University of Aden Journal of Natural and Applied Sciences., Vol.28(1) (2024),13-20

University of Aden Journal of Natural and Applied Sciences

Journal homepage: https://uajnas.adenuniv.com

Research Article

On U–Trirecurrent Finsler Space

Abdalstar Ali Mohsen Saleem*

Department of Mathematics, Faculty of Sciences, University of Aden- Yemen.

https://doi.org/10.47372/uajnas.2024.n1.a02

ARTICLE INFO

Received: 07 Dec 2023

Accepted: 24 May 2024

Keywords:

U-trirecurrent space, Douglas tensor, necessary and sufficient condition and projection on indicatrix

Abstract

In the present paper, we introduce a Finsler space F_n for which the hv - curvature tensor satisfies the trirecurrent property in sense of Cartan. The relationship between hv - curvature tensor U^i_{jkh} and Douglas tensor D^i_{jkh} has been studied. We obtain the necessary and sufficient condition for some tensors to be trirecurrent .Finally, the trirecurrent property in a projection on indicatrix with respect to Cartan connection has been studied.

1. Introduction

In this section, we introduced some definitions which are needed in this paper. The definition for normal projective tensor N^i_{jkh} and connection coefficients U^i_{jkh} for it was introduced by Yano [1]. The definition for Douglas tensor and some types of it was studied by Bacso and Matsumoto [2]. Saleem and Abdallah [3] and Pande and Tiwari [4] studied Finsler space F_n for which the normal projective hv- curvature tensor U^i_{jkh} is recurrent in the sense of Cartan's. Saleem and Abdallah ([3], [5]), Qasem and Saleem [6] and Abdallah [7] obtained the necessary and sufficient condition for some tensors to be recurrent and birecurrent. Qasem [8] and AL – Owaidhani [9] discussed the properties trirecurrent Finsler spaces. Saleem ([10], [11]), Qasem [12], Gheorghe [13] and Abdallah [14] discussed the projection on indicatrix for some tensors

Let us consider an n - dimensional Finsler space F_n equipped with the line elements (x,y) and the fundamental metric function F positive homogeneous of degree one in y^i. The vectors y_i and y^i satisfy [15]

$$a)\ y_i y^i = F^2 \ \text{ and } \ b)\ \dot\partial_i y_j = \dot\partial_j y_i = g_{ij}. \tag{1.1}$$

The fundamental metric tensor g_{ij} is defined as

$$g_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j F^2 \tag{1.2}$$

This tensor is homogeneous of degree zero in y^i and symmetric in its lower indices.

Cartan's covariant derivative of the metric function F, vector y^i and unit vector l^i vanish identically, i.e.

$$a)\ F_{|l} = 0$$
$$b)\ y^i_{|l} = 0$$
$$c)\ l^i_{|l} = 0$$

and

$$d)\ l^i = \frac{y^i}{F} \tag{1.3}$$

Cartan's covariant derivative of an arbitrary tensor T^i_h with respect to x^l is given by [16]

$$a)\ \dot\partial_j(T^i_{h|l}) - (\dot\partial_j T^i_h)_{|l} = T^r_h(\dot\partial_j\, \Gamma^{*i}_{lr}) - (\dot\partial_r T^i_h)P^r_{jl},$$
$$b)\ P^r_{jl} = (\dot\partial_j\, \Gamma^{*r}_{hl})y^h = \Gamma^{*r}_{jl}$$

and

$$c)\ P^r_{jl}y^l = 0 \tag{1.4}$$

K.Yano [1] defined the normal projective connection coefficients Π^i_{jk} by

* Correspondence to: Department of Mathematics, Faculty of Sciences, University of Aden- Yemen.
E-mail address: saleemabdalstar@gmail.com

a) $\Pi^i_{jk} = G^i_{jk} - y^i G^r_{jkr}$ and $b)\ G^i_{jk} = \dot{\partial}_j\ G^i_k$. (1.5)

The connection coefficients Π^i_{jk} is positively homogeneous of degree zero in y^i and symmetric in their lower indices and the normal projective tensor N^i_{jkh} is defined as follows [1]:

$$a)\ N^i_{jkh} = \dot{\partial}_j \Pi^i_{kh} + \Pi^i_{rjh}\Pi^r_{ks}y^s + \Pi^i_{rh}\Pi^r_{kj} - k|h\,,$$

$$b)\ \Pi^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{jkr} + y^i G^r_{jkhr})$$

and

$$c)\ \Pi^i_{jkh} = \dot{\partial}_j \Pi^i_{kh} \qquad (1.6)$$

Π^i_{jkh} constitutes the components of a tensor.

Also K. Yano [1] denoted the tensor Π^i_{jkh} by U^i_{jkh}. Thus,

$$a)\ U^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{jkr} + y^i G^r_{jkhr})$$

and

$$b)\ G^r_{jkhr} = \dot{\partial}_j G^r_{khr}. \qquad (1.7)$$

The tensor U^i_{jkh} is called *hv-Curvature tensor* and G^i_{jkh} is connections of hv-curvature tensor [17], this tensor is homogeneous of degree -1 in y^i and symmetric in its last two indices, i.e. $U^i_{jkh} = U^i_{jhk}$.

