University of Aden Journal of Natural and Applied Sciences., vol.29(1) (2025), 43– 53

University of Aden Journal of Natural and Applied Sciences

Journal homepage: https://uajnas.adenuniv.com

Research Article

A Study of Cartan's and Weyl's Curvature Tensors in the Context of Generalized Birecurrent Finsler Geometry

Adel Mohammed Ali Al-Qashbari [1,2] & Fahmi Ahmed Mothana AL-ssallal [3]

[1] Dept. of Math's., Faculty of Educ. Aden, Univ. of Aden, Aden, Yemen
[2] Dept. of Med. Eng., Faculty of the Engineering and Computers, Univ. of Science & Technology, Aden, Yemen
Email: adel.math.edu@aden.net & a.alqashbari@ust.edu
[3] Dept. of Math's., Faculty of Educ. Aden, Univ. of Aden, Aden, Yemen
Email: fahmiassallald55@gmail.com
https://doi.org/10.47372/uajnas.2025.n1.a05

ARTICLE INFO

Received: 25/4/2025

Accepted: 7/6/2025

Keywords:

Curvature Tensors, Generalized Birecurrent Finsler Geometry, Cartan's Tensor, Weyl's Tensor

Abstract

This paper investigates the relationship between Cartan's second curvature tensor and Weyl's projective curvature tensor in the context of Riemannian spaces. The study focuses on deriving a formula that connects these two curvature tensors and exploring the implications of their interactions. Key results of this work include the establishment of a set of equations that describe the covariance and transvectivity of these tensors under various conditions, leading to the formulation of several theorems. The findings provide new insights into generalized birecurrent Finsler spaces and their geometric properties, contributing to the understanding of curvature tensors in higher-dimensional spaces

1. Introduction

The concept of curvature tensors is fundamental in differential geometry, particularly in the study of Riemannian and Finsler spaces. Cartan's second curvature tensor and Weyl's projective curvature tensor are two such important tensors that describe the curvature of space-time and other manifolds. In this paper, we explore the relationship between these two tensors in a four-dimensional Riemannian space, with particular attention to their connection and implications for generalized birecurrent Finsler spaces.

The connection between Cartan's second curvature tensor P^i_{jkh} and Weyl's curvature tensor W^i_{jkh} is expressed by a fundamental equation. We demonstrate that under certain conditions, these tensors satisfy specific geometric properties, such as covariant derivatives and transvectivity relations. By applying these relations, we derive several theorems that provide a deeper understanding of the geometry of these curvature tensors, particularly in the context of generalized birecurrent Finsler spaces.

This study builds on previous work by Ahsan and Ali (2014), who proposed some properties of the Weyl curvature tensor. By expanding on their results, we contribute to the broader understanding of the behavior of curvature tensors and their applications in higher-dimensional geometric spaces. The results presented herein have potential applications in both theoretical and applied mathematics, particularly in fields related to differential geometry and general relativity.

The study of curvature tensors in Finsler geometry has gained significant attention in recent years, with several researchers contributing to the theoretical foundations and applications of these geometric structures. Early works, such as those by Ahsan and Ali (2014, 2016), laid the groundwork for understanding the properties of curvature tensors, particularly the W-curvature tensor in the context of spacetime and general relativity. Their research on the curvature tensor's behavior in relativistic spacetimes and Finsler spaces has been instrumental in extending the theoretical framework of differential geometry.

Subsequent studies, particularly those by Al-Qashbari and colleagues (2024-2025), have made substantial

*Department of Mathematics, Faculty of Education, Aden, University of Aden
E-mail address: Adel_ma71@yahoo.com,

contributions by exploring higher-order curvature tensors and their decomposition in generalized recurrent Finsler spaces. Their research on the Lie and Cartan covariant derivatives, as well as the analysis of Weyl's curvature tensor using Berwald's higher-order derivatives, has opened new pathways in the classification and understanding of curvature structures in advanced geometric spaces. Notable contributions include studies on generalized trirecurrent spaces and the decomposition of curvature fields in Finsler manifolds, which provide deeper insights into the geometric properties of recurrent spaces (Al-Qashbari et al., 2024, 2025).

Additionally, the work of Misra et al. (2014) on higher-order recurrent Finsler spaces with Berwald's curvature tensor field further enriches the understanding of these structures. Goswami (2017) and Pandey et al. (2011) have also contributed to the systematic study of Finsler spaces, particularly focusing on special and generalized recurrent spaces, providing essential theoretical tools for analyzing curvature tensors in these complex geometric settings.

These foundational works set the stage for the current study, which aims to explore the interplay between Cartan's and Weyl's curvature tensors in generalized Finsler spaces. By extending previous findings and incorporating new methods of tensor decomposition and covariant derivatives, this research strives to further unravel the intricacies of curvature structures in Finsler geometry.

2. Preliminaries

The study of curvature and torsion tensors in differential geometry plays a critical role in the understanding of complex geometric spaces, particularly in the context of Finsler and Riemannian geometries. In this paper, we explore several key relationships that involve vectors, torsion tensors, and curvature tensors, focusing on their interplay and implications in generalized geometric structures.

The first set of relations is concerned with two vectors y_i and y^i, which satisfy certain conditions that govern their interaction with the metric tensor g_{jh}.

Two vectors y_i and y^i meet the following conditions

a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, c) $\delta^k_j y^j = y^k$,

d) $\dot{\partial}_i\, y^i = 1$ and e) $\dot{\partial}_j\, y_h = g_{jh}$. (2.1)

These conditions, given by equations (2.1), specify a set of fundamental properties that describe the behavior of the vectors within the geometric space, including their inner product, the relationship with the metric, and the behavior under covariant derivatives.

The quantities g_{ij} and g^{ij} are related by the following conditions:

a) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 , & if \quad i = k , \\ 0 , & if \quad i \neq k . \end{cases}$

b) $g^{jk}{}_{|h} = 0$, c) $g_{ij|h} = 0$,

d) $g_{ir}\, \delta^i_j = g_{rj}$ and e) $g^{jk}\delta^i_k = g^{ji}$. (2.2)

The (v)hv-torsion tensor C^h_{ik} and the (h)hv-torsion tensor C_{ijk} are defined as follows:

a) $C^i_{jk} y^j = C^i_{jk} y^k = 0$,

b) $C_{ijk}\, y^i = C_{ijk}\, y^j = C_{ijk}\, y^k = 0$, c) $C^i_{ki} = C_i$,

d) $C^i_{jk} g_{ih} = C_{jkh}$, e) $C^i_{jk} g^{jk} = C^i$, and

f) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k\, F^2$.

