# On a Generalized $\mathcal{B}_m$U -Recurrent Finsler Space

**Abdalstar Ali Mohsen Saleem**
Dept. Of Math., Faculty of Education-Yafea, Univ.of Aden, Khormakssar, Yemen
Abdulsstar 1972 @ gmail.com
DOI: https://doi.org/10.47372/uajnas.2019.n2.a15


## Abstract

A Finsler space $F_n$ for which the h(v) - curvature tensor $U^i_{jkh}$ satisfies the condition $\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh})$, where $\lambda_m$ and $\mu_m$ are non-zero covariant vector fields and $\mathcal{B}_m$ is covariant derivative of first order in the sense of Berwald (Berwald's covariant differential operator ). In the present paper, satisfying this condition will be called *a generalized* $\mathcal{B}_m U$*-recurrent space*. The tensor $G^r_{rkh}$, the h(v)-torsion tensor $U^i_{kh}$, the G- Ricci tensor $G_{jk}$ and the U-Ricci tensor $U_{jk}$ are non-vanishing. Under certain conditions, a generalized $\mathcal{B}_m U$ - recurrent space becomes a generalized recurrent tensor. Also, we discuss the decomposing of the h(v) - curvature tensor $U^i_{jkh}$ in Finsler space.


**Key words:** Finsler space, generalized $\mathcal{B}_m U$ - recurrent space, generalized recurrent tensor, decompositions of tensor.

## Introduction

K. Yano [14] defined the normal projective connection $\Pi^i_{jk}$ by

(1.1) $\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{jkr}$.

R. B. Misra and F. M. Meher [4] considered a space equipped with normal projective connection $\Pi^i_{jk}$ whose curvature tensor $N^i_{jkh}$ is recurrent with respect to normal projective connection $\Pi^i_{jk}$ and they called it *RNP-Finsler space*. P.N. Pandey and V.J. Diwivedi [10] studied *RNP*-Finsler space and obtained many identities in *RNP*-Finsler space, most of these identities are also true in a recurrent Finsler space, with respect to Berwald's connection coefficients $G^i_{jk}$. P.N. Pandey ([6], [7], [8]) obtained a relation between the normal projective curvature tensor $N^i_{jkh}$ and Berwald curvature tenser $H^i_{jkh}$ and defined *NPR-Finsler space*, which is characterized by the recurrent of normal projective curvature tenser $N^i_{jkh}$ with respect to Berwald's connection coefficients $G^i_{jk}$. F.Y. A. Qasem [11] obtained several results concerning the h(v) - curvature tensor $U^i_{jkh}$ in such space.

Let us consider a set of quantities $g_{ij}$ defined by

(1.2) $g_{ij}(x,y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j F^2(x,y)$.

The quantities $g_{ij}$ constitute the components of covariant tensor of the type (0,2). Clearly, this shows that the tensor $g_{ij}(x,y)$ is positively homogeneous of degree zero in $y^i$ and symmetric in i and j. According to Euler's theorem on homogeneous functions, the vectors $y_i$ and $y^i$ satisfy the following estimates

(1.3) $a)\ y_i y^i = F^2$, $b)\ g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i$ and $c)\ g_{ij} y^i = y_j$.

By differentiating equation (1.2) partially with respect to $y^k$, we get a new tensor $C_{ijk}$ defined by

(1.4) $C_{ijk} = \frac{1}{2}\dot{\partial}_i g_{jk}$.

The tensor $C_{ijk}$ is positively homogeneous of degree -1 in $y^i$ and symmetric in all its indices and is called *(h)hv-torsion tensor*.

Berwald covariant derivative of the metric function $F$ and vector $y^i$ vanish identically, i.e.

(1.5) $a)\ \mathcal{B}_k F = 0$ and $b)\ \mathcal{B}_k y^i = 0$.

Berwald covariant derivative of the metric tensor $g_{ij}$ does not vanish and is given by

(1.6) $\mathcal{B}_k g_{ij} = -2C_{ijk|h}y^h = -2y^h\mathcal{B}_h C_{ijk}$.