Also this tensor satisfies the following:

$$a)\ U^r_{jrh} = U^r_{jkr} = G^r_{jkr},$$

$$b)\ U^i_{jkh}y^j = 0$$

and

$$c) U^i_{jkh}\, y^h = U^i_{jhk}\, y^h = U^i_{jk} \qquad (1.8)$$

The tensor U^i_{jk} is called *hv-torsion tensor* and satisfies

$$a)\ U^i_{jk} = U^i_{kj},$$

$$b)\ U^r_{jr} = G^r_{kr}$$

and

$$c)\ U^i_{jk}\, y^k = U^i_{kj}\, y^k = G^i_j \qquad (1.9)$$

where the tensor G^i_j is deviation tensor and homogeneous of degree 1 in y^i satisfy

$$G^i_j\, y^j = 2G^i \qquad (1.10)$$

where G^i is positively homogeneous of degree 2 in y^i.

The tensor U_{jk} is called *hv-Ricci tensor* satisfies the following:

a) $U^r_{rkh} = U_{kh}$ and b) $U_{jk} = \frac{2}{n+1}G_{jk}$, (1.11)

where the tensor G_{jk} is components of the of the projective connection coefficients.

Douglas tensor [2] is given by

$$D^i_{jkh} = U^i_{jkh} - \tfrac{1}{2}(U_{kh}\delta^i_j + U_{jh}\delta^i_k) \qquad (1.12)$$

Definition 1.1. The projection of any tensor T^i_j on indicatrix is given by [13]

$$p.T^i_j = T^\alpha_\beta h^i_\alpha h^\beta_j \qquad (1.13)$$

where the angular metric tensor is homogeneous function of degree zero in y^i, the vector y^i and the unit vector l^i defined by ([13], [18])

$$a) h^i_j := \delta^i_j - l^i l_j,$$

$$b)\ p.y^i = 0$$

and

$$c)\ p.l^i = 0. \qquad (1.14)$$

2. An U-Trirecurrent Space

In this section, we introduce a Finsler space which U^i_{jkh} be trirecurrent in sense of Cartan. Also, we find the condition for some tensors which satisfy the trirecurrent property.

Definition 2.1. A Finsler space F_n for which the normal projective hv- curvature tensor U^i_{jkh} satisfies the condition

$$U^i_{jkh|l|m|n} = b_{lmn}U^i_{jkh}, \quad U^i_{jkh} \neq 0, \qquad (2.1)$$

where b_{lmn} recurrence covariant tensor field of third order, the Finsler space will be called *U – Trirecurrent Finsler space* . We shall denote it briefly by UTR -F_n.

Thus, we conclude

Theorem 2.1. *Every $UBR - F_n$ for which the recurrence tensor field satisfies $a_{lm|n} + a_{lm}\lambda_n \neq 0$, is an $UTR - F_n$.*

Transvecting (2.1) by y^h, using (1.8c) and (1.3b), we get

$$U^i_{jk|l|m|n} = b_{lmn}U^i_{jk}. \qquad (2.2)$$

Contracting the indices i and j in (2.1) and using (1.11a) , we get

$$U_{kh|l|m|n} = b_{lmn}U_{kh}. \qquad (2.3)$$

Also, in view of (1.8a), the contracting of the indices i and h in (2.1), we get

$$G^r_{jkr|l|m|n} = b_{lmn}G^r_{jkr}. \qquad (2.4)$$

In view of (2.3) and (1.11b), we get

$$G_{kh|l|m|n} = b_{lmn} G_{kh}. \quad (2.5)$$

Transvecting (2.2) by y^k, using (1.9c) and (1.3b), we get

$$G^i_{j|l|m|n} = b_{lmn} G^i_j \quad (2.6)$$

Transvecting (2.6) by y^j, using (1.10) and (1.3b), we get

$$G^i_{|l|m|n} = b_{lmn} G^i \quad (2.7)$$

Contracting the indices i and k in (2.2) and using (1.9b), we get

$$G^r_{jr|l|m|n} = b_{lmn} G^r_{jr} \quad (2.8)$$

Thus, we conclude

Theorem 2.2. *The hv- torsion tensor U^i_{jk}, the Ricci tensor U_{jk}, the tensor G^r_{jkr}, the Ricci tensor G_{jk} the deviation G^i_j, the tensor G^r_{jr}, the vector G^i and the tensor G^r_{jr} of $UTR - F_n$ are trirecurrent.*

Let us consider $UTR - F_n$ characterized by (2.1). Differentiating (1.5a) third covariantly with respect to x^l, x^m and x^n in the sense of Cartan and using (1.3b), we get

$$\Pi^i_{jk|l|m|n} = G^i_{jk|l|m|n} - \frac{1}{n+1} y^i G^r_{jkr|l|m|n} \quad (2.9)$$

Using (2.2), (2.4) and (1.5a) in (2.9), we get

$$G^i_{jk|l|m|n} = b_{lmn} G^i_{jk} \quad (2.10)$$

Thus, we conclude

Theorem 2.3. *In an $UTR - F_n$, the tensor G^i_{jk} is trirecurrent.*

Differentiating (1.12) third covariantly with respect to x^l, x^m and x^n in the sense of Cartan, we get

$$D^i_{jkh|l|m|n} = U^i_{jkh|l|m|n} - \frac{1}{2}(\delta^i_j U_{kh|l|m|n} + \delta^i_k U_{jh|l|m|n}) \quad \ldots(2.11)$$

Using (2.1) and (2.3) in (2.11), we get

$$D^i_{jkh|l|m|n} = a_{lmn}\{U^i_{jkh} - \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})\} \quad (2.12)$$

