On a Generalized βH – Trirecurrent Finsler Space

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Abstract

In this paper, we introduced a Finsler space for which the h − curvature tensor H^i_{jkh} (curvature tensor of Berwald) satisfies the condition

$$\beta_\ell\beta_m\beta_n H^i_{jkh} = c_{\ell mn}H^i_{jkh} + d_{\ell mn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - 2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell})$$
$$-2y^r w_{\ell n}\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - 2y^r \mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}), H^i_{jkh} = 0,$$

where C_{jkm} is (h) hv − tortion tensor, $\beta_\ell\beta_m\beta_n$ is Berwald's covariant differential operator of the third order with respect to x^n, x^m and x^ℓ, successively, $\beta_\ell\,\beta_r$ is Berwald's covariant differential operator of the second order with respect to x^ℓ and x^r, successively, β_r is Berwald's covariant differential operator of the first order with respect to x^r, $c_{\ell mn}$ and $d_{\ell mn}$ are non – zero covariant tensors field of third order, b_{mn} and $w_{\ell n}$ are non – zero covariant tensors field of second order and μ_ℓ is non – zero covariant vector field. We called this space *a generalized βH – trirecurrent space*. The aim of this paper is to develop some properties of a generalized βH – trirecurrent space by obtaining Berwald's covariant derivative of the third order for the (h)v – torsion tensor H^i_{kh} and the deviation tensor H^i_k , the curvature vector H_k and the scalar curvature H are investigated.

Key words: Finsler space, generalized βH – trirecurrent space, Ricci tensor.

1. Introduction

Pandey P.N., Saxena S.S. and Goswami A. [3] introduced and studied a generalized H–recurrent Finsler space. F.Y.A.Qasem [4] introduced and discussed generalized H− birecurrent curvature tensor and W.H.A. Hadi [1] studied the generalized – birecurrent for some tensors and studied some special spaces in this space.

Let F_n be an n − dimensional Finsler spaces equipped with the metric function F satisfies conditions [5], the tensor C_{ijk} is positively homogeneous of degree -1 in $y^i$and symmetric in all its indices and is called *(h)hv − torsion tensor* [2]. According to Euler's theorem on homogeneous functions, this tensor satisfies the following:

(1.1) $\quad C_{ijk}y^i = C_{jki}y^i = C_{kij}y^i = 0.$

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [5]

(1.2) $\quad \mathcal{B}_k T^i_j := \partial_k T^i_j - \left(\dot\partial_r T^i_j\right)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$

Berwald's covariant derivative of y^i vanish as identically [5], i.e.

(1.3) $\quad \mathcal{B}_k y^i = 0.$

In view of (1.2), the second covariant derivative of an arbitrary vector field X^i with respect to x^h in the sense of Berwald [5],

(14) $\quad \mathcal{B}_h\mathcal{B}_k X^i = \partial_k\left(\mathcal{B}_h X^i\right) - \left(\partial_s\mathcal{B}_h X^i\right)G^s_k - \left(\mathcal{B}_r X^i\right)G^r_{hk} + (\mathcal{B}_h X^r)G^i_{rk}.$

Using (1.4) and taking skew − symmetric part, with respect to the indices k and h, we get the commutation formula for Berwald's covariant differentiation as follows [5]:

(1.5) $\mathcal{B}_h\mathcal{B}_k X^i - \mathcal{B}_k\mathcal{B}_h X^i = X^r H^i_{rkh} - (\dot{\partial}_r X^i) H^r_{kh}$

where

(1.6) (a) $H^i_{jkh} \coloneqq \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rjh} G^r_k - h/k$ and (b) $H^i_{kh} \coloneqq \partial_h G^i_k + G^r_k G^i_{rh} - h/k.$

Remark 1.1. $-h/k$ means the subtraction from the former term by interchanging the indices h and k.

The tensors H^i_{jkh} and H^i_{kh} , as defined above, are called $h-$*curvature tensor* ($h-$ *curvature tensor of Berwald*) and *h(hv)* $-$ *torsion tensor* are positively homogeneous of degree zero and one in y^i , respectively.