Berwald covariant derivative of an arbitrary tensor $T_h^i$ with respect to $x^k$ is given by

(1.7) $a)\ \mathcal{B}_k T_h^i = \partial_k T_h^i + T_h^r G_{rk}^i - T_r^i G_{sk}^r - (\dot{\partial}_r T_h^i) G_{hk}^r$

and the commutation formula for the operators $\dot{\partial}_j$ and $\mathcal{B}_k$ are given by [12]

$b)\ \dot{\partial}_j \mathcal{B}_k T_h^i - \mathcal{B}_k \dot{\partial}_j T_h^i = T_h^r \mathrm{G}_{jkr}^i - T_r^i \mathrm{G}_{jkh}^r$.

## Normal Projective Connection Coefficients

K. Yano [14] defined the normal projective connection coefficients $\Pi_{jk}^i$ defined by (1.1). The connection coefficients $\Pi_{jk}^i$ is positively homogeneous of degree zero in $y^i$ and symmetric in their lower indices. The normal projective tensor $N_{jkh}^i$ is given by

(2.1) $N_{jkh}^i = \dot{\partial}_j \Pi_{kh}^i + \Pi_{rjh}^i \Pi_{ks}^r y^s + \Pi_{rh}^i \Pi_{kj}^r - k|h$ ,

where

$$\Pi_{jkh}^i = \mathrm{G}_{jkh}^i - \frac{1}{n+1}(\delta_k^i \mathrm{G}_{jhr}^r + y^i \mathrm{G}_{jkhr}^r)$$

and

$$\Pi_{jkh}^i = \dot{\partial}_j \Pi_{kh}^i ,$$

where $\Pi_{jkh}^i$ constitutes the components of a tensor.

Also, K. Yano denoted this tensor by $U_{jkh}^i$. We shall follow K. Yano and denote the tensor $\Pi_{jkh}^i$ by $U_{jkh}^i$. Thus,

(2.2) $U_{jkh}^i = G_{jkh}^i - \frac{1}{n+1}(\delta_k^i \mathrm{G}_{jhr}^r + y^i \mathrm{G}_{jkhr}^r)$

and

(2.3) $G_{ljkr}^r = \dot{\partial}_l G_{jkr}^r$ .

The tensor $U_{jkh}^i$ is called *h(v) - curvature tensor* [2] and $G_{jkh}^i$ is connection of h(v) -curvature tensor [1] . This tensor is homogeneous of degree -1 in $y^i$. Also, this tensor satisfies the following:

(2.4) $U_{jki}^i = G_{jki}^i$ ,

(2.5) $U_{jkh}^i y^h = U_{jhk}^i y^h = U_{jk}^i$

and

(2.6) $U_{jkh}^i y^j = 0$.

The tensor $U_{kh}$ is called *h(v)-Ricci tensor* and satisfies the following [5]:

(2.7) $U_{ikh}^i = U_{kh}$

and

(2.8) $U_{kh} = \frac{2}{n+1} G_{kh}$ ,

where the tensor $G_{kh}$ is components of the projective connection coefficients [1]. The symmetric tensor $U_{jk}^i$ is called *h(v)-torsion tensor* and satisfies [5]

(2.9) $U_{jk}^i = U_{kj}^i$,

(2.10) $U_{jk}^i = \frac{1}{n+1} y^i G_{jk}$

and

(2.11) $U_{jk}^i y^k = 0$.

Douglas tensor ( [2], [3], [9]) is given by

(2.12) $D_{jkh}^i = U_{jkh}^i - \frac{1}{2}(\delta_j^i U_{kh} + \delta_k^i U_{jh})$.

This tensor satisfies the following:

(2.13) $D_{jkh}^i y^j = D_{kjh}^i y^j = D_{khj}^i y^j = 0$

and

(2.14) $D_{rkh}^r = 0$.

A Finsler space is called *a recurrent Finsler space* if it's the h(v)- curvature tensor $U^i_{jkh}$ satisfies ([11], [13])

(2.15) $\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh}, \quad U^i_{jkh} \neq 0,$

where $\lambda_m$ is non-zero covariant vector field.

