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Research Article

Generalized Q-Humbert Function

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Abstract

In this paper, we introduce generalized $\boldsymbol{q}$-Humbert function, some properties of this function, $\boldsymbol{q}$-difference equations and recurrence relations are derived.

1. Introduction

The theory of the quantum q-calculus has recently been applied in many areas of pure and applied mathematics as well as in such other disciplines as (for example) biology, physics, electrochemistry, economics, probability theory and statistics. Many special functions of mathematical physics have been shown to admit generalizations to a base q, which are usually referred to as q-special functions. The q-special functions have important roles in many branches of mathematics and mathematical physics (see, for example, [1,2,3,4,5, 6,7,8,9,10, 13, 14, 15]).

For the convenience of the reader, we first provide a summary of the mathematical notations and some basic definitions of the q-calculus for $|q| < 1$, which we need to use in this work.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [4,6,7,13]

$$[\alpha]_q = \frac{1-q^\alpha}{1-q},\ \ 0 < |q| < 1; q \in C - \{1\};\ \ \alpha \in C, \tag{1.1}$$

where

$$\lim_{q\to 1}[\alpha]_q = \lim_{q\to 1}\frac{1-q^\alpha}{1-q} = \alpha.$$

The q-analogue of $n!$ is then defined by

$$[n]_q! = \begin{cases} 1 & , \quad n = 0 \\ [n]_q[n-1]_q \dots [2]_q[1]_q\,, & \quad n \in N \end{cases} \tag{1.2}$$

Or

$$[n]_q! = \prod_{k=1}^{n}[k]_q = \prod_{k=1}^{n}\left(\frac{1-q^k}{1-q}\right) = \frac{(q;q)_n}{(1-q)^n},\quad n \in N. \tag{1.3}$$

The q-shifted factorials are defined by [9]

$$(a;q)_n = \begin{cases} 1 & , \quad n = 0 \\ \prod_{k=0}^{n-1}(1 - aq^k)\,, & \quad n \in N \end{cases} \tag{1.4}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}\quad ; n,k \in Z, \tag{1.5}$$

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k, \tag{1.6}$$

where $a, q \in R$ such that $0 < q < 1$.

Exton [4] presented the whole family of basic q-exponential function as:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} q^{\mu n(n-1)}\frac{z^n}{[n]_q!}, \tag{1.7}$$

Where

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}.$$

The one parameter family of q-exponential function

$$E_q^{(a)}(z) = \sum_{n=0}^{\infty} q^{an^2/2}\frac{z^n}{(q;q)_n},$$

With $a \in R$ has been considered in [5]. Consequently, in the limit when $q \to 1$, we have

$$\lim_{q\to 1} E_q^{(a)}\big((1-q)z\big) = e^z.$$

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $\frac{a}{2}$, we get

$$E\left(\frac{a}{2}, \frac{x}{1-q}; q\right) = E_q(x, a),$$

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where

$$E_q(x,a) = \sum_{n=0}^{\infty} q^{a\binom{n}{2}} \frac{x^n}{(q;q)_n}. \quad (1.8)$$

The above q-function can be rewritten by the formula

$$D_q E_q(x,a) = \frac{1}{1-q} E_q(q^a x, a). \quad (1.9)$$

Where the Jackson q-difference operator D_q is defined by [7,13]

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}, \quad (1.10)$$

and satisfies the product rule

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x). \quad (1.11)$$

$$D_q\big(f_1(x).f_2(x).f_3(x)\big) = D_q\{f_1(x)\}.f_2(x).f_3(x) + f_1(qx)D_q\{f_2(x)\}.f_3(x) + f_1(qx).f_2(qx)D_q\{f_3(x)\}. \quad (1.12)$$

