (1.21a), (1.10b), (1.19b) and (1.9c), we get

(2.19) $$\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big)_{|m_1|m_2|\ldots|m_n} = \lambda_{m_1m_2\ldots m_n}\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big) + \mu_{m_1m_2\ldots m_n}\big(\delta^i_h\, y_k - \delta^i_k\, y_h\big) .$$

Transvecting (2.19) by y_i , using (1.12a), (1.21b), and (1.7b), we get

(2.20) $$\big(-P_{jkh} - P_{jks}\, G^s_h + y_i\,\Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} + P_{jhk} + P_{jhs}\, G^s_k - y_i\,\Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\big)_{|m_1|m_2|\ldots|m_n} = \lambda_{m_1m_2\ldots m_n}\big(-P_{jkh} - P_{jks}G^s_h + y_i\,\Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} + P_{jhk} + P_{jhs}G^s_k - y_i\,\Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\big) + \mu_{m_1m_2\ldots m_n}\big(y_h\, g_{jk} - y_k\, g_{jh}\big) .$$

Further, transvecting (2.19) by g_{ir} , using (1.11a), (1.21c), (1.5), and (1.7b), we get

(2.21) $$\big(P_{krh} + P_{krs}\, G^s_h + \Gamma^*_{mrk}\, G^m_h - P_{hrk} - P_{hrs}\, G^s_k - \Gamma^*_{mrh}\, G^m_k\big)_{|m_1|m_2|\ldots|m_n} = \lambda_{m_1m_2\ldots m_n}\big(P_{krh} + P_{krs}\, G^s_h + \Gamma^*_{mrk}\, G^m_h - P_{hrk} - P_{hrs}\, G^s_k - \Gamma^*_{mrh}\, G^m_k\big) + \mu_{m_1m_2\ldots m_n}\big(g_{hr}\, y_k - g_{kr}\, y_h\big) .$$

The equation (2.19) can be written as

(2.22) $$\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big)_{|m_1|m_2|\ldots|m_n} = \lambda_{m_1m_2\ldots m_n}\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big) + \mu_{m_1m_2\ldots m_n}\big(\delta^i_h\, y_k - \delta^i_k\, y_h\big) .$$

Thus, we conclude

Theorem 2.9. In GR^h-$n^{th}RF_n$, the h-covariant derivative of the nth order for the tensor $\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big)$, $\big(-P_{jkh} - P_{jks}G^s_h + y_i\,\Gamma^{*i}_{mk}\,\Gamma^{*m}_{jh} + P_{jhk} + P_{jhs}G^s_k - y_i\,\Gamma^{*i}_{mh}\,\Gamma^{*m}_{jk}\big)$,
$\big(P_{krh} + P_{krs}\, G^s_h + \Gamma^*_{mrk}\, G^m_h - P_{hrk} - P_{hrs}\, G^s_k - \Gamma^*_{mrh}\, G^m_k\big)$ and
$\big(P^i_{kh} + P^i_{ks}\, G^s_h + \Gamma^{*i}_{mk}\, G^m_h - P^i_{hk} - P^i_{hs}\, G^s_k - \Gamma^{*i}_{mh}\, G^m_k\big)$,
are given by the conditions (2.19), (2.20), (2.21),and (2.22), respectively.

3. Necessary and Sufficient Condition

We know that Cartan's third curvature tensor R^i_{jkh} and Cartan's fourth curvature tensor K^i_{jkh} are connected by the equation (1.15).

Using (1.18a) in (1.15), we get

(3.1) $$R^i_{jkh} = K^i_{jkh} + C^i_{jr}\, H^r_{kh} .$$

Taking the h-covariant derivative for equation (3.1) with respect to x^m to nth order, we get

(3.2) $$R^i_{jkh|m_1|m_2|\ldots|m_n} = K^i_{jkh|m_1|m_2|\ldots|m_n} + \big(C^i_{jr}\, H^r_{kh}\big)_{|m_1|m_2|\ldots|m_n} .$$

Using the condition (2.1) in (3.2), we get

$$\lambda_{m_1m_2\ldots m_n} R^i_{jkh} + \mu_{m_1m_2\ldots m_n}\big(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\big) = K^i_{jkh|m_1|m_2|\ldots|m_n} + \big(C^i_{jr}\, H^r_{kh}\big)_{|m_1|m_2|\ldots|m_n} .$$

