Variations on uncertainty principle inequalities for Weinstein operator

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Abstract

The aim of this paper is to prove new variations of uncertainty principles for Weinstein operator. The first of these results is variation of Heisenberg-type in equality for Weinstein transform that is for $s > 0$. Then, there exists a constant $C(\alpha, s)$, such that for all $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$

$$\||x|^{2s} f\|_{L^1_\alpha(\mathbb{R}^d_+)} \||\xi|^s \mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} \geq C(\alpha, s)\|f\|_{L^1_\alpha(\mathbb{R}^d_+)} \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

The second result is variation of Donoho-Strak's uncertainty principle for Weinstein transform, Let $S, \Sigma \subset \mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$. If f is (ε_1, α)-timelimited on T and (ε_2, α)-bandlimited on Σ, then $\mu_\alpha(S)\mu_\alpha(\Sigma) \geq (1-\varepsilon_1)^2(1-\varepsilon_2^2)$.

The third result is variation of the local uncertainty for Weinstein and Weinstein-Gabor transform.

Key words: Weinstein operator; Heisenberg's uncertainty inequality, time frequency-concentration.

Introduction

A Fourier uncertainty principle is an inequality or uniqueness theorem concerning the joint localization of a function and its Fourier transform. The most familiar form is the Heisenberg-Pauli-Weil uncertainty inequality [19]. This leads to the classical formulation in the form of the lower bound of the product of the dispersions of a function f and its Fourier transform $\mathcal{F}(f)$

$$\||x|f\|_{L^2(\mathbb{R}^d)} \||\xi|\mathcal{F}(f)\|_{L^2(\mathbb{R}^d)} \geq \frac{d}{2}\|f\|^2_{L^2(\mathbb{R}^d)}, \qquad (1.1)$$

with equality if, and only if, f is a multiple of a suitable Gaussian function. The Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by:

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(x)\, e^{-i\langle x,\xi\rangle} dx, \qquad (1.2)$$

here, we will denote by $|x|$ and $< x, y >$ the usual norm and scalar product on $\mathbb{R}^d$, and it is extend from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way.

There are many variations and generalizations of this inequality. Each of which suggests a special class of formualtion of uncertainty principle. One of these is Donoho and Stark [7], who obtained a quantitative version of the uncertainty principle about the essential supports. Precisely, suppoes a nonozero function $f \in L^2(\mathbb{R}^d)$ is ε_1 –concentrated on $S \subset \mathbb{R}^d$ and its Fourier transform is ε_2 –concentrated (i.e. f is ε_2 –bandlimited) on $\Sigma \subset \mathbb{R}^d$, then

$$|S||\Sigma| \geq (1-\varepsilon_1-\varepsilon_2)^2. \qquad (1.3)$$

More recently, in [11,12], the authors considered an uncertainty inequality of the form

$$\||X|^2 f\|_{L^1(\mathbb{R}^d)} \||\xi|\mathcal{F}(f)\|^2_{L^2(\mathbb{R}^d)} \geq c\|f\|_{L^1(\mathbb{R}^d)} \|f\|^2_{L^2(\mathbb{R}^d)}. \qquad (1.4)$$

In order to describe our paper, we first need to introduce some notations.

The unit sphere of $\mathbb{R}^d_+$ is denoted by $S^{d-1}_+ = S^{d-1} \cap \mathbb{R}^d_+$, then

$$W_{d,\alpha} := \int_{S^{d-1}_+} x_d^{2\alpha+1}\, d\sigma_d(x) = \frac{\pi^{\frac{d-1}{2}}\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{d+1}{2}\right)},$$

where $W_{d,\alpha}$ is the ball area and σ_d is the normalized surface measure on S^{d-1}_+.

For a radial function $f \in L^1_\alpha(R^d_+)$, the function f is defined on $\mathbb{R}^d_+$ such that $f(x) = \tilde{f}(|x|)$, for all $x \in \mathbb{R}^d_+$, is integrable with respect to the measure $r^{2\alpha+d}dr$. More precisely, we have

$$\int_{R^d_+} f(x)\,d\mu_\alpha(x) = a_{d,\alpha}\int_0^\infty \tilde{f}(r) r^{2\alpha+d}dr, \qquad (1.5)$$

where

$$a_{d,\alpha} = \frac{W_{d,\alpha}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)} = \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)}.$$