In the earlier works [13,16], the Humbert function $J_{n,m}(x)$ is defined by means of the generating function:

$$exp\left(\frac{x}{3}\left(u+t-\frac{1}{ut}\right)\right) = \sum_{n,m=-\infty}^{\infty} J_{n,m}(x)\, u^m t^n. \quad (1.13)$$

or, more generally, as follows:

$$J_{n,m}(x) = \left(\frac{x}{3}\right)^{m+n} \frac{1}{\Gamma(m+1)\Gamma(n+1)}\ {}_0F_2\left(-;m+1,n+1;-\frac{x^3}{27}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)\Gamma(n+k+1)}\left(\frac{x}{3}\right)^{m+n+3k}. \quad (1.14)$$

In [13] Srivastava and Shehata introduced a family of new q-extensions of the Humbert function of the form

$$J_{n,m}^{(1)}(x/q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[m+k]_q![n+k]_q![k]_q!}\left(\frac{x}{3}\right)^{m+n+3k}. \quad (1.15)$$

In the next sections, we introduce the generalized q-Humbert function, study q-difference equations and some of its recurrence relations.

2. The Generalized q-Humbert Function and Its Generating Function

We introduce the generalized q-Humbert function by the following:

$$J_{m,n}(x;a;q) = \frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n} \sum_{k=0}^{\infty}(-1)^{(a+1)k} \frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{1+m};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k}. \quad \ldots(2.1)$$

Now, we get generating function of the generalized q-Humbert function in the form of the following theorem:

Theorem 2.1. Let $a \in Z^+, x \in R,\ u,t \in \mathbb{C}/\{0\},\ 0<|q|<1,\ q \in C$, the following generating function for the generalized q-Humbert function $J_{n,m}(x;a;q)$ holds true:

$$E_q\left[\frac{xu}{3},a\right].E_q\left[\frac{xt}{3},a\right].E_q\left[\frac{(-1)^{a+1}x}{3ut},a\right] = \sum_{m,n=-\infty}^{\infty} J_{m,n}(x;a;q)\, u^m t^n. \quad (2.2)$$

Proof. Let us denote the left hand side of (2.2) by W and by using (1.8), we have

$$W = \sum_{r=0}^{\infty}\frac{q^{a\binom{r}{2}}}{(q;q)_r}\left(\frac{xu}{3}\right)^r \sum_{s=0}^{\infty}\frac{q^{a\binom{s}{2}}}{(q;q)_s}\left(\frac{xt}{3}\right)^s \sum_{k=0}^{\infty}\frac{q^{a\binom{k}{2}}}{(q;q)_k}\left(\frac{(-1)^{a+1}x}{3ut}\right)^k$$

$$= \sum_{r,s,k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]}}{(q;q)_r(q;q)_s(q;q)_k}\left(\frac{x}{3}\right)^{r+s+k} u^{r-k}t^{s-k}.$$

Substitution r by $m+k$ and s by $n+k$, we get

$$W = \sum_{n,m=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{m+n+3k} u^m t^n.$$

By using relation (1.6) in above relation, we obtain

$$W = \sum_{m,n=-\infty}^{\infty}\frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{1+m};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k} u^m t^n.$$

By equating the coefficients of $u^m t^n$ with the right hand sides of (2.2), we obtain the required relation (2.1).

Corollary 2.1. Putting $a=0$ and taking the limit as $q \to 1$ and setting $x \to (1-q)x$ in equation (2.1) and in view of equations (1.15) and (1.14).

Proof. Since,

$$J_{m,n}((1-q)x;0;q) = \frac{1}{(q;q)_m(q;q)_n}\left(\frac{(1-q)x}{3}\right)^{m+n} \times \sum_{k=0}^{\infty}(-1)^{(0+1)k}\frac{q^{0\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{1+m};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{(1-q)x}{3}\right)^{3k}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{[m+k]_q![n+k]_q![k]_q!}\left(\frac{x}{3}\right)^{m+n+3k},$$

Hence,

$$J_{m,n}((1-q)x;0;q) = J_{m,n}(x;q). \quad (2.3)$$

And

$$\lim_{q\to1} J_{m,n}\big((1-q)x;0;q\big) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(m+k+1)\Gamma(n+k+1)}\left(\frac{x}{3}\right)^{m+n+3k}$$

$$\therefore \lim_{q\to1} J_{m,n}\big((1-q)x;0;q\big) = J_{n,m}(x). \quad (2.4)$$

respectively.