By using the condition (3.1), the above equation implies to

(3.3) $$\lambda_{m_1m_2\ldots m_n} K^i_{jkh} + \lambda_{m_1m_2\ldots m_n}\big(C^i_{jr}\, H^r_{kh}\big) + \mu_{m_1m_2\ldots m_n}\big(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\big) = K^i_{jkh|m_1|m_2|\ldots|m_n} + \big(C^i_{jr}\, H^r_{kh}\big)_{|m_1|m_2|\ldots|m_n}$$

This shows that

(3.4) $$K^i_{jkh|m_1|m_2|\ldots|m_n} = \lambda_{m_1m_2\ldots m_n} K^i_{jkh} + \mu_{m_1m_2\ldots m_n}\big(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\big)$$

if and only if

(3.5) $$\left(C^i_{jr}\, H^r_{kh} \right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \left(C^i_{jr}\, H^r_{kh} \right) \quad .$$

Thus, we conclude

Theorem 3.1. In GR^h-$n^{th}RF_n$, Cartan's fourth curvature tensor K^i_{jkh} is generalized nth- recurrent if and only if the tensor $\left(C^i_{jr}\, H^r_{kh} \right)$ is recurrent of nth order.

Contracting the indices i and h in (3.3), using (1.28a), (1.6) and (1.7b), we get

(3.6) $$\lambda_{m_1 m_2 \dots m_n} K_{jk} + \lambda_{m_1 m_2 \dots m_n} \left(C^p_{jr}\, H^r_{kp} \right) + \mu_{m_1 m_2 \dots m_n} (n-1)\, g_{jk}$$
$$= K_{jk|m_1|m_2|\dots|m_n} + \left(C^p_{jr}\, H^r_{kp} \right)_{|m_1|m_2|\dots|m_n} \quad .$$

This shows that

$$K_{jk|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} K_{jk} + \mu_{m_1 m_2 \dots m_n} (n-1)\, g_{jk} \quad ,$$

if and only if

$$\left(C^p_{jr}\, H^r_{kp} \right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \left(C^p_{jr}\, H^r_{kp} \right) \quad .$$

Thus, we conclude

Theorem 3.3. In GR^h-$n^{th}RF_n$, Ricci tensor K_{jk} (of Cartan's fourth curvature tensor K^i_{jkh}) is non-vanishing if and only if the tensor $\left(C^p_{jr}\, H^r_{kp} \right)$ is nth- recurrent.

Transvecting (3.6) by y^k, using (1.12b), (1.28b), (1.24) and (1.9c), we get

$$\lambda_{m_1 m_2 \dots m_n} K_j + \lambda_{m_1 m_2 \dots m_n} \left(C^p_{jr}\, H^r_p \right) + \mu_{m_1 m_2 \dots m_n} (n-1)\, y_j$$
$$= K_{j|m_1|m_2|\dots|m_n} + \left(C^p_{jr}\, H^r_p \right)_{|m_1|m_2|\dots|m_n} \quad .$$

This shows that

$$K_{j|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} K_j + \mu_{m_1 m_2 \dots m_n} (n-1)\, y_j$$

if and only if

$$\left(C^p_{jr}\, H^r_p \right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \left(C^p_{jr}\, H^r_p \right) \quad .$$

Thus, we conclude

Theorem 3.4. In GR^h-$n^{th}RF_n$, the curvature vector K_j (of Cartan's fourth curvature tensor K^i_{jkh}) is non-vanishing if and only if the tensor $\left(C^p_{jr}\, H^r_p \right)$ is nth-recurrent.

4. Some Rules of Tensors in $\mathbf{GR^h}$- $\mathbf{N^{th}}R\ \mathbf{F_n}$

We know that the associate tensor R_{ijkh} of Cartan's third curvature tensor R^i_{jkh} is satisfies the identity

(4.1) $$R_{ijhk} + R_{ihkj} + R_{ikjh} + \left(C_{ijs}\, K^s_{rhk} + C_{ihs}\, K^s_{rkj} + C_{iks}\, K^s_{rjh} \right) y^r = 0 \quad .$$

In view of the condition (1.18a), the identity (4.1) becomes

(4.2) $$R_{ijhk} + R_{ihkj} + R_{ikjh} + C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0 \quad .$$