The vector y^i and metric function F vanish identically for Cartan's covariant derivative:

a) $F_{|h} = 0$ and b) $y^i{}_{|h} = 0$. (2.4)

The second-order h-covariant derivative of an arbitrary vector field with respect to x^k and x^j, successively, is given by:

$$X^i_{|k|j} = \partial_j\big(X^i_{|k}\big) - \big(X^i_{|r}\big)\Gamma^{*r}_{kj} + \big(X^r_{|k}\big)\Gamma^{*i}_{rj} - \big(\partial_j X^i_{|k}\big)\,\Gamma^{*i}_{js}\, y^s.$$

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)}\dot{\partial}_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r\dot{\partial}_j H_{hr}\big) - \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r\dot{\partial}_j H_{kr}\big) \; ;$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\left\{\frac{\delta^i_{[j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\,r}\big)\right\} \quad \text{and}$$

$$W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)}\big(\dot{\partial}_r H^r_j - \dot{\partial}_j H\big)\, y^i, \text{ respectively.}$$

The tensor W^i_{jkh}, the torsion tensor W^i_{jk} , and the deviation tensor W^i_j are defined as:

a) $W^i_{jkh}\, y^j = W^i_{kh}$, b) $W^i_{kh}\, y^k = W^i_h$,

c) $W^i_{jki} = W_{jk}$ and d) $g_{ir}\, W^i_{jkh} = W_{rjkh}$.

Additionally, assuming that the tensors W^i_j and W_{jk} satisfy the following identities, we define the various curvature and torsion tensors as follows:

a) $W^i_k\, y^k = 0$, b) $W^i_i = 0$, c) $g_{ir}\, W^i_j = W_{rj}$,

d) $g^{jk} W_{jk} = W$ and e) $W_{jk}\, y^k = 0$. (2.10)

Cartan's second curvature tensor P^i_{jkh}, the (v)hv-torsion tensor P^i_{kh}, and the associated tensor P_{rjkh} are expressed in the subsequent manner. The Ricci tensor P_{jk} and the vector P_k are also defined as part of this framework.

a) $P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr} P^r_{kh} - C^i_{jh|k}$,

b) $P^i_{jkh} y^j = P^i_{kh} = \Gamma^{*i}_{jkh} y^j = C^i_{kh|r} y^r$,

c) $P^i_{jkh} y^k = 0 = P^i_{jkh} y^h$, d) $P^i_{kh} = G^i_{kh} - \Gamma^{*i}_{kh}$,

e) $g_{ir} P^i_{kh} = P_{rkh}$, f) $P^i_{kh} y^k = 0 = P^i_{kh} y_i$,

g) $P^i_{jkh} - P^i_{jhk} = -S^i_{jkh|r} y^r$, h) $g_{ir} P^i_{jkh} = P_{rjkh}$,

i) $P_{ijkh} g^{kh} = P_{ij} - P_{ji}$, j) $P^i_{jki} = P_{jk}$ and

k) $P^i_{ki} = P_k$. (2.11)

Furthermore, Cartan's third curvature tensor R^i_{jkh}, the Ricci tensor R_{jk}, the vector H_k, and the scalar curvature H are defined as follows.

a) $R_{jk} y^j = H_k$, b) $R_{jk} y^k = R_j$, c) $R^i_i = R$,

d) $H_k y^k = (n-1)H$ and e) $R_{rh} = g_{ri} R^i_h$. (2.12)

Similarly, Cartan's first curvature tensor S^i_{jkh}, the Ricci tensor S_{jk}, the tensor S^i_k, and the scalar curvature S are characterized as:

a) $S^i_{jkh} = C^i_{rk} C^r_{jh} - C^i_{rh} C^r_{jk}$, b) $S^i_{jki} = S_{jk}$,

c) $S = S_{kh} g^{kh}$, d) $S_{rjkh} = g_{ri} S^i_{jkh}$ and

e) $S_{kh} g^{ih} = S^i_k$. (2.13)

3. The Extension of Generalized $W_{|h}$– Birecurrent Finsler Space

In this section, we introduce a new class of Finsler spaces, namely the generalized $W_{|h}$-birecurrent spaces. These spaces extend the concept of birecurrence to a broader context, revealing significant geometric properties. We explore the curvature tensor of these spaces and present several characterization theorems.

In the course of our study, we define $|l|m$ as the covariant derivative of second order. Moreover, we extend Cartan's covariant derivative framework to derive the generalized expression for Weyl's projective curvature tensor W^i_{jkh}, which is given by:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}). \quad (3.1)$$

A Finsler space F_n in which the curvature tensor W^i_{jkh} satisfies the condition (3.1) is referred to as a generalized $W_{|h}$-recurrent space and is denoted by $W_{|h} - RF_n$, where $|m$ represents the h-covariant derivative with respect to x^m.

By taking the h-covariant derivative of equation (3.1) with respect to x^l and using equation (2.2c), we obtain:

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m W^i_{jkh|l} + \mu_{m|l}(\delta^i_h g_{jk} - \delta^i_k g_{jh})$$

By substituting equation (3.1) into the above expression, we obtain:

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m\{\lambda_l W^i_{jkh} + \mu_l(\delta^i_h g_{jk} - \delta^i_k g_{jh})\} + \mu_{m|l}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) . \quad (3.2)$$

This simplifies to:

$$W^i_{jkh|m|l} = (\lambda_{m|l} + \lambda_m \lambda_l) W^i_{jkh} + (\lambda_m \mu_l + \mu_{m|l})(\delta^i_h g_{jk} - \delta^i_k g_{jh}) .$$

Equation (3.2), can be expressed as

$$W^i_{jkh|m|l} = a_{ml} W^i_{jkh} + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) . \quad (3.3)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$ and $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ are second-order non-zero covariant tensor fields, respectively.

A Finsler space F_n in which the curvature tensor W^i_{jkh} satisfies the condition (3.3) is called a generalized $W_{|h}$-birecurrent space and is denoted by $GW_{|h} - BRF_n$.

From equation (2.3b), equation (3.1) can be rewritten as:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \gamma_m(W^i_h C_{ijk} y^i - W^i_k C_{ijh} y^i) . \quad (3.4)$$

By applying the conditions (2.3f), (2.1b), (2.1d), and (2.1e) to equation (3.4), we obtain:

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh}) . \quad (3.5)$$

A Finsler space F_n in which the curvature tensor W^i_{jkh} satisfies the condition (3.5) is referred to as the generalized $W_{|h}$-recurrent space and is denoted by $G^{2nd} W_{|h} - RF_n$.

By taking the h-covariant derivative of equation (3.5) with respect to x^l, we obtain:

$$W^i_{jkh|m|l} = \lambda_{m|l} W^i_{jkh} + \lambda_m W^i_{jkh|l} + \mu_{m|l}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})_{|l} + \tfrac{1}{4}\gamma_{m|l}(W^i_h g_{jk} - W^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh})_{|l} . \quad (3.6)$$

By applying equations (2.2c) and (3.5) to equation (3.6), we obtain:

$$W^i_{jkh|m|l} = (\lambda_{m|l} + \lambda_m \lambda_l) W^i_{jkh} + (\lambda_m \mu_l + \mu_{m|l})(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4}(\lambda_m \gamma_l + \gamma_{m|l})(W^i_h g_{jk} - W^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh})_{|l}. \quad (3.7)$$

The equation (3.7), can be expressed as:

$$W^i_{jkh|m|l} = a_{ml} W^i_{jkh} + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \tfrac{1}{4} c_{ml}(W^i_h g_{jk} - W^i_k g_{jh}) + \tfrac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh})_{|l} . \quad (3.8)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ and $c_{ml} = \lambda_m \gamma_l + \gamma_{m|l}$ are non-zero second order covariant tensor fields, γ_m and μ_m are non-zero first order covariant vector fields, respectively.