Corollary 2.2. Let n and m are integers, then the function $J_{m,n}(x;a;q)$ satisfies the following relations

$$J_{-m,n}(x;a;q) = (-1)^{(a+1)m} J_{m,m+n}(x;a;q). \quad (2.5)$$

$$J_{m,-n}(x;a;q) = (-1)^{(a+1)n} J_{m+n,n}(x;a;q). \quad (2.6)$$

Proof. From the definition of the generalized q-Humbert functions $J_{m,n}(x;a;q)$, we have

$$\because J_{-m,n}(x;a;q) = \sum_{k=m}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{-m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q;q)_{k-m}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{-m+n+3k},$$

replacing $s=k-m$, we find

$$J_{-m,n}(x;a;q) = \sum_{s=0}^{\infty}(-1)^{(a+1)(s+m)}\frac{q^{a\left[\binom{s}{2}+\binom{m+n+s}{2}+\binom{m+s}{2}\right]}}{(q;q)_s(q;q)_{m+n+s}(q;q)_{m+s}}\left(\frac{x}{3}\right)^{2m+n+3s}$$

$$= (-1)^{(a+1)m} J_{m,m+n}(x;a;q),$$

which the required relation (2.5).

The equation (2.6) can be proved in a like manner.

Corollary 2.3. The function $J_{m,n}(x;a;q)$ satisfies the following properties

$$\left.\begin{aligned} J_{-m,-n}(x;a;q) &= (-1)^{(a+1)m} J_{m,m-n}(x;a;q) \\ &= (-1)^{(a+1)n} J_{n-m,n}(x;a;q) \end{aligned}\right\}, \tag{2.7}$$

where n and m are integers.

Proof. From the definition of q-Humbert functions $J_{m,n}(x;a;q)$, we have

$$J_{-m,-n}(x;a;q) = \sum_{k=max\{m,n\}}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{k-m}{2}+\binom{k-n}{2}+\binom{k}{2}\right]}}{(q;q)_{k-m}(q;q)_{k-n}(q;q)_k} \left(\frac{x}{3}\right)^{-m-n+3k}. \tag{2.8}$$

Upon setting $s = k - m$ in the equation (2.8), we get

$$\begin{aligned} J_{-m,-n}(x;a;q) &= \sum_{s=0}^{\infty} (-1)^{(a+1)(m+s)} \frac{q^{a\left[\binom{s}{2}+\binom{m-n+s}{2}+\binom{m+s}{2}\right]}}{(q;q)_s(q;q)_{m-n+s}(q;q)_{m+s}} \left(\frac{x}{3}\right)^{2m-n+3s} \\ &= (-1)^{(a+1)m} J_{m,m-n}(x;a;q). \end{aligned}$$

Also, upon setting $s = k - n$ in the equation (2.8), we obtain

$$\begin{aligned} J_{-m,-n}(x;a;q) &= \sum_{s=0}^{\infty} (-1)^{(a+1)(n+s)} \frac{q^{a\left[\binom{n-m+s}{2}+\binom{s}{2}+\binom{n+s}{2}\right]}}{(q;q)_{n-m+s}(q;q)_s(q;q)_{n+s}} \left(\frac{x}{3}\right)^{2m-n+3s} \\ &= (-1)^{(a+1)n} J_{n-m,n}(x;a;q). \end{aligned}$$

Corollary 2.4. For k any positive even number, the function $J_{m,n}(x;a;q)$ satisfies the following relation:

$$J_{m,n}(-x;a;q) = (-1)^{m+n} J_{m,n}(x;a;q). \tag{2.9}$$

Proof. Since,

$$\begin{aligned} J_{m,n}(-x;a;q) &= \frac{1}{(q;q)_m(q;q)_n}\left(-\frac{x}{3}\right)^{m+n} \sum_{k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{1+m};q)_k(q^{1+n};q)_k(q;q)_k}\left(-\frac{x}{3}\right)^{3k} \\ &= (-1)^{m+n} \frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n} \sum_{k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{1+m};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k} \end{aligned}$$

Which in view of (2.1) yields relation (2.9).