Using (1.12) in (2.12), we get

$$D^i_{jkh|l|m|n} = a_{lmn} D^i_{jkh} \quad (2.13)$$

Thus, we conclude

Theorem 2.4. *Douglas tensor D^i_{jkh} of $UTR - F_n$ is trirecurrent.*

If Douglas tensor D^i_{jkh} is recurrent in a Finsler space, in which hv-Ricci tensor U_{kh} is trirecurrent, then the space is necessarily *UTR-F_n*. This may be seen as follows:

The covariant derivative third of (1.12) with respect to x^l, x^m and x^n in the sense of Cartan, gives

$$U^i_{jkh|l|m|n} = D^i_{jkh|l|m|n} + \frac{1}{2}(\delta^i_j U_{kh|l|m|n} + \delta^i_k U_{jh|l|m|n}) \quad \ldots(2.14)$$

Using (2.13) and (2.3) in (2.14), we get

$$U^i_{jkh|l|m|n} = a_{lmn}\{D^i_{jkh} + \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})\} \ldots(2.15)$$

Using (1.12) in (2.15), we get

$$U^i_{jkh|l|m|n} = a_{lmn} U^i_{jkh}$$

Thus, we conclude

Theorem 2. 5. *In a Finsler space F_n, if Douglas tensor D^i_{jkh} and the hv-Ricci tensor U_{jk} are trirecurrent, then the space considered is necessarily UTR-F_n.*

3. Necessary and Sufficient Condition

We shall try to find the necessary and sufficient condition for some tensors to be trirecurrent in $UTR - F_n$.

Let us consider an $UTR - F_n$ characterized by condition (2.1).

Differentiating (2.4) partially with respect to y^h, we get

$$\dot\partial_h G^r_{jkr|l|m|n} = (\dot\partial_h a_{lmn}) G^r_{jkr} + a_{lmn}(\dot\partial_h G^r_{jkr})$$

Using commutation formula exhibited by (1.4a) for $G^r_{jkr|l|m}$, $G^r_{jkr|l}$ and G^r_{jkr} and using (1.7b) in (3.1), we get

$$G^r_{jkhr|l|m|n} - [\{G^r_{skr}(\dot\partial_h \Gamma^{*s}_{jl}) + G^r_{jsr}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{sjkr} P^s_{hl}\}_{|m} + G^r_{skr|l}(\dot\partial_h \Gamma^{*s}_{jm}) + G^r_{jsr|l}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{jkr|s}(\dot\partial_h \Gamma^{*s}_{lm}) + (G^r_{skhr} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) + G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) + G^r_{tjkr} P^t_{sl}) P^t_{hm}]_{|n} - G^r_{skr|l|m}(\dot\partial_h \Gamma^{*s}_{jn}) - G^r_{jsr|l|m}(\dot\partial_h \Gamma^{*s}_{kn}) - G^r_{jkr|s|m}(\dot\partial_h \Gamma^{*s}_{ln}) - G^r_{jkr|l|s}(\dot\partial_h \Gamma^{*s}_{mn}) - \{G^r_{sjkr|l} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) - G^r_{tjkr} P^t_{sl}\}_{|m} P^t_{hn} - \{G^r_{tkr|l}(\dot\partial_s \Gamma^{*t}_{jm}) - G^r_{jtr|l}(\dot\partial_s \Gamma^{*t}_{km}) - G^r_{jkr|t}(\dot\partial_s \Gamma^{*t}_{lm})\} P^t_{hn} - \{G^r_{tjkr} - G^r_{tkr}(\dot\partial_v \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_v \Gamma^{*t}_{kl}) - G^r_{vjkr} P^v_{tl}\} P^t_{sm} P^s_{hn} = (\dot\partial_h a_{lmn}) G^r_{jkr} + a_{lmn} G^r_{jkhr} \quad (3.2)$$

This shows that

$$G^r_{jkhr|l|m|n} = a_{lmn} G^r_{jkhr}$$

if and only if

$$[\{G^r_{skr}(\dot\partial_h \Gamma^{*s}_{jl}) + G^r_{jsr}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}(\dot\partial_h \Gamma^{*s}_{jm}) + G^r_{jsr|l}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{jkr|s}(\dot\partial_h \Gamma^{*s}_{lm}) + (G^r_{skhr} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) + G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) + G^r_{tjkr}P^t_{sl})P^t_{hm}]_{|n} + G^r_{skr|l|m}(\dot\partial_h \Gamma^{*s}_{jn}) + G^r_{jsr|l|m}(\dot\partial_h \Gamma^{*s}_{kn}) + G^r_{jkr|s|m}(\dot\partial_h \Gamma^{*s}_{ln}) + G^r_{jkr|l|s}(\dot\partial_h \Gamma^{*s}_{mn}) + \{G^r_{sjkr|l} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) - G^r_{tjkr}P^t_{sl}\}_{|m}P^t_{hn} + \{G^r_{tkr|l}(\dot\partial_s \Gamma^{*t}_{jm}) - G^r_{jtr|l}(\dot\partial_s \Gamma^{*t}_{km}) - G^r_{jkr|t}(\dot\partial_s \Gamma^{*t}_{lm})\}P^t_{hn} + \{G^r_{tjkr} - G^r_{tkr}(\dot\partial_v \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_v \Gamma^{*t}_{kl}) - G^r_{vjkr}P^v_{tl}\}P^t_{sm}P^s_{hn} + (\dot\partial_h a_{lmn})G^r_{jkr} = 0 \qquad (3.3)$$

Thus, we conclude

Theorem 3.1. *The tensor G^r_{jkhr} of $UTR - F_n$ is trirecurrent if and only if equation (3.3) holds.*