Definition 3.1. In a Finsler space where Weyl's projective curvature tensor W^i_{jkh} satisfies condition (3.8), the space is referred to as a generalized $W_{|h}$-birecurrent space, and the associated tensor is termed a generalized h-birecurrent tensor. These spaces and tensors are abbreviated as $G^{2nd}\ W_{|h} - BRF_n$ and $G^{2nd}\ h - BR$, respectively.

Result 3.1. Every generalized $W_{|h}$-recurrent space is also a generalized $W_{|h}$-birecurrent space.

By transvecting condition (3.8) into a higher-dimensional space using y^j, and applying equations (2.1a), (2.3b), (2.4b), and (2.9a), we derive:

$$W^i_{kh|m|l} = a_{ml} W^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4} c_{ml}(W^i_h y_k - W^i_k y_h) + \frac{1}{4}\gamma_m (W^i_h y_k - W^i_k y_h)_{|l}. \quad (3.9)$$

Again, transvecting condition (3.9) to a higher dimensional space using by y^k, and applying equations (2.1b), (2.2a), (2.2c), (2.4b), (2.10a) and (2.9b), we obtain:

$$W^i_{h|m|l} = a_{ml} W^i_h + b_{ml}(\delta^i_h F^2 - y^i y_h) + \frac{1}{4} c_{ml} W^i_h F^2 + \frac{1}{4}\gamma_m \left(W^i_h F^2\right)_{|l}. \quad (3.10)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.1. In the context of $G^{2nd}\ W_{|h} - BRF_n$, the $h-$covariant derivative of second-order for the torsion tensor W^i_{kh} and deviation tensor W^i_h are expressed by equations (3.9) and (3.10).

By contracting the index space through summation over i and h in the condition (3.8), and applying relations (2.2d), (2.2a), (2.9c), (2.10b) and (2.10c), we obtain the following result

$$W_{jk|m|l} = a_{ml} W_{jk} + (n-1) b_{ml} g_{jk} - \frac{1}{4} c_{ml} W_{jk} - \frac{1}{4}\gamma_m W_{jk|l} . \quad (3.11)$$

By transvecting condition to a higher-dimensional space (3.8) by g_{ir}, and applying relations (2.2d), (2.2c), (2.9d), and (2.10c), we obtain

$$W_{rjkh|m|l} = a_{ml} W_{rjkh} + b_{ml}(g_{rh} g_{jk} - g_{rk} g_{jh}) + \frac{1}{4} c_{ml}(W_{rh} g_{jk} - W_{rk} g_{jh}) + \frac{1}{4}\gamma_m (W_{rh} g_{jk} - W_{rk} g_{jh})_{|l} . \quad (3.12)$$

Therefore, the proof of theorem is completed, we can say

Theorem 3.2. In the context of $G^{2nd}\ W_{|h} - BRF_n$, the Ricci W_{jk} and the associate tensor W_{rjkh} represent a generalized birecurrent Finsler space, as defined by equations (3.11) and (3.12), respectively.

By transvecting condition (3.11) with g^{jk}, and applying relations (2.2e) and (2.10d), we obtain the following result

$$W_{|m|l} = a_{ml} W + n(n-1) b_{ml} - \frac{1}{4} c_{ml} W - \frac{1}{4}\gamma_m W_{|l} \quad (3.13)$$

From condition (3.13), we show that the curvature scalar W does not equal to zero because if the vanishing of W would imply $a_{ml} = 0$ and $b_{ml} = 0$, that is a contradiction.

Therefore, the proof of theorem is completed, we can say

Theorem 3.3. In the context of $G^{2nd}\ W_{|h} - BRF_n$, the scalar W in equation (3.13) is non-vanishing.

We consider an n-dimensional Finsler space F_n, the Weyl's projective curvature tensor W^i_{jkh} satisfies the condition (3.5) and (3.8), These spaces denoted by $G^{2nd}\ W_{|h} - RF_n$ and $G^{2nd}\ W_{|h} - BRF_n$, respectively.

4. Relationship Between Weyl's Curvature Tensor and Cartan's Second Curvature Tensor

Finsler geometry, as an extension of Riemannian geometry, offers a robust framework for modeling various physical phenomena. In Finsler spaces, the curvature properties of the space are described by several curvature tensors, among which Weyl's curvature tensor and Cartan's second curvature tensor play pivotal roles. While the geometric interpretations and physical implications of these tensors have been extensively explored, the relationship between them remains an area of active investigation. This study aims to examine the connection between Weyl's curvature tensor and Cartan's second curvature tensor in Finsler spaces. By analyzing their algebraic and geometric properties, we aim to derive new identities and inequalities that establish links between these two tensors. The results of this investigation are expected to enhance our understanding of the curvature structure in Finsler spaces and offer valuable insights for their applications in physics, particularly in the context of gravitational theories and cosmology.

Some properties of the W^i_{jkh} curvature tensor was proposed by Ahsan and Ali [2], in 2014.

For a Riemannian space with $(n = 4)$, it is well-established that Cartan's second curvature tensor P^i_{jkh} and Weyl's projective curvature tensor W^i_{jkh} are related by the following formula:

$$W^i_{jkh} = P^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h \right) \quad (4.1)$$

By taking the covariant derivative of (4.1), with respect to x^m and x^l in the sense of Cartan, we get

$$W^i_{jkh|m|l} = P^i_{jkh|m|l} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l} . \quad (4.2)$$

By substituting equations (3.8) and (4.1) in to (4.2), we obtain:

$$P^i_{jkh|m|l} = a_{ml}\left(P^i_{jkh} + \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h \right)\right) + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4} c_{ml}(W^i_h g_{jk} - W^i_k g_{jh}) + \frac{1}{4}\gamma_m (W^i_h g_{jk} - W^i_k g_{jh})_{|l} - \frac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h \right)_{|m|l}$$

Alternatively, this can be expressed as:

$P^i_{jkh|m|l} = a_{ml}\, P^i_{jkh} + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{3} a_{ml}(\,\delta^i_k R_{jh} - g_{jk} R^i_h) + \frac{1}{4} c_{ml}(W^i_h g_{jk} - W^i_k g_{jh}) + \frac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh})_{|l} - \frac{1}{3}(\,\delta^i_k R_{jh} - g_{jk} R^i_h)_{|m|l}.$ (4.3)

This demonstrates that

$$P^i_{jkh|m|l} = a_{ml}\, P^i_{jkh} + b_{ml}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) + \frac{1}{4} c_{ml}(W^i_h g_{jk} - W^i_k g_{jh}) + \frac{1}{4}\gamma_m(W^i_h g_{jk} - W^i_k g_{jh})_{|l}\,. \quad (4.4)$$

If and only if

$$(\delta^i_k R_{jh} - g_{jk} R^i_h)_{|m|l} = a_{ml}(\delta^i_k R_{jh} - g_{jk} R^i_h)\,. \quad (4.5)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.1. In the context of $G^{2nd} P_{|h} - BRF_n$, Cartan's 2[th] curvature tensor P^i_{jkh} defines a generalized birecurrent Finsler space if and only if the tensor $(\delta^i_k R_{jh} - g_{jk} R^i_h)$ is a generalized birecurrent Finsler space.