## Generalized $\mathcal{B}_m U$-Recurrent Space

Let us consider a Finsler space $F_n$ for which the normal projective curvature tensor $U^i_{jkh}$ satisfies the following condition

(3.1) $\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}), \quad U^i_{jkh} \neq 0,$

where $\lambda_m$ and $\mu_m$ are non-zero covariant vector field and satisfying the condition (3.1) will be called a *generalized $\mathcal{B}_m U$- recurrent space*.

Transvecting (3.1) by $y^h$ and using (1.5b), (2.5) and (1.3c), we get

(3.2) $\mathcal{B}_m U^i_{jk} = \lambda_m U^i_{jk} + \mu_m(\delta^i_j y_k + \delta^i_k y_j).$

Thus, the following theorem

**Theorem 3.1.** *In generalized $\mathcal{B}_m U$-recurrent space, Berwald covariant derivative of the h(v) - torsion tensor $U^i_{kh}$ is given by (3.2).*

In view of (2.4), contracting the indices *i* and *h* in (3.1), we get

(3.3) $\mathcal{B}_m G^r_{jkr} = \lambda_m G^r_{jkr} + \mu_m(g_{kj} + g_{jk}).$

Contracting the indices *i* and *j* in (3.1) and using (2.7), we get

(3.4) $\mathcal{B}_m U_{kh} = \lambda_m U_{kh} + (n+1)\mu_m g_{kh}.$

In view of (3.4) and using (2.8), we get

(3.5) $\mathcal{B}_m G_{jk} = \lambda_m G_{jk} + \frac{1}{2}(n+1)^2 \mu_m g_{jk}.$

Thus, the following theorem

**Theorem 3.2.** *The U - Ricci tensor $U_{jk}$, the tensor $G^r_{jkr}$ and the G - Ricci tensor $G_{jk}$ of a generalized $\mathcal{B}_m U$-recurrent space are non – vanishing.*

Differentiating (3.3) partially with respect to $y^h$ in the sense of Berwald and using (1.4), we get

(3.6) $\dot{\partial}_h \mathcal{B}_m G^r_{jkr} = (\dot{\partial}_h \lambda_m) G^r_{jkr} + \lambda_m \dot{\partial}_h G^r_{jkr} + (\dot{\partial}_h \mu_m)(g_{kj} + g_{jk})$
$+2\mu_m(C_{hkj} + C_{hjk}).$

Using commutation formula exhibited by (1.7b) for $G^r_{jkr}$ in (3.6) and using (2.3), we get

(3.7) $\mathcal{B}_m G^r_{hjkr} - G^r_{skr} G^s_{hmj} - G^r_{jsr} G^s_{hmk} = (\dot{\partial}_h \lambda_m) G^r_{jkr} + \lambda_m G^r_{jkhr}$
$+\dot{\partial}_h \mu_m(g_{kj} + g_{jk}) + 2\mu_m(C_{hjk} + C_{hjk}).$

Therefore,

(3.8) $\mathcal{B}_m G^r_{hjkr} = \lambda_m G^r_{hjkr} + \mu_m(C_{hjk} + C_{hjk}),$

if and only if

(3.9) $G^r_{skr} G^s_{hmj} + G^r_{jsr} G^s_{hmk} + (\dot{\partial}_h \lambda_m) G^r_{jkr} + \dot{\partial}_h \mu_m(g_{kj} + g_{jk})$
$+\mu_m(C_{hjk} + C_{hjk}) = 0.$

Thus, the following theorem

**Theorem 3.3.** *In generalized $\mathcal{B}_m U$-recurrent space, Berwald covariant derivative of the tensor $G^r_{jkhr}$ is given by (3.8), if and only if (3.9) holds.*

Differentiating (2.2) covariantly with respect to $x^m$ in the sense of Berwald and using (1.5b), we get

(3.10) $\mathcal{B}_m U^i_{jkh} = \mathcal{B}_m G^i_{jkh} - \frac{1}{n+1}(\delta^i_j \mathcal{B}_m G^r_{jkr} + y^i \mathcal{B}_m G^r_{jkhr}).$