3. The q-difference Equations of The Function $J_{m,n}(x;a;q)$

Theorem (3.1). The q-Humbert function $J_{m,n}(x;a;q)$ satisfies the following relation:

$$\begin{aligned} \frac{d_q}{d_q x} J_{m,n}(x;a;q) &= \frac{q^{\frac{2am}{3}} u}{3(1-q) q^{\frac{an}{3}}} J_{m,n}\left(q^{\frac{a}{3}} x;a;q\right) \\ &+ \frac{q^{\frac{(2-a)m}{3}} t}{3(1-q) q^{\frac{(1-2a)n}{3}}} J_{m,n}\left(q^{\frac{1+a}{3}} x;a;q\right) + \\ &\frac{(-1)^{a+1} q^{\frac{(1-a)m}{3}}}{ut(1-q) q^{\frac{(a-1)n}{3}}} J_{m,n}\left(q^{\frac{2+a}{3}} x;a;q\right). \end{aligned} \tag{3.1}$$

Proof. Differentiating (2.2) with respect to x and using relations (1.9) and (1.12), yields

$$\begin{aligned} \sum_{n,m=-\infty}^{\infty} \frac{d_q}{d_q x} J_{m,n}(x;a;q)\, u^m t^n &= \frac{u}{3(1-q)} E_q\left[\frac{q^a x u}{3}, a\right] . E_q\left[\frac{xt}{3}, a\right] . E_q\left[\frac{(-1)^{a+1} x}{3ut}, a\right] \\ &+ \frac{t}{3(1-q)} E_q\left[\frac{qxu}{3}, a\right] . E_q\left[\frac{q^a xt}{3}, a\right] . E_q\left[\frac{(-1)^{a+1} x}{3ut}, a\right] \\ &+ \frac{(-1)^{a+1}}{ut(1-q)} E_q\left[\frac{qxu}{3}, a\right] . E_q\left[\frac{qxt}{3}, a\right] . E_q\left[\frac{(-1)^{a+1} q^a x}{3ut}, a\right]. \end{aligned}$$

Using the relation (1.8), we get

$$\begin{aligned} \sum_{n,m=-\infty}^{\infty} \frac{d_q}{d_q x} J_{m,n}(x;a;q)\, u^m t^n = {} & \frac{u}{3(1-q)} \sum_{r,s,k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]+ar}}{(q;q)_r(q;q)_s(q;q)_k} \left(\frac{x}{3}\right)^{r+s+k} u^{r-k} t^{s-k} \\ &+ \frac{t}{3(1-q)} \sum_{r,s,k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]+r+as}}{(q;q)_r(q;q)_s(q;q)_k} \left(\frac{x}{3}\right)^{r+s+k} u^{r-k} t^{s-k} \\ &+ \frac{(-1)^{a+1}}{ut(1-q)} \sum_{r,s,k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]+r+s+ak}}{(q;q)_r(q;q)_s(q;q)_k} \left(\frac{x}{3}\right)^{r+s+k} u^{r-k} t^{s-k}. \end{aligned} \tag{3.2}$$

Replacing r by $m + k$ and s by $n + k$ in right hand sides. in (3.2), we get

$$\begin{aligned} \sum_{n,m=-\infty}^{\infty} \frac{d_q}{d_q x} J_{m,n}(x;a;q)\, u^m t^n &= \frac{u}{3(1-q)} \sum_{n,m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}+a(m+k)}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{x}{3}\right)^{m+n+3k} u^m t^n \\ &+ \frac{t}{3(1-q)} \sum_{n,m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{(a+1)k} \frac{q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}+m+an+(1+a)k}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{x}{3}\right)^{m+n+3k} u^m t^n \\ &+ \frac{(-1)^{a+1}}{ut(1-q)} \sum_{n,m=-\infty}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{(a+1)k} q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}+m+n+(2+a)k}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{\sqrt{2A}x}{3}\right)^{m+n+3k} u^m t^n. \end{aligned}$$