For $r > 0$, we will denote by $B_r = \{x \in R^d_+ : |x| < r\}$ the 'ball' in $\mathbb{R}^d_+$ of center 0 and radius r and the characteristic function of a set A will be χ_A, so that

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}.$$

Throughout this paper, α is a real number, $\alpha > -\frac{1}{2}$. We consider Weinstein operator studied by Ben Nahia and Ben Salem [1.2] defined on $\mathbb{R}^{d-1}X(0,\infty)$ by

$$\Delta_W = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d} = \Delta_{d-1} + \mathcal{L}_\alpha,$$

for $d > 2$, the operator Δ_W is the *Laplace-Beltrami* operator on the Riemanian space $\mathbb{R}^{d-1}X(0,\infty)$ equipped with the metric[1]

$$ds^2 = x_d^{\frac{4\alpha+2}{d-2}} \sum_{i=1}^{d} dx_i^2.$$

Weinstein operator has several application in pure and applied mathematics especially in fluid mechanics [4,19].

For $1 \le p < \infty$, we denote by the Lebesgue space consisting of measurable functions f on $\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times \mathbb{R}_+$ equipped with the norm

$$\|f\|_{L^p_\alpha(\mathbb{R}^d_+)} = \left(\int_{\mathbb{R}^d_+} |f(x',x_d)|^p d\mu_\alpha(x',x_d)\right)^{1/p}, \quad 1 \le p < \infty$$

$$\|f\|_{L^\infty_\alpha(\mathbb{R}^d_+)} = \operatorname{ess}\sup_{x\in\mathbb{R}^d_+} |f(x)| < \infty,$$

where for $x = (x_1, \cdots, x_{d-1}, x_d) = (x', x_d)$and

$$d\mu_\alpha(x) = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)} dx' dx_d = \frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}} 2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)} dx_1, \cdots, dx_d.$$

For $f \in L^1_\alpha(\mathbb{R}^d_+)$, Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi',\xi_d) = \int_{\mathbb{R}^d_+} f(x',x_d) e^{-i\langle x',\xi'\rangle} j_\alpha(x_d\xi_d)(x,\xi) d\mu_\alpha(x',x_d),$$

where $j_\alpha$is the spherical Bessel function :

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k}, \quad z \in \mathbb{C},$$

extends uniquely to isometric isomorphism on $L^2_\alpha(\mathbb{R}^d_+)$, that is

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}, \qquad (1.6)$$

and we have

$$\mathcal{F}_W{}^{-1}(f)(\xi) = \mathcal{F}_W(f)(-\xi',\xi_d), \qquad \xi = (\xi',\xi_d) \in \mathbb{R}^d_+.$$

Moreover, if $f \in L^1_\alpha(\mathbb{R}^d_+)$, then

$$\|\mathcal{F}_W(f)\|_{L^\infty_\alpha(\mathbb{R}^d_+)} \leq \|f\|_{L^1_\alpha(\mathbb{R}^d_+)}. \quad (1.7)$$

We recall the generalized translation operator $\tau_x,\ x \in \mathbb{R}^d_+$ associated with Weinstein operator $\Delta_W$is defined for a continuo function f on $\mathbb{R}^d_+$even with respect to the last variable by

$$\tau_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^\pi f\left(x'+y';\sqrt{x_d^2+y_d^2+2x_d^2y_d^2 cos\theta}\right)(sin\theta)^{2\alpha}d\theta,$$

$y \in \mathbb{R}^d_+$. Where$(x'+y' = x_1+y_1,\cdots,x_{d-1}+y_{d-1})$.

Also, we denote by $L^p_{\omega_\alpha}, 1 \leq p < \infty$, the space of measurable functions f on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ with respect to the measure $d\omega_\alpha(x,y) = d\mu_\alpha(x)d\mu_\alpha(y)$such that

$$\|F\|_{L^p_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)} = \left(\int_{\mathbb{R}^d_+} |F(x,y)|^p d\omega_\alpha(x,y)\right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|F\|_{L^\infty_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)} = \operatorname{ess\,sup}_n |F(x,y)|.$$

For any function g $\in L^2_\alpha(\mathbb{R}^d_+)$ and any $y \in \mathbb{R}^d_+$, we denote the modulation of g by y as

$\mathcal{M}_y g := g_y = \mathcal{F}_W\left(\sqrt{\tau_y|g|^2}\right)$,

due to Plancheral theorem and the invariance of the measure $\mu_\alpha$under the generalized translation $\tau_x$we have for all $g \in L^2_\alpha(\mathbb{R}^d_+)$

$\|g_y\|_{L^2_\alpha(\mathbb{R}^d_+)} = \|g\|_{L^2_\alpha(\mathbb{R}^d_+)}$.