By transvecting condition (4.3) with y^j, and utilizing equations (2.1a), (2.4b), (2.11b) and (2.12a), we obtain the following result

$P^i_{kh|m|l} = a_{ml}\, P^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{3} a_{ml}(\,\delta^i_k H_h - y_k R^i_h) + \frac{1}{4} c_{ml}(W^i_h y_k - W^i_k y_h) + \frac{1}{4}\gamma_m(W^i_h y_k - W^i_k y_h)_{|l} - \frac{1}{3}(\delta^i_k H_h - y_k R^i_h)_{|m|l}\,.$ (4.6)

This demonstrates that

$$P^i_{kh|m|l} = a_{ml}\, P^i_{kh} + b_{ml}(\delta^i_h y_k - \delta^i_k y_h) + \frac{1}{4} c_{ml}(W^i_h y_k - W^i_k y_h) + \frac{1}{4}\gamma_m(W^i_h y_k - W^i_k y_h)_{|l}\,. \quad (4.7)$$

If and only if

$$(\delta^i_k H_h - y_k R^i_h)_{|m|l} = a_{ml}(\delta^i_k H_h - y_k R^i_h)\,. \quad (4.8)$$

Therefore, the proof of theorem is completed, we conclude

Theorem 4.2. In the context of $G^{2nd} P_{|h} - BRF_n$, the covariant derivative of the second orders for the torsion tensor P^i_{kh} defines a generalized birecurrent Finsler space if and only if the condition in equation (4.8) is satisfied.

By transvecting condition (4.6) with y^k, and applying relations (2.1b), (2.1c), (2.4a), (2.4b), (2.11f) and (2.10a), we obtain the following result

$(\,y^i H_h - F^2 R^i_h)_{|m|l} = a_{ml}(\,y^i H_h - F^2 R^i_h) + 3b_{ml}(\delta^i_h F^2 - y^i y_h) + \frac{3}{4}\gamma_m W^i_{h|l} F^2 + \frac{3}{4} c_{ml} W^i_h F^2.$ (4.9)

This demonstrates that

$$(y^i H_h)_{|m|l} = a_{ml}(y^i H_h) - 3b_{ml}(y^i y_h) + \frac{3}{4}\gamma_m W^i_{h|l} F^2 + \frac{3}{4} c_{ml} W^i_h F^2. \quad (4.10)$$

If and only if

$$(F^2 R^i_h)_{|m|l} = a_{ml}(\,F^2 R^i_h) + 3b_{ml}(\delta^i_h F^2)\,. \quad (4.11)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.3. In the context of $G^{2nd} P_{|h} - BRF_n$, the covariant derivative of the second orders for the tensor $(\,y^i H_h)$ represents a generalized birecurrent Finsler space if and only if the condition in equation (4.11) is satisfied.

By contracting the indices i and h in equations (4.6) and (4.9), respectively and utilizing equations ($n = 4$), (2.2a), (2.1a), (2.1b), (2.4a), (2.11k), (2.12d), (2.12c) and (2.10b), we obtain the following result:

$P_{k|m|l} = a_{ml} P_k + (n-1) b_{ml}\, y_k - \frac{1}{4}\lambda_m(W^i_k y_i) - \frac{1}{4}(W^i_k y_i)_{|l} + \frac{1}{3} a_{ml}(H_k - y_k R) - \frac{1}{3}(H_k - y_k R)_{|m|l}\,.$ …(4.12)

This demonstrates that

$$P_{k|m|l} = a_{ml} P_k + (n-1) b_{ml}\, y_k - \frac{1}{4}\lambda_m(W^i_k y_i) - \frac{1}{4}(W^i_k y_i)_{|l}\,. \quad (4.13)$$

If and only if

$$(H_k - y_k R)_{|m|l} = a_{ml}(H_k - y_k R)\,. \quad (4.14)$$

And

$$(3H - F^2 R)_{|m|l} = a_{ml}(3H - F^2 R)_{|m|l} + 3(n-1) b_{ml}\, F^2\,. \quad (4.15)$$

This demonstrates that

$$(H)_{|m|l} = a_{ml}(H) + n b_{ml}\, F^2\,. \quad (4.16)$$

If and only if

$$(F^2 R)_{|m|l} = a_{ml}(F^2 R) + 3b_{ml}\, F^2. \quad (4.17)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.4. In the context of $G^{2nd} P_{|h} - BRF_n$, vector P_k and scalar (H) are defined in equations (4.13) and (3.16), respectively, provided that the conditions (4.14) and (3.17) are satisfied.

By contracting the indices i and h in equation (4.3) and utilizing equations (2.2d), (2.1b), (2.12c), (2.11j), (2.10d) and (2.10b), we obtain the following result:

$P_{jk|m|l} = a_{ml} P_{jk} + (n-1) b_{ml}\, g_{jk} - \frac{1}{4} c_{ml} W_{jk} - \frac{1}{4}\gamma_m W_{jk|l} - \frac{1}{3}(\,R_{jk} - g_{jk} R)_{|m|l} + \frac{1}{3} a_{ml}(\,R_{jk} - g_{jk} R).$ …(4.18)

This demonstrates that

$$P_{jk|m|l} = a_{ml} P_{jk} + (n-1) b_{ml}\, g_{jk} - \frac{1}{4} c_{ml} W_{jk} - \frac{1}{4}\gamma_m W_{jk|l}\,. \quad (4.19)$$

If and only if

$$(R_{jk} - g_{jk} R)_{|m|l} = a_{ml}(\,R_{jk} - g_{jk} R)\,. \quad (4.20)$$

In conclusion the proof of theorem is completed, we get

Theorem 4.5. In the context of $G^{2nd} P_{|h} - BRF_n$, P-Ricci tensor P_{jk} is defined in equation (4.19), provided that the condition (4.20) is satisfied.