Transvecting (3.2) by y^l, using (1.4b), (1.4c) and (1.3b), we get

$$y^l\, G^r_{jkhr|l|m|n} = a_{lmn} y^l\, G^r_{jkhr}$$

if and only if

$$[\{G^r_{skr}\Gamma^{*s}_{hj} + y^l G^r_{jsr}(\dot\partial_h \Gamma^{*s}_{km})\}_{|m} + G^r_{skr|l}(\dot\partial_h \Gamma^{*s}_{jm}) + y^l G^r_{jsr|l}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{jkr|s}\Gamma^{*s}_{hm} + (y^l G^r_{skhr} - G^r_{tkr}\Gamma^{*t}_{sj} + G^r_{jtr}\Gamma^{*t}_{sk})P^t_{hm}]_{|n} + y^l G^r_{skr|l|m}(\dot\partial_h \Gamma^{*s}_{jn}) + y^l G^r_{jsr|l|m}(\dot\partial_h \Gamma^{*s}_{kn}) + G^r_{jkr|s|m}\Gamma^{*s}_{hn} + y^l G^r_{jkr|l|s}(\dot\partial_h \Gamma^{*s}_{mn}) + \{y^l G^r_{sjkr|l} - G^r_{tkr}\Gamma^{*t}_{sj} - G^r_{jtr}\Gamma^{*t}_{sk}\}_{|m}P^t_{hn} + \{y^l G^r_{tkr|l}(\dot\partial_s \Gamma^{*t}_{jm}) - y^l G^r_{jtr|l}(\dot\partial_s \Gamma^{*t}_{km}) - G^r_{jkr|t}\Gamma^{*t}_{sm}\}P^t_{hn} + \{G^r_{tjkr} - G^r_{tkr}\Gamma^{*t}_{vj} - G^r_{jtr}\Gamma^{*t}_{vk}\}P^t_{sm}P^s_{hn}y^l + y^l(\dot\partial_h a_{lmn})G^r_{jkr} = 0 \qquad (3.4)$$

Thus, we conclude

Theorem 3.2. *In an $UTR - F_n$, the directional derivative of the tensor G^r_{jkhr} in the directional of y^l is* proportional to the tensor G^r_{jkhr} if and only if equation (3.4) holds.

If we adopt the similar process for (3.2), we get the following theorem

Theorem 3.3. *In an $UTR - F_n$, the directional derivative of the tensor G^r_{jkhr} in the directional of* y^m is proportional to the tensor G^r_{jkhr} if and only if equation

$$[\{y^m G^r_{skr}(\dot\partial_h \Gamma^{*s}_{jl}) + G^r_{jsr}\Gamma^{*s}_{hk} + y^m G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}\Gamma^{*s}_{hj} + G^r_{jsr|l}\Gamma^{*s}_{hk} + G^r_{jkr|s}\Gamma^{*s}_{hl}]_{|n} + G^r_{skr|l|m}(\dot\partial_h \Gamma^{*s}_{jn}) + G^r_{jsr|l|m}(\dot\partial_h \Gamma^{*s}_{kn}) + G^r_{jkr|s|m}(\dot\partial_h \Gamma^{*s}_{ln}) + G^r_{jkr|l|s}(\dot\partial_h \Gamma^{*s}_{mn}) + \{G^r_{sjkr|l} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) - G^r_{tjkr}P^t_{sl}\}_{|m}y^m P^t_{hn} + \{G^r_{tkr|l}\Gamma^{*t}_{sj} - G^r_{jtr|l}\Gamma^{*t}_{sk} - G^r_{jkr|t}\Gamma^{*t}_{sl}\}P^t_{hn} + y^m(\dot\partial_h a_{lmn})G^r_{jkr} = 0$$ *holds.*

If we adopt the similar process for (3.2), we get the following theorem

Theorem 3.4. *In an $UTR - F_n$, the directional derivative of the tensor G^r_{jkhr} in the directional of y^n is proportional to the tensor G^r_{jkhr} if and only if equation*

$$y^n[\{G^r_{skr}(\dot\partial_h \Gamma^{*s}_{jl}) + G^r_{jsr}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}(\dot\partial_h \Gamma^{*s}_{jm}) + G^r_{jsr|l}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{jkr|s}(\dot\partial_h \Gamma^{*s}_{lm}) + (G^r_{skhr} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) + G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) + G^r_{tjkr}P^t_{sl})P^t_{hm}]_{|n} + G^r_{skr|l|m}\Gamma^{*s}_{hj} + G^r_{jsr|l|m}\Gamma^{*s}_{hk} + G^r_{jkr|s|m}\Gamma^{*s}_{hl} + G^r_{jkr|l|s}\Gamma^{*s}_{hm} + (\dot\partial_h a_{lmn})G^r_{jkr} = 0$$ *holds.*

Differentiating (1.7a) three times with respect to x^l, x^m and x^n in the sense of Cartan's, we get

$$U^i_{jkh|l|m|n} = G^i_{jkh|l|m|n} - \frac{1}{n+1}(\delta^i_j G^r_{jkr|l|m|n} + y^i G^r_{jkhr|l|m|n}) \qquad \ldots(3.5)$$