Weinstein-Gabor transform is defined as follows :

Let g be in $L^2_\alpha(\mathbb{R}^d_+)$, for a function $f \in L^2_\alpha(\mathbb{R}^d_+)$, we define its Weinstein-Gabor transform by

$$\mathcal{G}_g f(x,y) = \int_{\mathbb{R}^d_+} f(s)\overline{\tau_{-x}g_y(s)}\, d\mu_\alpha(s) = f *_W \overline{\mathcal{F}_W^{-1}\left(\sqrt{\tau_y|g|^2}\right)(x)}. \quad (1.8)$$

Here $*_W$denotes the convolution product associated with Weinstein operator given by

$$f *_W g(x) = \int_{\mathbb{R}^d_+} f(y)\overline{\tau_{-x}(g)(y)}\, d\mu_\alpha(y).$$

Weinstein Gabor transform satisfies the following properties :

1- For any f,g in$L^2_\alpha(\mathbb{R}^d_+)$,

$$\|\mathcal{G}_g f\|_{L^\infty_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)} \leq \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|_{L^2_\alpha(\mathbb{R}^d_+)}. \quad (1.9)$$

2- For any f,g in $L^2_\alpha(\mathbb{R}^d_+)$, we have the following Plancheral-type formula

$$\|\mathcal{G}_g f\|_{L^2_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)} = \|f\|_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|_{L^2_\alpha(\mathbb{R}^d_+)}. \quad (1.10)$$

For more details see [14].

Our first result is variation of Heisenbergs uncertainty for Weinstein transform that is, for $s > 0$. Then, there exists a constant $C(\alpha,s)$, such that for all $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$

$$\||x|^{2s}f\|_{L^1_\alpha(\mathbb{R}^d_+)}\||\xi|^s\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} \geq C(\alpha,s)\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

The second result is variation of Donoho-stark's uncertainty principle for Weinstein operator .

Let $S,\Sigma \subset \mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$. If f is (ε_1,α)-timelimited on T and (ε_2,α)-bandlimited on Σ, then

$$\mu_\alpha(S)\mu_\alpha(\Sigma) \geq (1-\varepsilon_1)^2(1-\varepsilon_2^2).$$

The last chapter is devoted to the study of variation of the local uncertainty inequality for the Weinstein and Weinstein-Gabor transform respectively as follows:

■ Let $S,\Sigma \subset \mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$. Then, there exists a constant $C(\alpha,s)$ such that

$$\|\mathcal{F}_W f\|_{L^2_\alpha(\Sigma)} \leq C(\alpha,s)\sqrt{\mu_\alpha(\Sigma)}\|f\|^{\frac{2s}{2s+\alpha+(d+1)/2}}_{L^2_\alpha(\mathbb{R}^d_+)}\||x|^{2s}f\|^{\frac{\alpha+(d+1)/2}{2s+\alpha+(d+1)/2}}_{L^2_\alpha(\mathbb{R}^d_+)}.$$

■ Let $s > 0$ and $X \subset \mathbb{R}_+^d \times \mathbb{R}_+^d$ such that $0 < \omega_\alpha(X) < \infty$. Then for all functions $f, g \in L_\alpha^2(\mathbb{R}_+^d)$

$$\left\|G_g f\right\|_{L^2_{\omega_\alpha}(X)}^2 \leq C_{(\alpha,s)}^{-1}\omega_\alpha(X)\left\||x|^s G_g f\right\|_{L^2_{\omega_\alpha}(\mathbb{R}_+^d\times\mathbb{R}_+^d)}\left\||\xi|^s G_g f\right\|_{L^2_{\omega_\alpha}(\mathbb{R}_+^d\times\mathbb{R}_+^d)}.$$

Uncertainty Principles for Weinstein Transform

Now, we shall prove the above mentioned of variations on uncertainty principles inequalities for Weinstein operator. To do so, we start by the following definition.