Berwald construcated the tensors $H^i_{jkh}$and H^i_{kh} from the tensor H^i_k called by him as *deviation tensor* , according to

(1.7) $H^i_h \coloneqq 2\partial_h G^i + \partial_s G^i_h y^s + 2G^i_{hs} G^s - G^i_s G^s_h$.

The h(hv) – torsion tensor and the deviation tensor satisfy the following [5]:

(1.8) (a) $H^i_{jkh} y^j = H^i_{kh}$, (b) $H^i_{jk} y^j = H^i_k$ and (c) $\dot{\partial}_j H^i_k = H^i_{jk}$.

The H – Ricci tensor , the curvature vector and the scalar curvature satisfy the following [5]:

(1.9) (a) $H^i_{jki} = H_{jk}$, (b) $H^i_{ki} = H_k$ and (c) $H^i_i = H$.

2. Generalized βH – Trirecurrent Space

A Finsler space for which Berwald curvature tensor H^i_{jkh} satisfies the generalized recurrence property, i.e. characterized by the equation [3]

(2.1) $\beta_n H^i_{jkh} = \lambda_n H^i_{jkh} + \mu_n(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $H^i_{jkh} = 0,$

where β_n is the differential operator with respect to x^n in the sense of Berwald, λ_n and μ_n are non $-$ zero covariant vectors field and isvcalled *the recurrence vectors field*, such space known *as generalized H – recurrent Finsler space*.

Taking the β – covariant derivative for (2.1) with respect to x^m, we get

$$\beta_m\beta_n H^i_{jkh} = (\beta_m\lambda_n) H^i_{jkh} + \lambda_n(\beta_m H^i_{jkh}) + (\beta_m\mu_n)(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \mu_n\beta_m(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

In view of the equation (2.1), we can write the above equation as

$$\beta_m\beta_n H^i_{jkh} = (\beta_m\lambda_n + \lambda_n\lambda_m) H^i_{jkh} + (\lambda_n\mu_m + \beta_m\mu_n)(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r\mu_n\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$$

which can be written as [4]

(2.2) $\beta_m\beta_n H^i_{jkh} = a_{mn} H^i_{jkh} + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r\mu_n\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}),$

where $a_{mn} = \beta_m\lambda_n + \lambda_n\lambda_m$ and $b_{mn} = \lambda_n\mu_m + \beta_m\mu_n$ are non $-$ zero covariant tensors field of second order and $\beta_m\beta_n$ is the differential operator with respect to x^n and x^m , successively, such space known as *generalized* βH *– birecurrent space.*

Remark 2.1. The expression β – covariant derivative stands the covariant derivative in the sense of Berwald.

Taking the β – covariant derivative for (2.2) with respect to x^ℓ, we get

$$\begin{aligned}\beta_\ell\beta_m\beta_n H^i_{jkh} = (\beta_\ell a_{mn}) H^i_{jkh} + a_{mn}(\beta_\ell H^i_{jkh}) + (\beta_\ell b_{mn})(\delta^i_k g_{jh} - \delta^i_h g_{jk}) \\ + b_{mn}\beta_\ell(\delta^i_k g_{jh} - \delta^i_h g_{jk}) - 2y^r(\beta_\ell\mu_n)\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) \\ -2y^r\mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}).\end{aligned}$$

In view of the equation (2.1), we can write the above equation as

$$\beta_\ell\beta_m\beta_n H^i_{jkh} = (\beta_\ell a_{mn} + a_{mn}\lambda_\ell)H^i_{jkh} + (\beta_\ell b_{mn} + a_{mn}\mu_\ell)(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$
$$-2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell}) - 2y^r(\beta_\ell\mu_n)\beta_r(\delta^i_k g_{jh} - \delta^i_h g_{jk})$$
$$-2y^r\mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$$

which can be written as

(2.3) $$\beta_\ell\beta_m\beta_n H^i_{jkh} = c_{\ell mn}H^i_{jkh} + d_{\ell mn}\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) - 2y^r b_{mn}\beta_r(\delta^i_k C_{jh\ell} - \delta^i_h C_{jk\ell})$$
$$-2y^r w_{\ell n}\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) - 2y^r\mu_n\beta_\ell\beta_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}) ,$$