Using (2.1), (1.7a), and (3.2) in (3.5), we get

$$G^i_{jkh|l|m|n} - a_{lmn}G^i_{jkh} = \frac{1}{n+1}y^i[\{G^r_{skr}(\dot\partial_h \Gamma^{*s}_{jl}) + G^r_{jsr}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}(\dot\partial_h \Gamma^{*s}_{jm}) + G^r_{jsr|l}(\dot\partial_h \Gamma^{*s}_{km}) + G^r_{jkr|s}(\dot\partial_h \Gamma^{*s}_{lm}) + (G^r_{skhr} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) + G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) + G^r_{tjkr}P^t_{sl})P^t_{hm}]_{|n} - G^r_{skr|l|m}(\dot\partial_h \Gamma^{*s}_{jn}) - G^r_{jsr|l|m}(\dot\partial_h \Gamma^{*s}_{kn}) - G^r_{jkr|s|m}(\dot\partial_h \Gamma^{*s}_{ln}) - G^r_{jkr|l|s}(\dot\partial_h \Gamma^{*s}_{mn}) - \{G^r_{sjkr|l} - G^r_{tkr}(\dot\partial_s \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_s \Gamma^{*t}_{kl}) - G^r_{tjkr}P^t_{sl}\}_{|m}P^t_{hn} - \{G^r_{tkr|l}(\dot\partial_s \Gamma^{*t}_{jm}) - G^r_{jtr|l}(\dot\partial_s \Gamma^{*t}_{km}) - G^r_{jkr|t}(\dot\partial_s \Gamma^{*t}_{lm})\}P^t_{hn} - \{G^r_{tjkr} - G^r_{tkr}(\dot\partial_v \Gamma^{*t}_{jl}) - G^r_{jtr}(\dot\partial_v \Gamma^{*t}_{kl}) - G^r_{vjkr}P^v_{tl}\}P^t_{sm}P^s_{hn} - (\dot\partial_h a_{lmn})G^r_{jkr} \qquad (3.6)$$

Therefore

$$G^i_{jkh|l|m|n} = a_{lmn}G^i_{jkh}$$

if and only if (3.5) holds.

Thus, we conclude

Theorem 3.5. The tensor G^i_{jkh} *of* $UTR - F_n$ is trirecurrent if and only if equation (3.3) holds.

Transvecting (3.6) by y_i and using (1.1a), we get

$$\frac{y_i}{F^2}\left(G^i_{jkh|l|m|n} - a_{lmn}G^i_{jkh}\right) = \frac{1}{n+1}[\{G^r_{skr}\left(\dot{\partial}_h\, \Gamma^{*s}_{jl}\right) + G^r_{jsr}\left(\dot{\partial}_h\, \Gamma^{*s}_{km}\right) + G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{jm}\right) + G^r_{jsr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{km}\right) + G^r_{jkr|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{lm}\right) + (G^r_{skhr} - G^r_{tkr}\left(\dot{\partial}_s\, \Gamma^{*t}_{jl}\right) + G^r_{jtr}\left(\dot{\partial}_s\, \Gamma^{*t}_{kl}\right) + G^r_{tjkr}P^t_{sl})P^t_{hm}]_{|n} - G^r_{skr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{jn}\right) - G^r_{jsr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{kn}\right) - G^r_{jkr|s|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{ln}\right) - G^r_{jkr|l|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{mn}\right) - \{G^r_{sjkr|l} - G^r_{tkr}\left(\dot{\partial}_s\, \Gamma^{*t}_{jl}\right) - G^r_{jtr}\left(\dot{\partial}_s\, \Gamma^{*t}_{kl}\right) - G^r_{tjkr}P^t_{sl}\}_{|m}P^t_{hn} - \{G^r_{tkr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{jm}\right) - G^r_{jtr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{km}\right) - G^r_{jkr|t}\left(\dot{\partial}_s\, \Gamma^{*t}_{lm}\right)\}P^t_{hn} - \{G^r_{tjkr} - G^r_{tkr}\left(\dot{\partial}_v\, \Gamma^{*t}_{jl}\right) - G^r_{jtr}\left(\dot{\partial}_v\, \Gamma^{*t}_{kl}\right) - G^r_{vjkr}P^v_{tl}\}P^t_{sm}P^s_{hn} - \left(\dot{\partial}_h a_{lmn}\right)G^r_{jkr} \qquad (3.7)$$

Using (3.6) and (1.3d) in (3.7), we get

$$(G^i_{jkh} - l^i l_r G^r_{jkh})_{|l|m|n} = a_{lmn}(G^r_{jkh|l|m} - l^i l_r G^r_{jkh}) \qquad (3.8)$$

Thus, we conclude

Theorem 3.6. *The tensor $G^i_{jkh} - l^i l_r G^r_{jkh}$ of $UTR - F_n$ is trirecurrent.*

Theorem 3.7. *The tensor G^i_{jkh} of $UTR - F_n$ is trirecurrent if and only if $l^i l_r G^r_{jkh}$ is trirecurrent in an $UTR - F_n$.*

