On a Generalized $B_m U$ - Recurrent Finsler Space …………….. Abdalstar Ali Mohsen Saleem

On a Generalized $B_m U$ - Recurrent Finsler Space
Abdalstar Ali Mohsen Saleem
Dept. Of Math., Faculty of Education-Yafea, Univ.of Aden, Khormakssar, Yemen
Abdulstar 1972 @ gmail.com
DOI: https://doi.org/10.47372/uajnas.2019.n2.a15

Abstract

A Finsler space $F_n$ for which the $h(v)$ - curvature tensor $U^i_{khh}$ satisfies the condition $B_m U^i_{khh} = \lambda_m U^i_{khh} + \mu_m (\delta^i_k g_{hh} + \delta^i_k g_{jh})$, where $\lambda_m$ and $\mu_m$ are non-zero covariant vector fields and $B_m$ is covariant derivative of first order in the sense of Berwald (Berwald's covariant differential operator). In the present paper, satisfying this condition will be called a generalized $B_m U$ - recurrent space. The tensor $G^r_{kh}$, the $h(v)$-torsion tensor $U^i_{kh}$, the G- Ricci tensor $G_{jk}$ and the U-Ricci tensor $U_{jk}$ are non-vanishing. Under certain conditions, a generalized $B_m U$ - recurrent space becomes a generalized recurrent tensor. Also, we discuss the decomposing of the $h(v)$ - curvature tensor $U^i_{khh}$ in Finsler space.

Key words: Finsler space, generalized $B_m U$ - recurrent space, generalized recurrent tensor, decompositions of tensor.

Introduction

K. Yano [14] defined the normal projective connection $\Pi^i_{jk}$ by

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} \gamma^i G^j_{kr}.$$

R. B. Misra and F. M. Meher [4] considered a space equipped with normal projective connection $\Pi^i_{jk}$ whose curvature tensor $N^i_{khh}$ is recurrent with respect to normal projective connection $\Pi^i_{jk}$ and they called it RNP-Finsler space. P.N. Pandey and V.J. Diwivedi [10] studied RNP-Finsler space and obtained many identities in RNP-Finsler space, most of these identities are also true in a recurrent Finsler space, with respect to Berwald’s connection coefficients $G^i_{jk}$. P.N. Pandey ([6], [7], [8]) obtained a relation between the normal projective curvature tensor $N^i_{khh}$ and Berwald curvature tensor $H^i_{khh}$ and defined NPR-Finsler space, which is characterized by the recurrent of normal projective curvature tensor $N^i_{khh}$ with respect to Berwald's connection coefficients $G^i_{jk}$. F.Y. A. Qasem [11] obtained several results concerning the $h(v)$ - curvature tensor $U^i_{khh}$ in such space.

Let us consider a set of quantities $g_{ij}$ defined by

$$g_{ij}(x,y) = \frac{1}{2} \delta^k_i \delta^j_\ell F^2(x,y).$$

The quantities $g_{ij}$ constitute the components of covariant tensor of the type $(0,2)$. Clearly, this shows that the tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in $y^i$ and symmetric in $i$ and $j$. According to Euler’s theorem on homogeneous functions, the vectors $y_i$ and $y^i$ satisfy the following estimates

$$a) y_i y^i = F^2, \quad b) g_{ij} = \delta^k_i y_j = \delta^k_j y_i \quad \text{and} \quad c) g_{ij} y^i = y_j.$$

By differentiating equation (1.2) partially with respect to $y^k$, we get a new tensor $C_{ijk}$ defined by

$$C_{ijk} = \frac{1}{2} \delta^l_i g_{jk}.$$

The tensor $C_{ijk}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in all its indices and is called $(h)v$-torsion tensor.