Definition: Let $0 < \varepsilon < 1$ and $s > 0$. Let $T, \mathcal{W} \subset \mathbb{R}_+^d$ be a pair of measurable subsets. For $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^2(\mathbb{R}_+^d)$

■ We say that f is (ε, α)-concentrated at $x = 0$ if

$$\left\||x|^{2s} f\right\|_{L_\alpha^1(\mathbb{R}_+^d)} \leq \varepsilon\|f\|_{L_\alpha^1(\mathbb{R}_+^d)}. \quad (2.11)$$

■ We say that f is (ε, α)-timelimited on T if

$$\|f\|_{L_\alpha^1(T^c)} \leq \varepsilon\|f\|_{L_\alpha^1(\mathbb{R}_+^d)}. \quad (2.12)$$

■ We say that $\mathcal{F}_W(f)$ is (ε, α)-concentrated at $\xi = 0$ if

$$\left\||x|^{s}\mathcal{F}_W(f)\right\|_{L_\alpha^2(\mathbb{R}_+^d)} \leq \varepsilon\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}. \quad (2.13)$$

■ We say that $\mathcal{F}_W(f)$ is (ε, α)-bandlimited on $\mathcal{W}$ if

$$\|\mathcal{F}_W(f)\|_{L_\alpha^2(\mathcal{W}^c)} \leq \varepsilon\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}. \quad (2.14)$$

A variation of Heisenberg uncertainty inequality for Weinstein transform

Variations for Fourier transform $\mathcal{F}$ have appeared in the literature in [11,15] to obtain an upper bound for the so-called *Laue* constant and in the survey paper [8] for an extensive discussion. We have seen a variation of Heisenberg's uncertainty principle (1.4) associated with certain transformations, here we establish an analogous of (1.4) for Weinstein transform.

In order to prove our result, we first need to find some notations and to investigate the Nash-type inequality and Carlson-type inequality for the Weinstein transform as follows:

Nash-type inequality: The classical Nash inequality in $\mathbb{R}^d$, may be stated as

$$\|f\|_{L^2(\mathbb{R}^d)}^{2+\frac{4}{d}} \leq C_d\|f\|_{L^1(\mathbb{R}^d)}^{\frac{4}{d}}\left\||x|\hat{f}\right\|_{L^2(\mathbb{R}^d)}^2,$$

for all functions $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^2(\mathbb{R}_+^d)$. This inequality has been first introduced by Nash [16] to obtain regularity properties on the solutions to parabolic partial differential equations, and the optimal constant C_d has been computed more recently in [5].

Proposition 2.1. (Nash-type inequality for $\mathcal{F}_W$)

Let $s > 0$. Then, there exists a constant $C_1(\alpha, s)$ such that for all $f \in L_\alpha^1(\mathbb{R}_+^d) \cap L_\alpha^2(\mathbb{R}_+^d)$

$$\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^{2+\frac{2s}{\alpha+(d+1)/2}} \leq C_1(\alpha, s)\|f\|_{L_\alpha^1(\mathbb{R}_+^d)}^{\frac{2s}{\alpha+(d+1)/2}}\left\||\xi|^s\mathcal{F}_W(f)\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2,$$

where $$C_1(\alpha, s) = \left(\frac{1}{s2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)}\right)^{\frac{s}{\alpha+(d+1)/2}}\left(\frac{2s+2\alpha+d+1}{2\alpha+d+1}\right)^{\frac{s+\alpha+(d+1)/2}{\alpha+(d+1)/2}}.$$

Proof: For $r > 0$, we will denote $\chi_r = \chi_{\{x:|x|<r\}}$ and $\overline{\chi}_r = 1 - \chi_r$. Then, using Plancherel's formula(1.6) we have

$$\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 = \|\mathcal{F}_W(f)\|_{L_\alpha^2(\mathbb{R}_+^d)}^2$$
$$= \|\mathcal{F}_W(f)\chi_r\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \|\mathcal{F}_W(f)\overline{\chi}_r\|_{L_\alpha^2(\mathbb{R}_+^d)}^2.$$

By using (1.5) and (1.7), we get

$$\|\mathcal{F}_W(f)\chi_r\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 = \int_{|\xi|<r}|\mathcal{F}_W(f)(\xi)|^2\, d\mu_\alpha(\xi)$$

$$\leq \frac{1}{2^{\alpha+\frac{d-1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)} \frac{r^{2\alpha+d+1}}{(2\alpha+d+1)} \|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)}$$

$$\leq \frac{r^{2\alpha+d+1}}{2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+3}{2}\right)} \|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)}.$$

$$\|\mathcal{F}_W(f)\overline{\chi}_r\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \int_{|\xi|\geq r} |\mathcal{F}_W(f)(\xi)|^2 \, d\mu_\alpha(\xi)$$

$$\leq r^{-2s} \int_{|\xi|\geq r} |\xi|^{2s} |\mathcal{F}_W(f)(\xi)|^2 \, d\mu_\alpha(\xi)$$

$$\leq r^{-2s} \||\xi|^s \mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

It follows

$$\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)} \leq \frac{r^{2\alpha+d+1}}{2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+3}{2}\right)} \|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)} + r^{-2s}\||\xi|^s\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)}.$$