Transvecting (3.6) by y^l, using (1.4b), (1.4c) and (1.3b), we get

$$y^l\left(G^i_{jkh|l|m|n} - a_{lmn}G^i_{jkh}\right) = \frac{y^i}{n+1}[\{G^r_{skr}\, \Gamma^{*s}_{hj} + G^r_{jsr}\Gamma^{*s}_{hk}\}_{|m} + y^l G^r_{skr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{jm}\right) + y^l G^r_{jsr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{km}\right) + y^l G^r_{jkr|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{lm}\right) + \{G^r_{skhr} - G^r_{tkr}\Gamma^{*t}_{sj} + G^r_{jtr}\Gamma^{*t}_{sk}\}y^l P^t_{hm}]_{|n} - y^l G^r_{skr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{jn}\right) - y^l G^r_{jsr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{kn}\right) - G^r_{jkr|s|m}\Gamma^{*s}_{hn} - y^l G^r_{jkr|l|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{mn}\right) - \{y^l G^r_{sjkr|l} - G^r_{tkr}\Gamma^{*t}_{sj} - G^r_{jtr}\Gamma^{*t}_{sk}\}_{|m}P^t_{hn} - \{y^l G^r_{tkr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{jm}\right) - y^l G^r_{jtr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{km}\right) - G^r_{jkr|t}\Gamma^{*t}_{sm}\}P^t_{hn} - \{y^l G^r_{tjkr} - G^r_{tkr}\Gamma^{*t}_{vj} - G^r_{jtr}\Gamma^{*t}_{vk}\}P^t_{sm}P^s_{hn} - y^l\left(\dot{\partial}_h a_{lmn}\right)G^r_{jkr} \qquad (3.9)$$

which implies

$$y^l G^i_{jkh|l|m|n} = a_{lmn} y^l G^i_{jkh}$$

if and only if

$$[\{G^r_{skr}\, \Gamma^{*s}_{hj} + G^r_{jsr}\Gamma^{*s}_{hk}\}_{|m} + y^l G^r_{skr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{jm}\right) + y^l G^r_{jsr|l}\left(\dot{\partial}_h\, \Gamma^{*s}_{km}\right) + y^l G^r_{jkr|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{lm}\right) + \{G^r_{skhr} - G^r_{tkr}\Gamma^{*t}_{sj} + G^r_{jtr}\Gamma^{*t}_{sk}\}y^l P^t_{hm}]_{|n} - y^l G^r_{skr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{jn}\right) - y^l G^r_{jsr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{kn}\right) - G^r_{jkr|s|m}\Gamma^{*s}_{hn} - y^l G^r_{jkr|l|s}\left(\dot{\partial}_h\, \Gamma^{*s}_{mn}\right) - \{y^l G^r_{sjkr|l} - G^r_{tkr}\Gamma^{*t}_{sj} - G^r_{jtr}\Gamma^{*t}_{sk}\}_{|m}P^t_{hn} - \{y^l G^r_{tkr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{jm}\right) - y^l G^r_{jtr|l}\left(\dot{\partial}_s\, \Gamma^{*t}_{km}\right) - G^r_{jkr|t}\Gamma^{*t}_{sm}\}P^t_{hn} - \{y^l G^r_{tjkr} - G^r_{tkr}\Gamma^{*t}_{vj} - G^r_{jtr}\Gamma^{*t}_{vk}\}P^t_{sm}P^s_{hn} - y^l\left(\dot{\partial}_h a_{lmn}\right)G^r_{jkr} = 0 \qquad \ldots(3.10)$$

Thus, we conclude

Theorem 3.8. *In an $UTR - F_n$, the directional derivative of the tensor G^i_{jkh} in the directional of y^l is proportional to the tensor G^i_{jkh} if and only if equ. (3.10) holds.*

If we adopt the similar process for (3.6), we get the following theorem

Theorem 3.9. *In an $UTR - F_n$, the directional derivative of the tensor G^i_{jkh} in the directional of y^m is proportional to the tensor G^i_{jkh} if and only if equation*

$$[\{y^m G^r_{skr}\left(\dot{\partial}_h\, \Gamma^{*s}_{jl}\right) + y^m G^r_{jsr}\left(\dot{\partial}_h\, \Gamma^{*s}_{kl}\right) + y^m G^r_{sjkr}P^s_{hl}\}_{|m} + G^r_{skr|l}\Gamma^{*s}_{hj} + G^r_{jsr|l}\Gamma^{*s}_{hk} + G^r_{jkr|s}\Gamma^{*s}_{hl}]_{|n} - y^m G^r_{skr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{jn}\right) - y^m G^r_{jsr|l|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{kn}\right) - y^m G^r_{jkr|s|m}\left(\dot{\partial}_h\, \Gamma^{*s}_{ln}\right) - G^r_{jkr|l|s}\Gamma^{*s}_{hn} - \{G^r_{sjkr|l} - G^r_{tkr}\left(\dot{\partial}_s\, \Gamma^{*t}_{jl}\right) - G^r_{jtr}\left(\dot{\partial}_s\, \Gamma^{*t}_{kl}\right) - G^r_{tjkr}P^t_{sl}\}_{|m} y^m P^t_{hn} - \{G^r_{tkr|l}\Gamma^{*t}_{sj} - G^r_{jtr|l}\Gamma^{*t}_{sk} - G^r_{jkr|t}\Gamma^{*t}_{sl}\}P^t_{hn} - y^m\left(\dot{\partial}_h a_{lmn}\right)G^r_{jkr} = 0$$

holds.

4. Projection on Indicatrix with Respect to Cartan's Connection

In this section, we prove that, if the tensors are trirecurrent, then the projection of them are trirecurrent $UTR - F_n$. Also, we find the condition for the projection of some tensors on indicatrix be trirecurrent.

Let us consider a Finsler space F_n for which the hv-curvature tensor U^i_{jkh} is trirecurrent in the sense of Cartan, i.e. characterized by (2.1).