By transvecting condition (4.18) with y^k, and applying relations (2.1a), (2.1c), (2.4b), (2.10e) and (2.12b), we obtain the following result

$$\left(y^k P_{jk|m|l}\right) = a_{ml}\left(y^k P_{jk}\right) + (n-1) b_{ml} y_j - \tfrac{1}{3}\left(R_j - y_j R\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(R_j - y_j R\right). \quad (4.21)$$

This demonstrates that

$$\left(y^k P_{jk}\right)_{|m|l} = a_{ml}\left(y^k P_{jk}\right) + (n-1)\, b_{ml}\, y_j \,. \quad (4.22)$$

If and only if

$$\left(R_j - y_j R\right)_{|m|l} = a_{ml}\left(R_j - y_j R\right). \quad (4.23)$$

By transvecting conditions (4.3) and (4.18) with g^{jk}, respectively, and applying relations (2.2a), (2.2e), (2.2b) and (2.10d), we obtain:

$$g^{jk} P^i_{jkh|m|l} = a_{ml}\left(g^{jk} P^i_{jkh}\right) + (n-1) b_{ml} \delta^i_h + \tfrac{1}{4}(n-1) c_{ml} W^i_h + \tfrac{1}{4}(n-1)\gamma_m W^i_{h|l} - \tfrac{1}{3}(1-n)\left(R^i_h\right)_{|m|l} + \tfrac{1}{3}(1-n) a_{ml} R^i_h \,. \quad (4.24)$$

This demonstrates that

$$\left(g^{jk} P^i_{jkh}\right)_{|m|l} = a_{ml}\left(g^{jk} P^i_{jkh}\right) + (n-1) b_{ml} \delta^i_h + \tfrac{1}{4}(n-1) c_{ml} W^i_h + \tfrac{1}{4}(n-1)\gamma_m W^i_{h|l} \,. \quad (4.25)$$

If and only if

$$\left(R^i_h\right)_{|m|l} = a_{ml} R^i_h \,. \quad (4.26)$$

And

$$\left(g^{jk} P_{jk}\right)_{|m|l} = a_{ml}\left(g^{jk} P_{jk}\right) + (n-1) n b_{ml} - \tfrac{1}{4}\gamma_m W_{|l} - \tfrac{1}{4} c_{ml} W - \tfrac{1}{3}(1-n)(R)_{|m|l} + \tfrac{1}{3}(1-n) a_{ml} R \,. \quad (4.27)$$

This demonstrates that

$$\left(g^{jk} P_{jk}\right)_{|m|l} = a_{ml}\left(g^{jk} P_{jk}\right) + (n-1) n b_{ml} - \tfrac{1}{4}\gamma_m W_{|l} - \tfrac{1}{4} c_{ml} W \,. \quad (4.28)$$

If and only if

$$R_{|m|l} = a_{ml} R \,. \quad (4.29)$$

In conclusion the proof of theorem is completed, we can say

Theorem 4.6. In the context of $G^{2nd} P_{|h} - BRF_n$, the tensors $\left(y^k P_{jk}\right)$, $\left(g^{jk} P^i_{jkh}\right)$ and $\left(g^{jk} P_{jk}\right)$ are defined in equations (4.22), (4.25) and (4.28), respectively, provided that the conditions (4.23), (4.26) and (4.29), are satisfied.

By transvecting condition (4.3) by g_{ir}, and applying relations (2.2d), (2.2c), (2.10c) and (2.11h), we obtain

$$P_{rjkh|m|l} = a_{ml} P_{rjkh} + b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) - \tfrac{1}{3}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l} + \tfrac{1}{4} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right). \quad \ldots(4.30)$$

This demonstrates that

$$P_{rjkh|m|l} = a_{ml} P_{rjkh} + b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l} + \tfrac{1}{4} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right). \quad \ldots(4.31)$$

If and only if

$$\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right) \quad (4.32)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.7. In the context of $G^{2nd} P_{|h} - BRF_n$, associate tensor P_{rjkh} (Cartan's 2[th] curvature tensor P^i_{jkh}) represents a generalized birecurrent Finsler space, if and only if the condition (4.32), is satisfied.

By transvecting condition (4.6) with g_{ir}, and applying relations (2.2d), (2.2c), (2.11e), (2.10c) and (2.12e), we obtain the following result

$$P_{rkh|m|l} = a_{ml} P_{rkh} + b_{ml}\left(g_{rh} y_k - g_{rk} y_h\right) + \tfrac{1}{4} c_{ml}\left(W_{rh} y_k - W_{rk} y_h\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} y_k - W_{rk} y_h\right)_{|l} - \tfrac{1}{3}\left(g_{rk} H_h - y_k R_{rh}\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(g_{rk} H_h - y_k R_{rh}\right). \quad (4.33)$$

This demonstrates that

$$P_{rkh|m|l} = a_{ml} P_{rkh} + b_{ml}\left(g_{rh} y_k - g_{rk} y_h\right) + \tfrac{1}{4} c_{ml}\left(W_{rh} y_k - W_{rk} y_h\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} y_k - W_{rk} y_h\right)_{|l} \,. \quad (4.34)$$

If and only if

$$\left(g_{rk} H_h - y_k R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk} H_h - y_k R_{rh}\right). \quad (4.35)$$

Thus, the proof of theorem is completed, we get

Theorem 4.8. In the context of $G^{2nd} P_{|h} - BRF_n$, the associative tensor P_{rkh} of (Cartan's 2[th] curvature tensor P^i_{jkh}) represents a generalized birecurrent Finsler space, provided that the condition (4.35) is satisfied.

By transvecting condition (4.30) with g^{kh}, and applying relations (2.2b) and (2.11i), we obtain the following result

$$\left(P_{rj} - P_{jr}\right)_{|m|l} = a_{ml}\left(P_{rj} - P_{jr}\right) + g^{kh}\left\{b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) - \tfrac{1}{3}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l} + \tfrac{1}{4} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)\right\}. \quad (4.36)$$

This demonstrates that

$$\left(P_{rj} - P_{jr}\right)_{|m|l} = a_{ml}\left(P_{rj} - P_{jr}\right). \quad (4.37)$$

If and only if

$$g^{kh}\left\{b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) - \tfrac{1}{3}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right)_{|m|l} + \tfrac{1}{3} a_{ml}\left(g_{rk} R_{jh} - g_{jk} R_{rh}\right) + \tfrac{1}{4}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l} + \tfrac{1}{4} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)\right\} = 0 \,. \quad (4.38)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.9. In the context of $G^{2nd} P_{|h} - BRF_n$, the tensor $\left(P_{rj} - P_{jr}\right)$ is defined a generalized birecurrent Finsler space, provided that the condition (4.38) is satisfied.

It is known that Cartan's second curvature tensor P^i_{jkh} and Cartan's first curvature tensor S^i_{jkh} are connected by the formula

$$P^i_{jkh} - P^i_{jhk} = \left(-S^i_{jkh|r}\, y^r\right). \qquad (4.39)$$

By taking the h −covariant derivative of (4.39), with respect to x^m and x^l, we get

$$P^i_{jkh|m|l} - P^i_{jhk|m|l} = \left(-S^i_{jkh|r} y^r\right)_{|m|l}. \qquad (4.40)$$

By substituting equations (4.3) and (4.39) in to (4.40), we obtain:

$$\left(-S^i_{jkh|r} y^r\right)_{|m|l} = a_{ml}\left(-S^i_{jkh|r} y^r\right) + \omega_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{2}\, c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \tfrac{1}{2}\,\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l} - \tfrac{2}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l} + \tfrac{2}{3} a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right). \qquad (4.41)$$

where $2b_{ml} = \omega_{ml}$.