On comparing the coefficients of $u^m t^n$ on both sides of the above equation, we obtain

$$\begin{aligned} \frac{d_q}{d_q x} J_{m,n}(x;a;q) &= \frac{q^{\frac{2am}{3}} u}{3(1-q) q^{\frac{an}{3}}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{q^{\frac{a}{3}} x}{3}\right)^{m+n+3k} \\ &+ \frac{q^{\frac{(2-a)m}{3}} t}{3(1-q) q^{\frac{(1-2a)n}{3}}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{q^{\frac{1+a}{3}} x}{3}\right)^{m+n+3k} \\ &+ \frac{(-1)^{a+1} q^{\frac{(1-a)m}{3}}}{ut(1-q) q^{\frac{(a-1)n}{3}}} \sum_{k=0}^{\infty} (-1)^k \frac{q^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k} \left(\frac{q^{\frac{2+a}{3}} x}{3}\right)^{m+n+3k}. \end{aligned} \tag{3.2}$$

Which is the required relation (3.1).

Theorem (3.2). The function sequence $J_{m,n}(x;a;q)$ satisfies the next recurrence relations**:**

$$[m+1]_q J_{m+1,n}(x;a;q) = \frac{q^{\frac{2am}{3}} x}{3(1-q) q^{\frac{an}{3}}} J_{m,n}\left(q^{\frac{a}{3}} x;a;q\right) + \frac{(-1)^a q^{\frac{(2-a)m}{3}} x}{3(1-q) q^{\frac{(1+a)n}{3}}} J_{m-2,n-1}\left(q^{\frac{1+a}{3}} x;a;q\right), \tag{3.3}$$

and

$$[n+1]_q J_{m,n+1}(x;a;q) = \frac{q^{\frac{2an}{3}} x}{3(1-q) q^{\frac{am}{3}}} J_{m,n}\left(q^{\frac{a}{3}} x;a;q\right) + \frac{(-1)^a q^{\frac{(2-a)n}{3}} x}{3(1-q) q^{\frac{(1+a)m}{3}}} J_{m-1,n-2}\left(q^{\frac{1+a}{3}} x;a;q\right). \tag{3.4}$$

Proof. Differentiating (2.2) with respect to u and using relations (1.9) and (1.11), we find

$$\sum_{n,m=-\infty}^{\infty}\frac{\partial_q}{\partial_q u}J_{n,m}(x;a;q)\,u^m t^n$$
$$=\frac{x}{3(1-q)}E_q\left[\frac{q^a xu}{3},a\right].E_q\left[\frac{xt}{3},a\right].E_q\left[\frac{(-1)^{a+1}x}{ut},a\right]$$
$$+\frac{(-1)^a x}{3(1-q)u^2 t}E_q\left[\frac{qxu}{3},a\right].E_q\left[\frac{xt}{3},a\right].E_q\left[\frac{(-1)^{a+1}q^a x}{ut},a\right].$$

From relation (1.8), we obtain

$$\sum_{n,m=-\infty}^{\infty}[m]_q J_{m,n}(x;a;q)\,u^{m-1}t^n$$
$$=\frac{x}{3(1-q)}\sum_{r,s,k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]+ar}}{(q;q)_r(q;q)_s(q;q)_k}\left(\frac{x}{3}\right)^{r+s+k}u^{r-k}t^{s-k}$$
$$+\frac{(-1)^a x}{3(1-q)u^2 t}\sum_{r,s,k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{r}{2}+\binom{s}{2}+\binom{k}{2}\right]+r+ak}}{(q;q)_r(q;q)_s(q;q)_k}\left(\frac{x}{3}\right)^{r+s+k}u^{r-k}t^{s-k}. \quad (3.5)$$

Replacing r and s by $m+k$ and $n+k$ respectively in the r. h. s. of (3.5), we get

$$\sum_{n,m=-\infty}^{\infty}[m+1]_q J_{m+1,n}(x;a;q)u^m t^n$$
$$=\frac{x}{3(1-q)}\sum_{n,m=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]+am+ak}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n$$
$$+\frac{(-1)^a x}{3(1-q)u^2 t}\sum_{n,m=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]+m+(1+a)k}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n.$$

Comparing of both sides, we get the relation (3.3).