Berwald covariant derivative of the metric function $F$ and vector $y^i$ vanish identically, i.e.

$$a) B_k F = 0 \quad \text{and} \quad b) B_k y^i = 0.$$

Berwald covariant derivative of the metric tensor $g_{ij}$ does not vanish and is given by
On a Generalized $B_mU$ -Recurrent Finsler Space .................. Abdalstar Ali Mohsen Saleem

\begin{equation}
B_k g_{ij} = -2C_{ijk|h} y^h = -2y^h B_h g_{ijk}.
\end{equation}

Berwald covariant derivative of an arbitrary tensor $T^l_h$ with respect to $x^k$ is given by

\begin{equation}
B_k T^l_h = \partial_k T^l_h + T^r_h G^l_{rk} - T^l_h G^r_{hk} - (\partial_{\gamma} T^l_h) G^\gamma_{hk},
\end{equation}

and the commutation formula for the operators $\partial_j$ and $B_k$ are given by \[12\]

\begin{itemize}
\item[a)] $\partial_j B_k T^l_h - B_k \partial_j T^l_h = T^l_h G^r_{jkr} - T^r_i G^l_{jkr}.$
\end{itemize}

**Normal Projective Connection Coefficients**

K. Yano \[14\] defined the normal projective connection coefficients $\Pi^i_{jk}$ defined by (1.1). The connection coefficients $\Pi^i_{jk}$ is positively homogeneous of degree zero in $y^i$ and symmetric in their lower indices. The normal projective tensor $N^i_{kjh}$ is given by

\begin{equation}
N^i_{kjh} = \partial_j \Pi^i_{kh} + \Pi^i_{jkh} \Pi^j_{k\gamma} y^\gamma + \Pi^i_{k\gamma} \Pi^j_{j\gamma} - k | h,
\end{equation}

where

\begin{equation}
\Pi^i_{jk&h} = G^i_{jk&h} - \frac{1}{n+1} (\delta^i_k G^r_{jhr} + y^i G^r_{jkr}),
\end{equation}

and

\begin{equation}
\Pi^i_{jk} = \partial_j \Pi^i_{kh}.
\end{equation}

The normal projective tensor $N^i_{kjh}$ constitutes the components of a tensor.

Also, K. Yano denoted this tensor by $U^i_{jk}$, we shall follow K. Yano and denote the tensor $\Pi^i_{jk}$ by $U^i_{jk}$, thus,

\begin{equation}
U^i_{jk} = G^i_{jk} - \frac{1}{n+1} (\delta^i_k G^r_{jhr} + y^i G^r_{jkr})
\end{equation}

and

\begin{equation}
G^i_{jkr} = \partial_t G^i_{jkr}.
\end{equation}

The tensor $U^i_{jk}$ is called $h(v)$ - curvature tensor \[2\] and $G^i_{jkr}$ connection of $h(v)$ - curvature tensor \[1\]. This tensor is homogeneous of degree -1 in $y^i$. Also, this tensor satisfies the following:

\begin{equation}
U^i_{jkl} = G^i_{jkl},
\end{equation}

\begin{equation}
U^i_{kjh} y^h = U^i_{jhk} y^h = U^i_{jk},
\end{equation}

\begin{equation}
U^i_{kjh} y^j = 0.
\end{equation}

The tensor $U^i_{khh}$ is called $h(v)$- Ricci tensor and satisfies the following \[5\]:

\begin{equation}
U^i_{khh} = U^i_{kh}.
\end{equation}

And

\begin{equation}
U^i_{khh} y^j = 0.
\end{equation}

The tensor $U^i_{kh}$ is components of the projective connection coefficients \[1\]. The symmetric tensor $U^i_{jk}$ is called $h(v)$ - torsion tensor and satisfies \[5\]

\begin{equation}
U^i_{jk} = U^i_{kj},
\end{equation}

\begin{equation}
U^i_{jk} = \frac{1}{n+1} y^i G^i_{jk},
\end{equation}

and

\begin{equation}
U^i_{jk} y^k = 0.
\end{equation}

Douglas tensor \[2, 3, 9\] is given by

\begin{equation}
D^l_{jkh} = U^l_{jkh} - \frac{1}{2} (\delta^l_j U^i_{kh} + \delta^l_k U^i_{jh}).
\end{equation}

This tensor satisfies the following:

\begin{equation}
D^l_{jkh} y^j = D^l_{kjh} y^j = D^l_{khj} y^j = 0,
\end{equation}

\begin{equation}
D^l_{rkh} = 0.
\end{equation}
On a Generalized $B_mU$-Recurrent Finsler Space ………………Abdalstar Ali Mohsen Saleem

A Finsler space is called a recurrent Finsler space if it's the $h(v)$-curvature tensor $U_{jk}^i$ satisfies (11), (13)

$$B_mU_{jk}^i = \lambda_m U_{jk}^i, \quad U_{jk}^i \neq 0,$$

where $\lambda_m$ is non-zero covariant vector field.