Using (3.1) in (3.10), we get

(3.11) $\lambda_m U^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}) = \mathcal{B}_m G^i_{jkh} - \frac{1}{n+1}(\delta^i_j \mathcal{B}_m G^r_{jkr} + y^i \mathcal{B}_m G^r_{jkhr}).$

Using (2.2), (3.3) and (3.7) in (3.11), we get

(3.12) $\mathcal{B}_m G^i_{jkh} - \lambda_m G^i_{jkh} - \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}) = \frac{1}{n+1} y^i\{G^r_{skr} G^s_{hmj}$
$+G^r_{jsr} G^s_{hmk} + (\dot{\partial}_h \lambda_m) G^r_{jkr} + \dot{\partial}_h \mu_m (g_{jk} + g_{kj}) + \mu_m (C_{jhk} + C_{jkh})$
$+\mu_m (g_{jk} + g_{kj})\}$.

This shows that

$$\mathcal{B}_m G^i_{jkh} = \lambda_m G^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}),$$

if and only if

(3.13) $G^r_{skr} G^s_{hmj} + G^r_{jsr} G^s_{hmk} + (\dot{\partial}_h \lambda_m) G^r_{jkr} + \dot{\partial}_h \mu_m (g_{jk} + g_{kj})$
$+\mu_m (C_{jhk} + C_{jkh}) + \mu_m (g_{jk} + g_{kj})$.

Thus, the following theorem

**Theorem 3.4.** *In generalized $\mathcal{B}_m U$-recurrent space, the curvature tensor $G^i_{jkh}$ is generalized recurrent if and only if equation* (3.13) *holds.*

Differentiating (2.12) covariantly with respect to $x^m$ in the sense of Brewed, we get

(3.14) $\mathcal{B}_m D^i_{jkh} = \mathcal{B}_m U^i_{jkh} - \frac{1}{2}(\delta^i_j \mathcal{B}_m U_{kh} + \delta^i_k \mathcal{B}_m U_{jh})$.

Using (3.1) and (3.4 ) in (3.14), we get

$$\mathcal{B}_m D^i_{jkh} = \lambda_m U^i_{jkh} - \frac{1}{2}(\delta^i_j \lambda_m U_{kh} + \delta^i_k \lambda_m U_{jh}) + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh})$$
$$-\frac{(n+1)}{2} \mu_m (\delta^i_j g_{kh} + \delta^i_k g_{jh})$$

which can be written as

(3.15) $\mathcal{B}_m D^i_{jkh} = \lambda_m \left\{U^i_{jkh} - \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})\right\} + \frac{(1-n)}{2} \mu_m (\delta^i_j g_{kh} + \delta^i_k g_{jh})$.

Using (2.12) in (3.15), we get

(3.16) $\mathcal{B}_m D^i_{jkh} = \lambda_m D^i_{jkh} + \frac{(1-n)}{2} \mu_m (\delta^i_j g_{kh} + \delta^i_k g_{jh})$.

Thus, the following theorem

**Theorem 3.5.** *In generalized $\mathcal{B}_m U$-recurrent space, Douglas tensor $D^i_{jkh}$ is generalized recurrent.*

If Douglas tensor $D^i_{jkh}$ is a generalized recurrent space, our space is necessarily generalized $\mathcal{B}_m U$-recurrent space, this may be seen as follows:

Taking covariant derivative of (2.12), with respect to $x^m$ in the sense of Brewed, gives

(3.17) $\mathcal{B}_m U^i_{jkh} = \mathcal{B}_m D^i_{jkh} + \frac{1}{2}(\delta^i_j \mathcal{B}_m U_{kh} + \delta^i_k \mathcal{B}_m U_{jh})$.

Using (3.4) and (3.16) in (3.17), we get

(3.18) $\mathcal{B}_m U^i_{jkh} = \lambda_m \left\{D^i_{jkh} + \frac{1}{2}(\delta^i_j U_{kh} + \delta^i_k U_{jh})\right\} + \mu_m (\delta^i_j g_{kh} + \delta^i_k g_{jh})$.