This demonstrates that

$$\left(-S^i_{jkh|r} y^r\right)_{|m|l} = a_{ml}\left(-S^i_{jkh|r} y^r\right) + \omega_{ml}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) + \tfrac{1}{2}\, c_{ml}\left(W^i_h g_{jk} - W^i_k g_{jh}\right) + \tfrac{1}{2}\,\gamma_m\left(W^i_h g_{jk} - W^i_k g_{jh}\right)_{|l}. \qquad (4.42)$$

If and only if

$$\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m|l} = a_{ml}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right). \qquad (4.43)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.10. In the context of $G^{2nd}\, P_{|h} - BRF_n$, the tensor $\left(-S^i_{jkh|r} y^r\right)$ represents a generalized birecurrent Finsler space, provided that the condition (4.43) is satisfied.

By contracting the indices i and h in equation (4.41), and utilizing equations (2.2d), (2.1b), (2.13b), (2.12c), (2.9c) and (2.10b), we obtain the following result:

$$\left(-S_{jk|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{jk|r} y^r\right) + (n-1)\omega_{ml}\, g_{jk} - \tfrac{1}{2}\, c_{ml} W_{jk} - \tfrac{1}{2}\,\gamma_m W_{jk|l} - \tfrac{2}{3}\left(R_{jk} - g_{jk} R\right)_{|m|l} + \tfrac{2}{3} a_{ml}\left(R_{jk} - g_{jk} R\right). \qquad (4.44)$$

This demonstrates that

$$\left(-S_{jk|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{jk|r} y^r\right) + (n-1)\omega_{ml} g_{jk} - \tfrac{1}{2}\, c_{ml} W_{jk} - \tfrac{1}{2}\,\gamma_m W_{jk|l}. \qquad (4.45)$$

If and only if

$$\left(R_{jk} - g_{jk}\, R\right)_{|m|l} = a_{ml}\left(R_{jk} - g_{jk}\, R\right). \qquad (4.46)$$

By transvecting condition (4.41) with g_{ir}, and applying relations (2.2d), (2.10c), (2.12e) and (2.13d), we obtain the following result

$$\left(-S_{rjkh|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{rjkh|r} y^r\right) + b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) + \tfrac{1}{2} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right) + \tfrac{1}{2}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l} - \tfrac{2}{3}\left(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\right)_{|m|l} + \tfrac{2}{3} a_{ml}\left(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\right). \qquad (4.47)$$

This demonstrates that

$$\left(-S_{rjkh|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{rjkh|r} y^r\right) + b_{ml}\left(g_{rh} g_{jk} - g_{rk} g_{jh}\right) + \tfrac{1}{2} c_{ml}\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right) + \tfrac{1}{2}\gamma_m\left(W_{rh} g_{jk} - W_{rk} g_{jh}\right)_{|l}. \qquad (4.48)$$

If and only if

$$\left(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk}\, R_{jh} - g_{jk}\, R_{rh}\right). \qquad (4.49)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.11. In the context of $G^{2nd}\, P_{|h} - BRF_n$, S−Ricci tensor $\left(-S_{jk|r} y^r\right)$ and the associate tensor $\left(-S_{rjkh|r} y^r\right)$ are defined in equations (4.45) and (4.48), respectively, if and only if the conditions in equations (4.46) and (4.49) are satisfied.

By transvecting condition (4.44) with g^{ik}, and applying relations (2.2b), (2.2a), (2.10e) and (2.13e), we obtain the following result

$$\left(-S^i_{j|r} y^r\right)_{|m|l} = a_{ml}\left(-S^i_{j|r} y^r\right) + (n-1)\omega_{ml}\,\delta^i_j - \tfrac{1}{2}\, c_{ml} W^i_j - \tfrac{1}{2}\gamma_m\left(W^i_j\right)_{|l} - \tfrac{2}{3}\left(R^i_j - \delta^i_j R\right)_{|m|l} + \tfrac{2}{3} a_{ml}\left(R^i_j - \delta^i_j R\right). \qquad (4.50)$$

This demonstrates that

$$\left(-S^i_{j|r} y^r\right)_{|m|l} = a_{ml}\left(-S^i_{j|r} y^r\right) + (n-1)\,\omega_{ml}\delta^i_j - \tfrac{1}{2}\, c_{ml} W^i_j - \tfrac{1}{2}\gamma_m\left(W^i_j\right)_{|l}. \qquad (4.51)$$

If and only if

$$\left(R^i_j - \delta^i_j R\right)_{|m|l} = a_{ml}\left(R^i_j - \delta^i_j R\right). \qquad (4.52)$$

By transvecting condition (4.44) with g^{jk}, and applying relations (2.2d), (2.2b), (2.10d) and (2.13c), we obtain the following result

$$\left(-S_{|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{|r} y^r\right) + (n-1)n\,\omega_{ml} - \tfrac{1}{2}\, c_{ml} W - \tfrac{1}{2}\,\gamma_m W_{|l} - g^{jk}\left\{\tfrac{2}{3}\left(R_{jk} - g_{jk}\, R\right)_{|m|l} - \tfrac{2}{3} a_{ml}\left(R_{jk} - g_{jk}\, R\right)\right\}. \qquad (4.53)$$

This demonstrates that

$$\left(-S_{|r} y^r\right)_{|m|l} = a_{ml}\left(-S_{|r} y^r\right) + (n-1)n\,\omega_{ml} - \tfrac{1}{2}\, c_{ml} W - \tfrac{1}{2}\,\gamma_m W_{|l}. \qquad (4.54)$$

If and only if

$$g^{jk}\left[\tfrac{2}{3}\left(R_{jk} - g_{jk}\, R\right)_{|m|l} - \tfrac{2}{3} a_{ml}\left(R_{jk} - g_{jk}\, R\right)\right] = 0. \qquad (4.55)$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.12. In the context of $G^{2nd}\, P_{|h} - BRF_n$, the tensor $\left(-S^i_{j|r} y^r\right)$ and the tensor $\left(-S_{|r} y^r\right)$ are defined in equations (4.51) and (4.54), respectively, if and only if the conditions in equations (4.52) and (4.55) are satisfied.

From the equation (2.11b), we get

$$P^i_{kh} = C^i_{kh|r}\, y^r \tag{4.56}$$

By taking the $h-$ covariant derivative of (4.56), with respect to x^m and x^l, we get

$$\left(P^i_{kh}\right)_{|m|l} = \left(C^i_{kh|r}\, y^r\right)_{|m|l}. \tag{4.57}$$

By substituting equations (4.7) and (4.56) in to (4.57), we obtain:

$$\left(C^i_{kh|r}\, y^r\right)_{|m|l} = a_{ml}\left(C^i_{kh|r} y^r\right) + b_{ml}\left(\delta^i_h y_k - \delta^i_k y_h\right) + \tfrac{1}{4} c_{ml}\left(W^i_h y_k - W^i_k y_h\right) + \tfrac{1}{4}\gamma_m\left(W^i_h y_k - W^i_k y_h\right)_{|l} + \tfrac{1}{3} a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right) - \tfrac{1}{3}\left(\delta^i_k H_h - y_k R^i_h\right)_{|m|l}. \quad \ldots(4.58)$$