where $c_{\ell mn} = \beta_\ell a_{mn} + a_{mn}\lambda_\ell$ and $d_{\ell mn} = \beta_\ell b_{mn} + a_{mn}\mu_\ell$ are non – zero covariant tensors field of third order $w_{\ell n} = \beta_\ell\mu_n$ is non – zero covariant tensor field of second order and $\beta_\ell\beta_m\beta_n$ is the differential operator with respect to x^n, x^m and x^ℓ, successiently.

Definition 2.1. A Finsler space F_n for which Berwald curvature tensor H^i_{jkh} satisfies the condition (2.3) and called *a generalized βH – trirecurrent space* and the tensor *a generalized β – trirecurrent*. We shall denote them briefly as *G βH – TR* and *Gβ – TR*, respectively.

Now, transvecting the condition (2.3) by y^j, in view of (1.3), and by using (1.8a) and (1.1), we get

(2.4) $$\beta_\ell\beta_m\beta_n H^i_{kh} = c_{\ell mn}H^i_{kh} + d_{\ell mn}\big(\delta^i_k y_h - \delta^i_h y_k\big)$$

Transvecting (2.4) by y^h , in view of (1.3) and by using (1.8b), we get

(2.5) $$\beta_\ell\beta_m\beta_n H^i_k = c_{\ell mn}H^i_k + d_{\ell mn}\big(\delta^i_k F^2 - y^i y_k\big).$$

Thus, we may conclude

Theorem 2.1. *In G βH – TR F_n , Berwald's covariant derivative of the third order for the h(v) – torsion tensor H^i_{kh} and the deviation tensor H^i_k given by the equations (2.4) and (2.5), respectively.*

Contracting the indices i and k in the condition (2.3) and using (1.9a), we get

(2.6) $$\beta_\ell\beta_m\beta_n H_{jh} = c_{\ell mn}H_{jh} + d_{\ell mn}(n-1)g_{jh} - 2b_{mn}(n-1)\beta_r C_{jh\ell} - 2y^r w_{\ell n}(n-1)\beta_r C_{jhm}$$
$$-2y^r\mu_n(n-1)\beta_\ell\beta_r C_{jhm} .$$

Thus, we may conclude

Theorem 2.2. *In G βH – TR F_n ,Berwald's covariant derivative of the third order for the H – Ricci tensor H_{jk} given by the equation (2.6) .*

Contracting the indices i and k in the equations (2.4) and (2.5), using(1.9b) and (1.9c), we get

(2.7) $$\beta_\ell\beta_m\beta_n H_h = c_{\ell mn}H_h + d_{\ell mn}(n-1)y_h$$

and

(2.8) $$\beta_\ell\beta_m\beta_n H = c_{\ell mn}H + d_{\ell mn}F^2.$$

The equations (2.7) and (2.8) show that the curvature vector H_h and the scalar curvature H can't vanish because the vanishing of them would imply $d_{\ell mn} = 0$, a contradiction.

Thus , we may conclude

Theorem 2.3. *In G βH – TR F_n , the curvature vector H_h and the curvature scalar H are non – vanishing.*

We know that [5]

(2.9) $$H^i_{ikh} = H_{hk} - H_{kh}.$$

Taking the β – covariant derivative of the third order with respect to x^n, x^m and x^ℓ, successively,for the equation (2.9), we get

$$\beta_\ell\beta_m\beta_n H^i_{ikh} = \beta_\ell\beta_m\beta_n H_{hk} - \beta_\ell\beta_m\beta_n H_{kh} .$$