**Generalized $B_mU$-Recurrent Space**

Let us consider a Finsler space $F_n$ for which the normal projective curvature tensor $U_{jk}^i$ satisfies the following condition

$$B_mU_{jk}^i = \lambda_m U_{jk}^i + \mu_m (\delta_j^i g_{kh} + \delta_k^i g_{jh}), \quad U_{jk}^i \neq 0,$$

where $\lambda_m$ and $\mu_m$ are non-zero covariant vector field and satisfying the condition (3.1) will be called a generalized $B_mU$-recurrent space.

Differentiating (3.2) by $y^h$ and using (1.5b), (2.5) and (1.3c), we get

$$B_m U_{jk}^i = \lambda_m U_{jk}^i + \mu_m (\delta_j^i y_k + \delta_k^i y_j).$$

Thus, the following theorem

**Theorem 3.1.** In generalized $B_mU$-recurrent space, Berwald covariant derivative of the $h(v)$-torsion tensor $U_{jk}^i$ is given by (3.2).

In view of (2.4), contracting the indices $i$ and $h$ in (3.1), we get

$$B_m G_{jkr}^i = \lambda_m G_{jkr}^i + \mu_m (g_{kj} + g_{jk}).$$

Contracting the indices $i$ and $j$ in (3.1) and using (2.7), we get

$$B_m U_{kh} = \lambda_m U_{kh} + (n + 1) \mu_m g_{kh}.$$ 

In view of (3.4) and using (2.8), we get

$$B_m G_{jk} = \lambda_m G_{jk} + \frac{1}{2} (n + 1)^2 \mu_m g_{jk}.$$ 

Thus, the following theorem

**Theorem 3.2.** The $U$-Ricci tensor $U_{jk}$, the tensor $G_{jkr}^i$ and the $G$-Ricci tensor $G_{jk}$ of a generalized $B_mU$-recurrent space are non-vanishing.

Differentiating (3.3) partially with respect to $y^h$ in the sense of Berwald and using (1.4), we get

$$\hat{\delta}_h B_m G_{jkr}^i = (\hat{\delta}_h \lambda_m) G_{jkr}^i + \lambda_m \hat{\delta}_h G_{jkr}^i + (\hat{\delta}_h \mu_m)(g_{kj} + g_{jk}) + 2 \mu_m (C_{hjk} + C_{hjk}).$$

Using commutation formula exhibited by (1.7b) for $G_{jkr}^i$ in (3.6) and using (2.3), we get

$$B_m G_{hjkr}^i = \lambda_m G_{hjkr}^i + \mu_m (C_{hjk} + C_{hjk}).$$

Therefore,

$$B_m G_{hjkr}^i = \lambda_m G_{hjkr}^i + \mu_m (C_{hjk} + C_{hjk}),$$

if and only if

$$G_{skr} G_{hjm}^s + G_{jrs} G_{hmk}^r + (\hat{\delta}_h \lambda_m) G_{jkr}^i + \hat{\delta}_h \mu_m (g_{kj} + g_{jk}) + \mu_m (C_{hjk} + C_{hjk}) = 0.$$ 

Thus, the following theorem

**Theorem 3.3.** In generalized $B_mU$-recurrent space, Berwald covariant derivative of the tensor $G_{jkr}^i$ is given by (3.8), if and only if (3.9) holds.