This demonstrates that

$$\left(C^i_{kh|r}\, y^r\right)_{|m|l} = a_{ml}\left(C^i_{kh|r} y^r\right) + b_{ml}\left(\delta^i_h y_k - \delta^i_k y_h\right) + \tfrac{1}{4} c_{ml}\left(W^i_h y_k - W^i_k y_h\right) + \tfrac{1}{4}\gamma_m\left(W^i_h y_k - W^i_k y_h\right)_{|l}. \tag{4.59}$$

If and only if

$$\left(\delta^i_k H_h - y_k R^i_h\right)_{|m|l} = a_{ml}\left(\delta^i_k H_h - y_k R^i_h\right). \tag{4.60}$$

By transvecting condition (4.58) with g_{ij}, and applying relations (2.2d), (2.3d) and (2.10c), we obtain the following result

$$\left(C_{jkh|r} y^r\right)_{|m|l} = a_{ml}\left(C_{jkh|r} y^r\right) + b_{ml}\left(g_{jh} y_k - g_{jk} y_h\right) + \tfrac{1}{4} c_{ml}\left(W_{jh} y_k - W_{jk} y_h\right) + \tfrac{1}{4}\gamma_m\left(W_{jh} y_k - W_{jk} y_h\right)_{|l} + \tfrac{1}{3} a_{ml}\left(g_{jk} H_h - y_k R_{jh}\right) - \tfrac{1}{3}\left(g_{jk} H_h - y_k R_{jh}\right)_{|m|l}. \tag{4.61}$$

This demonstrates that

$$\left(C_{jkh|r} y^r\right)_{|m|l} = a_{ml}\left(C_{jkh|r} y^r\right) + b_{ml}\left(g_{jh} y_k - g_{jk} y_h\right) + \tfrac{1}{4} c_{ml}\left(W_{jh} y_k - W_{jk} y_h\right) + \tfrac{1}{4}\gamma_m\left(W_{jh} y_k - W_{jk} y_h\right)_{|l} \tag{4.62}$$

If and only if

$$\left(g_{jk} H_h - y_k R_{jh}\right)_{|m|l} = a_{ml}\left(g_{jk} H_h - y_k R_{jh}\right). \tag{4.63}$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.13. In the context of $G^{2nd}\, P_{|h} - BRF_n$, the $h-$ covariant derivative of the second orders for the associate tensor $\left(C^i_{kh|r} y^r\right)$ and (h)hv-torsion tensor $\left(C_{jkh|r} y^r\right)$ are defined a generalized birecurrent Finsler space if and only if the conditions in equations (4.60) and (4.61) are satisfied, respectively.

By contracting the indices i and h in equation (4.58), and utilizing equations (2.2a), (2.1a), (2.3c), (2.1b), (2.12c) and (2.10b), we obtain the following result:

$$\left(C_{k|r} y^r\right)_{|m|l} = a_{ml}\left(C_{k|r} y^r\right) + b_{ml}(n-1)\, y_k + \tfrac{1}{3} a_{ml}\left(H_k - y_k R\right) - \tfrac{1}{3}\left(H_k - y_k R\right)_{|m|l}. \tag{4.64}$$

This demonstrates that

$$\left(C_{k|r} y^r\right)_{|m|l} = a_{ml}\left(C_{k|r} y^r\right) + b_{ml}(n-1) y_k. \tag{4.65}$$

If and only if

$$\left(H_k - y_k R\right)_{|m|l} = a_{ml}\left(H_k - y_k R\right). \tag{4.66}$$

By transvecting condition (4.58) with g^{kh}, and applying relations (2.2b), (2.3e), (2.10c) and (2.10e), we obtain the following result

$$\left(C^i_{|r} y^r\right)_{|m|l} = a_{ml}\left(C^i_{|r} y^r\right) + \tfrac{1}{3} a_{ml}\left(H^i - R^i\right) - \tfrac{1}{3}\left(H^i - R^i\right)_{|m|l}. \tag{4.67}$$

This demonstrates that

$$\left(C^i_{|r} y^r\right)_{|m|l} = a_{ml}\left(C^i_{|r} y^r\right). \tag{4.68}$$

If and only if

$$\left(H^i - R^i\right)_{|m|l} = a_{ml}\left(H^i - R^i\right). \tag{4.69}$$

Therefore, the proof of theorem is completed, we can say

Theorem 4.14. In the context of $G^{2nd}\, P_{|h} - BRF_n$, the tensor $\left(C_{k|r} y^r\right)$ and the tensor $\left(C^i_{|r} y^r\right)$ are defined in equations (4.65) and (4.68), respectively, provided that the conditions (4.66) and (4.69) are satisfied.

5. Conclusions

This paper investigates the relationship between Weyl's curvature tensor and Cartan's second curvature tensor, specifically focusing on their connection and implications in the context of Finsler geometry. The primary result shows that the two curvature tensors, W^i_{jkh} and P^i_{jkh}, are related through a specific formula, which connects projective and intrinsic curvatures in Riemannian spaces, particularly when $n = 4$.

By deriving the covariant derivatives of these tensors, we have established several critical results in Finsler geometry, including the general conditions for a space to be classified as a generalized birecurrent Finsler space. These conditions are expressed through a series of equations, and the conditions for when these spaces hold true are carefully derived from the curvature relations.

Moreover, through transvection and the application of various differential relations, we have provided detailed formulations of how these tensors interact with each other and with other geometrical objects, such as the Ricci tensor and scalar fields, under different configurations. The work also demonstrates how Cartan's second curvature tensor P^i_{jkh} defines a generalized birecurrent Finsler space, provided specific conditions are satisfied.

This paper also extends the analysis to the covariant derivatives of second-order torsion tensors, showing that these derivatives exhibit similar properties when the underlying space satisfies certain conditions. The final results culminate in a series of theorems that provide a complete characterization of the conditions under which these tensors represent generalized birecurrent Finsler spaces.

In conclusion, the research lays a solid foundation for understanding the geometric structure of spaces defined by Weyl's and Cartan's curvature tensors. Future work can explore further generalizations and applications of these results to more complex geometries, extending the scope of Finsler spaces in both theoretical and practical contexts.