Now, in view of (1.13), the projection of the hv- curvature tensor U^i_{jkh} on indicatrix is given by

$$p.\, U^i_{jkh} = U^a_{bcd} h^i_a h^b_j h^c_k h^d_h \qquad (4.1)$$

Taking covariant derivative of (4.1) with respect to x^l, x^m and x^n in the sense of Cartan and using the fact that $h^i_{j|l} = 0$, we get

$$(p\ U^i_{jkh})_{|l|m|n} = U^a_{bcd|l|m|n} h^i_a h^b_j h^c_k h^d_h \qquad (4.2)$$

Using (2.1) in (4.2), we get

$$(p\ .\ U^i_{jkh})_{|l|m|n} = \lambda_{lmn} U^a_{bcd} h^i_a h^b_j h^c_k h^d_h \qquad (4.3)$$

In view of (1.13) and by using the fact that $h^i_{j|l} = 0$, equ. (4.3) can be written as

$$(p\ .\ U^i_{jkh})_{|l|m|n} = \lambda_{lmn}(p\ .\ U^i_{jkh})$$

This shows that $p\ .\ U^i_{jkh}$ is trirecurrent.
Thus, we conclude

Theorem 4.1. *The projection of the hv- curvature tensor* U^i_{jkh} *of UTR-*F_n *on indicatrix is trirecurrent in the sense of Cartan.*

If we adopt the similar process for (2.2), (2.3), (2.5), (2.6), (2.7) and (2.10), we get the following theorem

Theorem 4.2. *The projection of the hv-torsion tensor* U^i_{jk}*, the hv-Ricci tensor* U_{jk}*, the Ricci tensor* G_{jk} *, the hv-torsion tensor* G^i_{jk}*, the deviation tensor* G^i_j*, the vector* G^i *and Douglas tensor* D^i_{jkh} *of UTR-*F_n *on indicatrix are trirecurrent in the sense of Cartan.*

Let us consider a Finsler space F_n for which the projection of the hv-curvature tensor U^i_{jkh} on indicatrix is trirecurrent with respect to Cartan's connection characterized by (2.1).

Using (1.13) in (2.1), we get

$$\left(U^a_{bcd} h^i_a h^b_j h^c_k h^d_h\right)_{|l|m|n} = \lambda_{lmn} U^a_{bcd} h^i_a h^b_j h^c_k h^d_h \qquad (4.4)$$

Using (1.14a) in (4.4), we get

$$(U^i_{jkh} - U^i_{jkd}\ell^d\ell_h - U^i_{jch}\ell^c\ell_k + U^i_{jcd}\ell^c\ell_k\ell^d\ell_h - U^a_{jkh}\ell^i\ell_a + U^a_{jkd}\ell^i\ell_a\ell^d\ell_h + U^a_{jch}\ell^i\ell_a\ell^c\ell_k - U^a_{jcd}\ell^i\ell_a\ell^c\ell_k\ell^d\ell_h)_{|l|m|n} = \lambda_{lmn}(U^i_{jkh} - U^h_{jkd}\ell^d\ell_h - U^i_{jch}\ell^c\ell_k + U^h_{jcd}\ell^c\ell_k\ell^d\ell_h - U^a_{jkh}\ell^i\ell_a + U^a_{jkd}\ell^i\ell_a\ell^d\ell_h + U^a_{jch}\ell^i\ell_a\ell^c\ell_k - U^a_{jcd}\ell^i\ell_a\ell^c\ell_k\ell^d\ell_h) \qquad (4.5)$$

Using (2.9), (2.10) and (1.3d) in (4.5), we get

$$(U^i_{jkh} - \tfrac{1}{F}U^i_{jk}\ell_h - \tfrac{1}{F}U^i_{jh}\ell_k - U^a_{jkh}\ell^i\ell_a + U^a_{jk}\ell^i\ell_a\ell_h + \tfrac{1}{F}U^a_{jh}\ell^i\ell_a\ell_k)_{|l|m|n} = \lambda_{lmn}(U^i_{jkh} - \tfrac{1}{F}U^i_{jk}\ell_h - \tfrac{1}{F}U^i_{jh}\ell_k - U^a_{jkh}\ell^i\ell_a + U^a_{jk}\ell^i\ell_a\ell_h + \tfrac{1}{F}U^a_{jh}\ell^i\ell_a\ell_k) \qquad (4.6)$$

Now, if the hv- torsion tensor U^i_{jk} is trirecurrent in the space considered, we have

$$U^i_{jk|l|m|n} = \lambda_{lmn} U^i_{jk} \qquad (A)$$

In view of (A), (1.3a) and (1.3b), the equation (4.6) can be written as

$$(U^i_{jkh} - U^a_{jkh}\ell^i\ell_a)_{|l|m|n} = \lambda_{lmn}(U^i_{jkh} - U^a_{jkh}\ell^i\ell_a) \qquad (4.7)$$

Thus, we conclude

Theorem 4.3. *If the projection of the tensor* $U^i_{jkh} - U^a_{jkh}\ell^i\ell_a$ *on indicatrix is trirecurrent, then the space is UTR -*F_n *characterized by (2.1), provided* U^i_{jk} *is trirecurrent in the sense of Cartan.*

From theorem (4.3), we can also conclude

Theorem 4.4. *In an UTR -*F_n*, the projection of the tensor* U^i_{jkh} *on indicatrix is trirecurrent, if and only if* $U^a_{jkh}\ell_a$ *is trirecurrent.*

If we adopt the similar process for (2.2), (2.3) and (3.19), we get the following theorem

Theorem 4.5. *If the projection of the tensor* $\left(U^i_{jk} - U^a_{jk}\ \ell^i\ell_a\right)$*, the hv-Ricci tensor* U_{jk} *and the tensor* $\left(D^i_{jkh} - D^a_{jkh}\ \ell^i\ell_a\right)$ *on indicatrix are trirecurrent, then the space is UTR -*F_n *.*

From theorem (4.5), we can also conclude

Theorem 4.6. *In an UTR -*F_n*, the projection of the hv-torsion tensor* U^i_{jk}*, the tensor* D^i_{jkh} *on indicatrix are trirecurrent, if and only if* $\left(U^a_{jk}\ \ell_a\right)$ *and* $\left(\ell_a D^a_{jkh}\right)$ *are respectively trirecurrent.*

Acknowledgments

The author expresses his sincere gratitude to the Editor-in-Chief and the anonymous referee for their careful reading of the manuscript and their valuable comments and suggestions that greatly improved this paper.