Differentiating (2.2) covariantly with respect to $x^m$ in the sense of Berwald and using (1.5b), we get

$$B_m U_{jk}^i = B_m G_{jkh}^i - \frac{1}{n+1} (\delta_j^i B_m G_{jkr}^i + y^i B_m G_{jkr}^i).$$

Using (3.1) in (3.10), we get

$$\lambda_m U_{jk}^i + \mu_m (\delta_j^i g_{kh} + \delta_k^i g_{jh}) = B_m G_{jkh}^i - \frac{1}{n+1} (\delta_j^i B_m G_{jkr}^i + y^i B_m G_{jkr}^i).$$

Using (2.2), (3.3) and (3.7) in (3.11), we get
On a Generalized $B_mU$ - Recurrent Finsler Space  ……………………Abdalstar Ali Mohsen Saleem

\begin{equation}
(3.12) \quad B_mG^i_{jkh} - \lambda_mG^i_{jkh} - \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}) = \frac{1}{n+1} y^i \{ G^s_{kr} G^g_{hmj} \\
+ G^r_{jrs} G^s_{hmk} + (\hat{\partial}_h \lambda_m) G^r_{jkr} + \hat{\partial}_h \mu_m(g_{jk} + g_{kj}) + \mu_m(C_{jkh} + C_{jkh}) \\
+ \mu_m(g_{jk} + g_{kj}) \}.
\end{equation}

This shows that

\[ B_mG^i_{jkh} = \lambda_mG^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}). \]

if and only if

\begin{equation}
(3.13) \quad G^r_{jrs} G^s_{hmk} + \delta^i_j \lambda_m (G^r_{jkr} + \hat{\partial}_h \mu_m (g_{jk} + g_{kj})) \\
+ \mu_m(C_{jkh} + C_{jkh}) + \mu_m(g_{jk} + g_{kj}).
\end{equation}

Thus, the following theorem

**Theorem 3.4.** In generalized $B_mU$-recurrent space, the curvature tensor $G^i_{jkh}$ is generalized recurrent if and only if equation (3.13) holds.

Differentiating (2.12) covariantly with respect to $x^m$ in the sense of Brewed, we get

\begin{equation}
(3.14) \quad B_mD^i_{jkh} = B_mU^i_{jkh} - \frac{1}{2} \delta^i_j B_mU_{kh} + \delta^i_k B_mU_{jh}.
\end{equation}

Using (3.1) and (3.4) in (3.14), we get

\[ B_mD^i_{jkh} = \lambda_mU^i_{jkh} - \frac{1}{2} \delta^i_j \lambda_m U_{kh} + \delta^i_k \lambda_m U_{jh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}) \\
- \frac{(n+1)}{2} \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).
\]

which can be written as

\begin{equation}
(3.15) \quad B_mD^i_{jkh} = \lambda_m \left\{ U^i_{jkh} - \frac{1}{2} \delta^i_j U_{kh} + \delta^i_k U_{jh} \right\} + \frac{(1-n)}{2} \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).
\end{equation}

Using (2.12) in (3.15), we get

\begin{equation}
(3.16) \quad B_mD^i_{jkh} = \lambda_mD^i_{jkh} + \frac{(1-n)}{2} \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).
\end{equation}

Thus, the following theorem

**Theorem 3.5.** In generalized $B_mU$-recurrent space, Douglas tensor $D^i_{jkh}$ is generalized recurrent.

If Douglas tensor $D^i_{jkh}$ is a generalized recurrent space, our space is necessarily generalized $B_mU$-recurrent space, this may be seen as follows:

Taking covariant derivative of (2.12), with respect to $x^m$ in the sense of Brewed, gives

\begin{equation}
(3.17) \quad B_mU^i_{jkh} = B_mD^i_{jkh} + \frac{1}{2} \delta^i_j B_mU_{kh} + \delta^i_k B_mU_{jh}.
\end{equation}

Using (3.4) and (3.16) in (3.17), we get

\begin{equation}
(3.18) \quad B_mU^i_{jkh} = \lambda_m \left\{ D^i_{jkh} + \frac{1}{2} \delta^i_j U_{kh} + \delta^i_k U_{jh} \right\} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).
\end{equation}

Using (2.12) in (3.18), we get

\[ B_mU^i_{jkh} = \lambda_mU^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).
\]

Thus, the following theorem

**Theorem 3.6.** In a Finsler space $F_n$, if Douglas tensor $D^i_{jkh}$ is a generalized recurrent and the $U$ - Ricci tensor $U_{jk}$ is given by (3.4), then the space considered is a generalized $B_mU$-recurrent space.