Similarly way differentiating (2.2) with respect to t, we find

$$\sum_{n,m=-\infty}^{\infty}[n+1]_q J_{m,n+1}(x;a;q)\,u^m t^n$$
$$=\frac{x}{3(1-q)}\sum_{n,m=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]+an+ak}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n$$
$$+\frac{(-1)^a x}{3(1-q)ut^2}\sum_{n,m=-\infty}^{\infty}\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]+n+(1+a)k}}{(q;q)_{m+k}(q;q)_{n+k}(q;q)_k}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n.$$

Now, on comparing of coefficients of $u^m t^n$, we obtain relation (3.4).

4. Recurrence Relations

The following q-recurrence relations of the function $J_{m,n}(x;a;q)$ holds true:

$$\frac{q^{a\left[\binom{m}{2}+\binom{n}{2}\right]}}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}=$$
$$\sum_{k=0}^{\infty}(-1)^{ak}\frac{q^{a\binom{k}{2}}}{(q;q)_k}\left(\frac{x}{3}\right)^k J_{m+k,n+k}(x;a;q), \quad (4.1)$$

$$J_{m,n}(x;a;q)=\frac{3}{x}\frac{q^{-am}(1-q^{m+1})}{(q;q)_{m+1}(q;q)_n}\left(\frac{x}{3}\right)^{m+1+n}J_{m+1,n}\left(q^{-a/3}x;a;q\right)$$
$$+(-1)^{(a+1)}\frac{q^{m+1-a(m+n)}}{(q;q)_{m+3}(q;q)_n}\left(\frac{x}{3}\right)^{m+3+n}J_{m+3,n}(q^{-a}x;a;q), \quad (4.2)$$

and

$$J_{m,n}(x;a;q)=$$
$$\frac{3}{x}\frac{q^{-an}(1-q^{n+1})}{(q;q)_m(q;q)_{n+1}}\left(\frac{x}{3}\right)^{m+n+1}J_{m,n+1}\left(q^{-a/3}x;a;q\right)$$
$$+(-1)^{(a+1)}\frac{q^{n+1-a(m+n)}}{(q;q)_m(q;q)_{n+3}}\left(\frac{x}{3}\right)^{m+n+3}J_{m,n+3}(q^{-a}x;a;q). \quad (4.3)$$

Proof. Using generating function of function $J_{m,n}(x;a;q)$ and definition expression for functions $E_q\left[\frac{xu}{3},a\right]$, $E_q\left[\frac{xt}{3},a\right]$ and $E_q\left[\frac{(-1)^{a+1}x}{3ut},a\right]$, we have

$$E_q\left[\frac{xu}{3},a\right].E_q\left[\frac{xt}{3},a\right]$$
$$=E_q\left[\frac{(-1)^a x}{3ut},a\right]\sum_{n,m=-\infty}^{\infty}J_{m,n}(x;a;q)\,u^m t^n.$$

Using the relation (1.6)

$$\sum_{m=0}^{\infty}\frac{q^{a\binom{m}{2}}}{(q;q)_m}\left(\frac{xu}{3}\right)^m\sum_{n=0}^{\infty}\frac{q^{a\binom{n}{2}}}{(q;q)_n}\left(\frac{xt}{3}\right)^n$$
$$=\sum_{k=0}^{\infty}\frac{q^{a\binom{k}{2}}}{(q;q)_k}\left(\frac{(-1)^a x}{3ut}\right)^k\sum_{n,m=-\infty}^{\infty}J_{m,n}(x;a;q)\,u^m t^n.$$