References

1. Abu-Donia, H., Shenawy, S. & Abdehameed, A. (2020). The w^*-curvature tensor on relativistic space-times. Kyungpook Mathematical Journal, 60, 185-195.
2. Ahsan, Z. & Ali, M. (2014). On some properties of w-curvature tensor. Palestine Journal of Mathematics, 3 (1), 61-69.
3. Ahsan, Z. & Ali, M. (2016). Curvature tensor for the spacetime of general relativity. Palestine Journal of Mathematics, 2,1-15.
4. Al-Qashbari, A.M. Abdallah A.A. and Baleedi, S.M. (2025): A Study of M-Projective Curvature Tensor $\overline{W}^i_{jkh}$ in $G\mathcal{B}K-5RF_n$ Via Lie Derivative, International Journal of Applied Science and Mathematical Theory, Vol.11, No1,1-9.
5. Al-Qashbari, A.M. Abdallah A.A. and Abdallah, F. (2025): Decomposition of Generalized Recurrent Tensor Fields of R^h −nth Order in Finsler Manifolds, Journal of Science and Technology, Vol.30, Issue 2, 99-105.
6. AL-Qashbari, A.M.A., Abdallah, A.A. & Nasr, K.S. (2025): Generalized of h-torsion and curvature structures in generalized recurrent space of third order by Cartan covariant derivatives, GPH- International Journal of Mathematics, Vol.8, Issue 2, 1-9.
7. Al-Qashbari, A.M. Abdallah, A.A. and Al-ssallal, F.A. (2024). Recurrent Finsler structures with higher-order generalizations defined by special curvature tensors, International Journal of Advanced Research in Science, Communication and Technology, Vol.4, No 1,68-75,
8. Al-Qashbari, A.M. and Al-ssallal, F.A. (2024). Study of curvature tensor by using Berwald's and Cartan's higher-order derivatives in Finsler space, Technological Applied and Humanitarian Academic Journal, Vol.1, No 1, 1-15,
9. Al-Qashbari, A.M. Haoues M. and Al-ssallal, F.A. (2024). A decomposition analysis of Weyl's curvature tensor via Berwald's first and second order derivatives in Finsler spaces, Journal of Innovative Applied Mathematics and Computational Science,4(2),201-213.
10. Al-Qashbari, A.M. Abdallah A.A. and Baleedi, S.M. (2025): Berwald covariant derivative and Lie derivative of conharmonic curvature tensors in generalized fifth recurrent Finsler space, GPH- International Journal of Mathematics, Vol.8, Issue 1, 24-32.
11. AL-Qashbari, A.M.A. & Qasem, F.Y.A. (2017). Study on generalized $\mathcal{B}$R-trirecurrent Finsler Space. Journal of Yemen Engineer,15, 79-89.
12. AL-Qashbari, A.M.A. (2020). On generalized for curvature tensors P^i_{jkh} of second order in Finsler space. University of Aden Journal of Natural and Applied Sciences, 24(1), 171-176.
13. AL-Qashbari, A.M.A. (2019). Some properties for Weyl's projective curvature tensors of generalized W^h-birecurrent in Finsler spaces. University of Aden Journal of Natural and Applied Sciences,23, (1), 181-189.
14. AL-Qashbari, A.M.A. (2020). Some identities for generalized curvature tensors in $\mathcal{B}$-Recurrent Finsler space. Journal of New Theory, ISSN:2149-1402, 32. 30-39.
15. AL-Qashbari, A.M.A. (2020). Recurrence decompositions in Finsler space. Journal of Mathematical Analysis and Modeling, ISSN:2709-5924, 1, 77-86.
16. AL-Qashbari, A.M.A. & AL-Maisary A.A.S. (2023). Study on generalized W^i_{jkh} of fourth order recurrent in Finsler space. Journal of Yemeni Engineer, Univ. Aden, 17 (2), 72-86.
17. AL-Qashbari, A.M.A., Abdallah, A.A. & Nasr, K.S. (2025): On generalized trirecurrent space by using Gh-covariant derivative in Finsler geometry, Journal of Mathematical Problems, Equations and Statistics, Vol.6, No.1, 91-100.
18. AL-Qashbari, A.M.A., Saleh, S. and Ibedou, I. (2024): On some relations of R-projective curvature tensor in recurrent Finsler space, Journal of Non-Linear Modeling and Analysis (JNMA), (China), Vol. 6, No.4, 1216-1227.
19. Baleedi, S.M.S. (2017). On certain generalized $\mathcal{B}K$-recurrent Finsler space (M. Sc. Thesis, University of Aden, Aden, Yemen).
20. Goswami, A. (2017). A study of certain types of special Finsler spaces in differential geometry: Systematic Review. Journal of Mathematics Applied in Science and Technology, 9, (1), 23-30. Issn. 0973-6344.
21. Misra, B., Misra, S. B., Srivastava, K. & Srivastava, R. B. (2014). Higher order recurrent Finsler spaces with Berwald's curvature tensor field. Journal of Chemical, Biological and Physical Sciences, 4, (1), 624-631.
22. Prada Y. N. and Nagaraja H. G. (2025). W_0^*-Curvature Tensor Characterizations on- Kenmotsu Metric Space. Int. J. Open Problems Compt. Math., Vol. 18, No. 2, 32-43.

23. Pandey, P.N., Saxena, S. & Goswani, A. (2011). On a generalized H-recurrent space. Journal of International Academy of Physical Sciences, 15, 201-211.

24. Rund, H. (1981). The differential geometry of Finsler spaces. Springer-Verlag, Berlin Göttingen-Heidelberg, 2nd Edit. (In Russian), Nauka, (Moscow).

25. Rawat, K.S. and Uniyal, N. (2012): Decomposition of curvature tensor fields in a Tachibana recurrent space, Ultra Scientist of Physical Sciences (An Inter. Jour. of Physical Sciences), 24(2), A, 424-428.

26. Opondo, M.A. (2021): Study of Projective curvature tensor W^i_{jkh} in bi-recurrent Finsler space, M. Sc. Dissertation, Kenyatta University, (Nairobi), (Kenya)

University of Aden Journal of Natural and Applied Sciences., vol.29(1) (2025) 43 – 53

مجلة جامعة عدن للعلوم الطبيعية والتطبيقية

Journal homepage: https://uajnas.adenuniv.com

بحث علمي

دراسة حول موترَي الانحناء لكارتان وويل في سياق الهندسة الفينسلرية ثنائية التكرار المعممة

عادل محمد علي القشبري[2,1] و فهمي أحمد مثنى السَّلال[3]

[1] قسم الرياضيات – كلية التربية عدن – جامعة عدن – عدن – اليمن
[2] قسم الهندسة الطبية – كلية الهندسة والحاسبات – جامعة العلوم والتكنولوجيا – عدن – اليمن
[3] قسم الرياضيات – كلية التربية عدن – جامعة عدن – عدن – اليمن

https://doi.org/10.47372/uajnas.2025.n1.a05

مفاتيح البحث

التسليم: 25/ ابريل/ 2025
القبول: 7 / يونيو / 2025

كلمات مفتاحية:
موترات الانحناء، الهندسة الفينسلرية ثنائية التكرار المعممة، موتر كارتان، موتر ويل.

الملخص

تتناول هذه الدراسة العلاقة بين موتر الانحناء الثاني لكارتان وموتر الانحناء الإسقاطي لويل في سياق الفضاءات الريمانية. وتركز الدراسة على اشتقاق صيغة تربط بين هذين الموترين، واستكشاف الآثار المترتبة على تفاعلهما. وتشمل النتائج الأساسية لهذا العمل صياغة مجموعة من المعادلات التي تصف التغاير والتحويل التبادلي (transvectivity) لهذين الموتَرين تحت شروط متنوعة، مما يؤدي إلى صياغة عدد من النظريات. وتُسهم هذه النتائج في تقديم رؤى جديدة حول الفضاءات الفينسلرية ثنائية التكرار المعممة وخصائصها الهندسية، مما يعزز من فهم موترات الانحناء في الفضاءات ذات الأبعاد العليا.