The h-covariant differentiation of the identity (4.2), with respect to x^m , for nth order gives

(4.3) $$R_{ijhk|m_1|m_2|\dots|m_n} + R_{ihkj|m_1|m_2|\dots|m_n} + R_{ikjh|m_1|m_2|\dots|m_n}$$
$$+\left(C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} \right)_{|m_1|m_2|\dots|m_n} = 0 \quad .$$

Using the condition (2.2) in (4.3), we get

$$\lambda_{m_1 m_2 \dots m_n} R_{ijhk} + \mu_{m_1 m_2 \dots m_n} \left(g_{ih}\, g_{jk} - g_{ik}\, g_{jh} \right) + \lambda_{m_1 m_2 \dots m_n} R_{ihkj}$$
$$+ \mu_{m_1 m_2 \dots m_n} \left(g_{ik}\, g_{hj} - g_{ij}\, g_{hk} \right) + \lambda_{m_1 m_2 \dots m_n} R_{ikjh}$$
$$+\mu_{m_1 m_2 \dots m_n} \left(g_{ij}\, g_{kh} - g_{ih}\, g_{kj} \right) + \left(C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} \right)_{|m_1|m_2|\dots|m_n} = 0$$

Since the metric tensor g_{jk} is symmetric, then the above equation implies to

$$\lambda_{m_1 m_2 \dots m_n} R_{ijhk} + \lambda_{m_1 m_2 \dots m_n} R_{ihkj} + \lambda_{m_1 m_2 \dots m_n} R_{ikjh} + \left(C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} \right)_{|m_1|m_2|\dots|m_n} = 0 \quad .$$

or

(4.4) $$\lambda_{m_1 m_2 \dots m_n} \left(R_{ijhk} + R_{ihkj} + R_{ikjh} \right) + \left(C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} \right)_{|m_1|m_2|\dots|m_n} = 0 \quad .$$

Using the condition (4.2) in (4.4), we get

(4.5) $$\left(C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} \right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \left(C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} \right) \quad .$$

Thus, we conclude

Theorem 4.1. In GR^h-$n^{th}RF_n$, the tensor $\left(C_{ijs} H^s_{hk} + C_{ihs} H^s_{kj} + C_{iks} H^s_{jh} \right)$ behaves as the nth-recurrent.

Transvecting (4.5) by y^h , using (1.12b) and (1.10a), we get

(4.6) $$\left(C_{ijs} H^s_{k} - C_{iks} H^s_{j} \right)_{|m_1|m_2|\dots|m_n} = \lambda_{m_1 m_2 \dots m_n} \left(C_{ijs} H^s_{k} - C_{iks} H^s_{j} \right) \quad .$$

Thus, we conclude

Theorem 4.2. In GR^h-$n^{th}RF_n$, the tensor $\left(C_{ijs} H^s_{k} - C_{iks} H^s_{j} \right)$ behaves as the nth-recurrent.

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دراسة الاشتقاق احادي المعاودة في فضاء فنسلر من الرتب العليا باستخدام مؤثر الانحناء لكارتان

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الملخص

في هذه الورقة قدمنا دراسة الاشتقاق أحادي المعاودة من الرتب العليا في فضاء فنسلر للمؤتر الرابع لكارتان R^i_{jkh} عن طريق استخدام اشتقاق كارتان من الرتبة النونية وعرفنا الفضاء للمشتقة النونية للمؤتر R^i_{jkh} كما يلي: $R^i_{jkh|m_1|m_2|\dots|m_n} = \lambda_{m_1m_2\dots m_n} R^i_{jkh} + \mu_{m_1m_2\dots m_n}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$ حيث الفضاء $R^i_{jkh} \neq 0$ الذي يحقق بعض النتائج والعلاقات الهامة لبعض المؤثرات في هذا الفضاء. وقدمنا بعض النظريات المتعلقة باشتقاق كارتان من الرتبة النونية لبعض المؤترات للمنحنيات تحت منحنى كارتان R^i_{jkh} مع إثباتاتها.

الكلمات المفتاحية: فضاء فنسلر، تعميم R الاشتقاق النوني للفضاء، الاشتقاق ذات الرتب العليا لكارتان، النوع الرابع لمؤثر كارتان R^i_{jkh} والنوع الثاني لمؤثر كارتان P^i_{jkh} .