Using (2.12) in (3.18), we get

$$\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}).$$

Thus, the following theorem

**Theorem 3.6.** *In a Finsler space $F_n$, if Douglas tensor $D^i_{jkh}$ is a generalized recurrent and the U - Ricci tensor $U_{jk}$ is given by (3.4), then the space considered isa generalized $\mathcal{B}_m U$-recurrent space.*

## Decompositions of h(v) - Curvature Tensor in Finsler Space

Let us consider the decomposition of the h(v) - curvature tensor $U^i_{jkh}$of a Finsler space is of the type (1,3) as follows :

(4.1) $U^i_{jkh} = y^i Y_{jkh}$

where $Y_{jkh}$ is non-zero tensor filed called *decomposition tensor field*.

We define

(4.2) $y^i \lambda_i = \sigma$

such $\lambda_i$ as recurrence vector and $\sigma$ is decomposition scalar.

In view of (4.1) and (2.2), we get

(4.3) $y^i Y_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + y^i G^r_{jkhr})$.

Transvecting (4.3) by $\lambda_i$ and using (4.2), we get

(4.4) $Y_{jkh} = \frac{\lambda_i}{\sigma} G^i_{jkh} - \frac{1}{n+1}(\frac{\lambda_j}{\sigma} G^r_{khr} + G^r_{jkhr})$.

Thus, the following theorem

**Theorem 4.1.** *If the h(v) - curvature tensor $U^i_{jkh}$ of a Finsler space is decomposable in the form (4.1), then the tensor $Y_{jkh}$ is defined by (4.4).*

In view of (4.1), (2.5) and (2.10), we get

(4.5) $Y_{jkh} y^h = \frac{1}{n+1} G_{jk}$ ,

since $y^i \neq 0$.

Transvecting (4.5) by $y^j$ and using (1.3a), we get

(4.6) $G_{jk} = (n+1)\, Y_{jk}$,

where $Y_{jkh} y^h = Y_{jk}$.

In view of (4.6) and (2.8), we get

(4.7) $U_{jk} = \frac{1}{2} Y_{jk}$ .

Thus, the following theorem

**Theorem 4.2.** *If the h(v) - curvature tensor $U^i_{jkh}$ of a Finsler space is decomposable in the form (4.1), then the U- Ricci tensor $G_{jk}$ and the G- Ricci tensor-$U_{jk}$ are defined by (4.6) and (4.7).*

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# حول تعميم فضاء فنسلر $\mathcal{B}_m$U – أحادي المعاودة

**عبد الستار علي محسن سليم**
قسم الرياضيات، كلية التربية/ يافع ،جامعة عدن ـ خور مكسر ـ اليمن
DOI: https://doi.org/10.47372/uajnas.2019.n2.a15

## الملخص

في هذه الورقة تم تقديم فضاء فنسلر الذي يحقق فيه الموتر التقوسي الشرط الآتي:

$$\mathcal{B}_m U^i_{jkh} = \lambda_m U^i_{jkh} + \mu_m(\delta^i_j g_{kh} + \delta^i_k g_{jh}),\ U^i_{jkh} \neq 0,$$

حيث $\lambda_m$ و $\mu_m$ هي متجهات متحدة الاختلاف, $\mathcal{B}_m$ هي مشتقة برواد وتم تسمية هذا الفضاء الذي يحقق الشرط أعلاه فضاء $\mathcal{B}_m U$ ـ أحادي المعاودة المعمم.

في هذه الورقة تم إيجاد المشتقة المتحدة الاختلاف بمفهوم برولاد للموتر التقوسي $U^i_{jkh}$. كما تم إثبات أن موتر رتشي, والكميه المتجهه كلها لا تنتهي في فضاء U ـ أحادي المعاودة المعمم. وتم إيجاد الشرط اللازم ليكون فضاء فنسلر $\mathcal{B}_m U$ ـ أحادي المعاودة المعمم عندما يكون كل من موتر تقوس دوجلاس أحادي المعاودة المعمم وموتر ريشتي معرفاً بالمعادلة (3.4).

**الكلمات المفتاحية**: فضاء فنسلر, فضاء$\mathcal{B}_m U$ ـ أحادي المعاودة المعمم , تعميم أحادي المعاودة وتحلل الموتر التقوسي.