By using (2.6), the above equation can be written

$$\beta_\ell\beta_m\beta_n H^i_{ikh} = c_{\ell mn} H_{hk} + d_{\ell mn}(n-1) g_{hk} - 2y^r b_{mn}(n-1)\beta_r C_{hk\ell} - 2y^r w_{\ell n}(n-1)\beta_r C_{hkm} - 2y^r \mu_n (n-1)\beta_\ell\beta_r C_{hkm} - [c_{\ell mn} H_{kh} + d_{\ell mn}(n-1) g_{kh} - 2y^r b_{mn}(n-1)\beta_r C_{kh\ell} - 2y^r w_{\ell n}(n-1)\beta_r C_{khm} - 2y^r \mu_n (n-1)\beta_\ell\beta_r C_{khm}].$$

By using the symmetric property of the h(hv) – torsion tensor C_{jkh} in it's all indies , the above equation can be written as

$$\beta_\ell\beta_m\beta_n H^i_{ikh} = c_{\ell mn}(H_{hk} - H_{kh}).$$

By using (2.9) in the above equation , we get

$$\beta_\ell\beta_m\beta_n (H_{hk} - H_{kh}) = c_{\ell mn}(H_{hk} - H_{kh}).$$

Thus , we may conclude

Theorem 2.4. *In G βH – TR F_n , the tensor $(H_{hk} - H_{kh})$ behaves as trirecurrent .*

We know that [5]

(2.10) $\quad H^i_{kh} = \frac{1}{3}\left(\dot\partial_k H^i_h - \dot\partial_h H^i_k\right).$

Taking the β – covariant derivative of third order with respect to x^n, x^m and x^ℓ, successively, for the equation (2.10), we get

$$\beta_\ell\beta_m\beta_n H^i_{kh} = \frac{1}{3}\beta_\ell\beta_m\beta_n\left(\dot\partial_k H^i_h - \dot\partial_h H^i_k\right) .$$

In view of (1.8c) , the above equation can be written as

$$\beta_\ell\beta_m\beta_n H^i_{kh} = \frac{1}{3}\left(\beta_\ell\beta_m\beta_n H^i_{kh} - \beta_\ell\beta_m\beta_n H^i_{hk}\right) .$$

By using (2.4) in the above equation, we get

$$\beta_\ell\beta_m\beta_n H^i_{kh} = \frac{1}{3}[\, c_{\ell mn} H^i_{kh} + d_{\ell mn}\left(\delta^i_k y_h - \delta^i_h y_k\right) - c_{\ell mn} H^i_{hk} - d_{\ell mn}\left(\delta^i_h y_k - \delta^i_k y_h\right)]$$

which can be written as

$$\beta_\ell\beta_m\beta_n H^i_{kh} = \frac{1}{3}\, c_{\ell mn}\left(H^i_{kh} - H^i_{hk}\right) + \frac{2}{3} d_{\ell mn}\left(\delta^i_k y_h - \delta^i_h y_k\right)$$

or

$$\beta_\ell\beta_m\beta_n H^i_{kh} = v_{\ell mn}\left(H^i_{kh} - H^i_{hk}\right) + w_{\ell mn}\left(\delta^i_k y_h - \delta^i_h y_k\right),$$

where $v_{\ell mn} = \frac{1}{3}\, c_{\ell mn}$ and $w_{\ell mn} = \frac{2}{3} d_{\ell mn}$.

Thus , we may conclude

Theorem 2.5. *In G βH – TR F_n , Berwald torsion tensor H^i_{kh} is generalized – trirecurrent tensor.*

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حول تعميم فضاء فنسلر βH– ثلاثي المعاودة

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المُلخص

تم تقديم تعريف هذا الفضاء من خلال اشتقاق المعادلة المميزة له لتعميم فضاء βH – ثنائي المعاودة وأطلقنا عليه الفضاء المعمم βH – ثلاثي المعاودة ورمزنا له بالرمز $G\beta H - TRF_n$، كما تم التوصل إلى بعض المبرهنات المختلفة بهذا الخصوص. وقد تم إثبات أن الموتر H – ريشي H_{jh} والمتجه التقوسي H_h هي موترات لا تنتهي في هذا الفضاء.

الكلمات المفتاحية: فضاء فنسلر، مشتقة متحدة الاختلاف بمفهوم بروالد من المرتبة الثالثة – موتر H – ريشي .