**Decompositions of $h(v)$ - Curvature Tensor in Finsler Space**

Let us consider the decomposition of the $h(v)$ - curvature tensor $U^i_{jkh}$ of a Finsler space is of the type (1.3) as follows :

\begin{equation}
(4.1) \quad U^i_{jkh} = y^i Y_{jkh}
\end{equation}

where $Y_{jkh}$ is non-zero tensor filed called *decomposition tensor field*.

We define

\begin{equation}
(4.2) \quad y^i \lambda_i = \sigma
\end{equation}

such $\lambda_i$ as recurrence vector and $\sigma$ is decomposition scalar.
On a Generalized $B_m U$ -Recurrent Finsler Space …………………Abdalstar Ali Mohsen Saleem

In view of (4.1) and (2.2), we get

\[(4.3) \quad y^i Y_{jkh} = G^i_{jkh} - \frac{1}{n+1} (\delta^i_j G^k_{hr} + y^i G^j_{xhr}).\]

Transvecting (4.3) by $\lambda_i$ and using (4.2), we get

\[(4.4) \quad Y_{jkh} = \frac{\lambda_i}{\sigma} G^i_{jkh} - \frac{1}{n+1} (\frac{\lambda_i}{\sigma} G^j_{xhr} + G^i_{xhr}).\]

Thus, the following theorem

**Theorem 4.1.** If the $h(v)$ - curvature tensor $U^i_{jkh}$ of a Finsler space is decomposable in the form (4.1), then the tensor $Y_{jkh}$ is defined by (4.4).

In view of (4.1), (2.5) and (2.10), we get

\[(4.5) \quad Y_{jkh} y^h = \frac{1}{n+1} G_{jr},\]

since $y^i \neq 0$.

Transvecting (4.5) by $y^j$ and using (1.3a), we get

\[(4.6) \quad G_{jk} = (n + 1) Y_{jk},\]

where $Y_{jkh} y^h = Y_{jk}$.

In view of (4.6) and (2.8), we get

\[(4.7) \quad U_{jk} = \frac{1}{2} Y_{jk}.\]

Thus, the following theorem

**Theorem 4.2.** If the $h(v)$ - curvature tensor $U^i_{jkh}$ of a Finsler space is decomposable in the form (4.1), then the $U$ - Ricci tensor $G_{jk}$ and the $G$ - Ricci tensor $U_{jk}$ are defined by (4.6) and (4.7).

References

حول تعميم فضاء فنسلر $B_mU$ - أحادي المعاودة

عبد الستار علي محسن سليم
قسم الرياضيات، كلية التربية، جامعة عدن - خور مكسر - اليمن
DOI: https://doi.org/10.47372/uajnas.2019.n2.a15

المستشف

في هذه الورقة تم تقديم فضاء فنسلر الذي يحقق فيه الموتر التقوسي الشرط الآتي:

$B_mU_{jkh} = \lambda U_{jkh} + \mu (g_{kh} + g_{jh}), U_{jkh} \neq 0,$

حيث $\lambda$ و $\mu$ هي متجهات متحدة الاختلاف. $B_mU$ هي فضاء يحقق هذا الفضاء الذي يحقق الشرط أعلاه في فضاء $U$ - أحادي المعاودة المعمم. يتم تمثله بواسطة $B_mU$ - أحادي المعاودة المعمم في هذه الورقة. ويعتبر الشكل المثبط المتقدم للفضاء الذي يحقق الشكل المثبط المتقدم للموتر التقوسي $U_{jkh}$. كما تم إثبات أن في هذه الورقة تم إيجاد الشكل المثبط المتقدم للفضاء الذي يحقق الشكل المثبط المتقدم للموتر التقوسي $U$. فضاء فنسلر $B_mU$ يتوافق مع الفضاء المبدع في الفضاء المبدع $B_mU$ - أحادي المعاودة المعمم، والموتر ريشتي والكلام المثبط المتقدم للفضاء الذي يحقق الشكل المثبط المتقدم للموتر التقوسي $B_mU$ - أحادي المعاودة المعمم، يتم تمثله بواسطة $B_mU$ - أحادي المعاودة المعمم، الفضاء المبدع،كلميات المفتاحية: فضاء فنسلر, فضاء $B_mU$, أحادي المعاودة المعمم, تعميم أحادي المعاودة وتحليل الموتر التقوسي.