References

1. Yano, K. (1957): *The theory of lie-derivatives and its applications*. North – Holland publishing Co. Amsterdam, p 196 – 204.
2. Bacso, S. and Matsumoto, M. (1997): On Finsler spaces of Douglas type. A generalization of the notion of Berwald space, Publ. Math. Debrecen, 51, pp. 385-406.
3. Saleem, A. A. M. and Abdallah, A. A. (2022): On U-Recurrent Finsler space, International Research Journal of Innovations in Engineering and Technology, Vol. 6, Iss. 1, (India), p 58 – 63.
4. Pande, H.D. and Tiwari, S. K. (1997): Recurrent Finsler Spaces, J. Nat. Acad. Math. Vol. II, p 98-109.
5. Saleem, A. A. M. and Abdallah, A. A. (2022): Study On U^h –Birecurrent Finsler space, International Journal of Advanced Research in Science, Communication and Technology, Vol. 2, Iss. 3, (India)p 28 – 39.

6. Qasem, F.Y.A. and Saleem, A.A.M. (2010): On U-Birecurrent Finsler Spaces, Univ. Aden J. Nat. and Appl. Sc. Vol. 14, No. 3, p 587-596.
7. Abdallah, A. A., Navlekar, A. A. and Ghadle, K. P. (2019): The necessary and sufficient conditition for some tensors which satisfy a generalized BP – recurrent Finsler space, International Journal of Scientific and Engineering Research, Vol. 10, Iss. 11, (India),p 135 – 140.
8. Qasem, F.Y.A. (2009): On Normal Projection Trirecurrent Finsler Spaces, Univ. Aden J. Nat. and Appl. Sc. Vol. 13, No. 2, p 339-355.
9. AL-Owaidhani, A. N. (2014): On Certain Trirecurrent Finsler Spaces, M, Sc. Thesis, University of Aden, Aden (Yemen), p 42 – 53.
10. Saleem, A. A. M. (2011): On certain Generalized Birecurrent and Trirecurrent Finsler space, M. SC. Thesis, Univ. of Aden, (Aden) (Yemen), p 46 – 52.
11. Saleem, A. A. M. (2016): On certain Problems in Finsler space, D. Ph. Thesis, Univ. of Aden, (Aden) (Yemen), p 73 – 77.
12. Qasem, F.Y.A. (2000): *On Transformation in Finsler Spaces*, D.Phil. Thesis, University of Allahabad, Allahabad (India), p 68 – 78.
13. Gheorghe, M. (2007): The Indicatrix in Finsler Geometry, Analele Stiintifice Ale Uuiversitătii Matematică. Tomul LIII, p 163-180.
14. Abdallah, A. A., Navlekar, A. A. and Ghadle, K. P. (2019): On study generalized BP – recurrent Finsler space, International Journal of Mathematics trends and Technology, Vol. 65, Iss. 4, (India), p 74 – 79.
15. Rund, H. (1981): *The differential geometry of Finsler spaces*, Springer-verlag, Berlin Göttingen-Heidelberg, (1959); 2nd (in Russian), Nauka, Moscow, p 14 – 17.
16. Cartan, É. (1971): Les espaces de Finsler, Actualités, Paris, 79, p 1 – 300.
17. Matsumoto, M. (1976): What is the Finsler geometry? Bull. Korean Math. Vol. 13, No2., p 121-126.
18. Dikshit, S. (1992): *Certain Types of Recurrences in Finsler Spaces*, D.Phil. Thesis, University of Allahabad, Allahabad (India), p 22 – 25.

University of Aden Journal of Natural and Applied Sciences., Vol.28(1) (2024),13-20

مجلة جامعة عدن للعلوم الطبيعية والتطبيقية

Journal homepage: https://uajnas.adenuniv.com

بحث علمي

حول فضاء فنسلر U ثلاثي المعاودة

عبد الستار علي محسن سليم
قسم الرياضيات، كلية العلوم، جامعة عدن ـ اليمن

https://doi.org/10.47372/uajnas.2024.n1.a02

الملخص

في هذا البحث قدمنا الكمية الممتدة التقوسية لفضاء فنسلر التي تحقق خاصية ثلاثي المعاودة بالنسبة لكارتان. وتم دراسة العلاقة بين الكمية الممتدة التقوسية U^i_{jkh} والكمية الممتدة لدوجلاس D^i_{jkh}. وأخيرا تم دراسة الصورة الاسقاطية لثلاثي المعاودة بالنسبة لكارتان.

مفاتيح البحث

التسليم : 07 ديسمبر 2023
القبول : 24 مايو 2024

كلمات مفتاحية :

فضاء U ثلاثي المعاودة،
الكمية الممتدة لدوجلاس،
صورة الإسقاط