Thus,

$$\sum_{m,n=0}^{\infty}\frac{q^{a\left[\binom{m}{2}+\binom{n}{2}\right]}}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}u^m t^n$$
$$=\sum_{k=0}^{\infty}(-1)^{ak}\frac{q^{a\binom{k}{2}}}{(q;q)_k}\left(\frac{x}{3}\right)^k\sum_{n,m=-\infty}^{\infty}J_{m,n}(x;a;q)\,u^{m-k}t^{n-k}$$
$$=\sum_{k=0}^{\infty}(-1)^{ak}\frac{q^{a\binom{k}{2}}}{(q;q)_k}\left(\frac{x}{3}\right)^k\sum_{n,m=0}^{\infty}J_{m+k,n+k}(x;a;q)\,u^m t^n.$$

Equating of the coefficients of $u^m t^n$ of the above equation, we obtain the required relation (4.1).

To prove (4.2), we know that

$$J_{m,n}(x;a;q)=\frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}$$
$$\times\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}(q^{1+m+k};q)}{(q^{1+m};q)_k(q^{1+m+k};q)(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k}.$$

By using relations (1.6) and (1.4), we obtain

$$J_{m,n}(x;a;q)=\frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}$$
$$\times\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}(1-q^{1+m+k})}{(q^{1+m};q)_{k+1}(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k}$$
$$=\frac{1}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+n}$$
$$\times\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}(1-q^{m+1}+q^{m+1}-q^{1+m+k})}{(q^{1+m};q)_{k+1}(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k}.$$

Also, applying the relation (1.6), we get

$$J_{m,n}(x;a;q)=\frac{3}{x}\frac{(1-q^{m+1})}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+1+n}$$
$$\times\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+1+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]-a(m+k)}}{(q^{1+m};q)_1(q^{m+2};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3k}$$
$$+\frac{q^{m+1}}{(q;q)_m(q;q)_n}\left(\frac{x}{3}\right)^{m+3+n}$$
$$\times\sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}(1-q^k)}{(q^{1+m};q)_1(q^{m+2};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{x}{3}\right)^{3(k-1)}.$$

By using equation (1.5), we find

$$J_{m,n}(x;a;q)=\frac{3}{x}\frac{q^{-am}(1-q^{m+1})}{(q;q)_{m+1}(q;q)_n}\left(\frac{x}{3}\right)^{m+1+n}J_{m+1,n}\left(q^{-a/3}x;a;q\right)$$
$$+\frac{q^{m+1}}{(q;q)_{m+3}(q;q)_n}\left(\frac{x}{3}\right)^{m+3+n}\sum_{k=1}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{m+4};q)_{k-1}(q^{1+n};q)_{k-1}(q;q)_{k-1}}\left(\frac{x}{3}\right)^{3(k-1)}.$$

Replacing k by $k+1$ in the second term in the right hand sides of above equation, we obtain

$$J_{m,n}(x;a;q) = \frac{3}{x}\frac{q^{-am}(1-q^{m+1})}{(q;q)_{m+1}(q;q)_n}\left(\frac{x}{3}\right)^{m+1+n} J_{m+1,n}\left(q^{-a/3}x;a;q\right) + (-1)^{(a+1)}\frac{q^{m+1-a(m+n)}}{(q;q)_{m+3}(q;q)_n}\left(\frac{x}{3}\right)^{m+3+n} \times \sum_{k=0}^{\infty}(-1)^{(a+1)k}\frac{q^{a\left[\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}\right]}}{(q^{m+4};q)_k(q^{1+n};q)_k(q;q)_k}\left(\frac{q^{-a}x}{3}\right)^{3k},$$

which the required relation (4.2).

Similarly, the relation (4.3), can be proved.

3. Conclusions

In the present investigation, we have introduced and studied the generalized q-Humbert function and its some properties and recurrence relations. Our investigation of this generalized q-Humbert function is potentially useful in motivating farther researches on this subject.

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