By minimizing the right-hand side of that last inequality over $r > 0$, we obtain

$$\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)} \leq$$

$$\left[\frac{2s+2\alpha+d+1}{2\alpha+d+1}\right]\left(\frac{1}{s2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)}\right)^{\frac{2s}{2s+2\alpha+d+1}} \|f\|_{L^1_\alpha(\mathbb{R}^d_+)}^{\frac{2s}{s+\alpha+(d+1)/2}} \||\xi|^s\mathcal{F}_W(f)\|_{L^2_\alpha(\mathbb{R}^d_+)}^{\frac{2\alpha+d+1}{s+\alpha+(d+1)/2}}.$$

Carlson-type inequality: Carlson [6] showed that

$$\int_0^\infty |f(x)| \, dx \leq \sqrt{\pi}\left(\int_0^\infty |f(x)|^2 \, dx\right)^{\frac{1}{4}}\left(\int_0^\infty x^2|f(x)|^2 \, dx\right)^{\frac{1}{4}}.$$

This inequality has been generalized, discussed and applied in several texts, in particular, Levin [35] showed that

$$\int_0^\infty |f(x)| \, dx \leq C\left(\int_0^\infty x^{p-1-\iota}|f(x)|^p \, dx\right)^s\left(\int_0^\infty x^{q-1+\tau}|f(x)|^q \, dx\right)^t, \qquad (2.15)$$

with $s = \frac{\tau}{p\tau+q\iota}$ and $t = \frac{\iota}{p\tau+q\iota}$

Inequality (2.15) holds for $p = q = 2$ and $\iota = \tau = 1$, while for $\tau = p = 2$ and $\iota = q = 1$, we have

$$\left(\int_0^\infty |f(x)| \, dx\right)^5 \leq C\left(\int_0^\infty |f(x)|^2 \, dx\right)^2\left(\int_0^\infty x^2|f(x)| \, dx\right). \qquad (2.16)$$

Now, we will prove a Carlson-type inequality (2.16) where Lebesgue measure is replaced by μ_α .

Proposition (2.2). (Carlson-type inequality for μ_α)

Let $s > 0$. Then, there exists a constant $C_2(\alpha, s)$ such that for all $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$

$$\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}^{1+\frac{2s}{\alpha+(d+1)/2}} \leq C_2(\alpha,s)\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}^{\frac{2s}{\alpha+(d+1)/2}}\||x|^{2s}f\|_{L^1_\alpha(\mathbb{R}^d_+)},$$

where

$$C_2(\alpha,s)=\left(\frac{1}{2^{\alpha+\frac{d+5}{2}}s^2\Gamma(\alpha+(d+3)/2)}\right)^{\frac{s}{\alpha+(d+1)/2}}\frac{(2s+\alpha+(d+1)/2)^{\frac{2s+\alpha+(d+1)/2}{\alpha+(d+1)/2}}}{\alpha+(d+1)/2}.$$

Proof: We know that

$$\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}=\|f\chi_r\|_{L^1_\alpha(\mathbb{R}^d_+)}+\|f\overline{\chi}_r\|_{L^1_\alpha(\mathbb{R}^d_+)},$$

then

$$\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\le\|f\chi_r\|_{L^1_\alpha(\mathbb{R}^d_+)}+r^{-2s}\||x|^{2s}f\|_{L^1_\alpha(\mathbb{R}^d_+)}.$$

Hence, by following Cauchy-Schwarz inequality and (1.5), we get

$$\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\le\left(\frac{1}{2^{\alpha+\frac{d+1}{2}}\Gamma\left(\alpha+\frac{d+1}{2}\right)}\right)^{\frac{1}{2}}r^{\alpha+(d+1)/2}\|f\|_{L^2_\alpha(\mathbb{R}^d_+)}+r^{-2s}\||x|^{2s}f\|_{L^1_\alpha(\mathbb{R}^d_+)}.$$

By minimizing the right-hand side of that inequality over $r>0$, we get

$$\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\le\left(\frac{1}{s^2 2^{\alpha+\frac{d+5}{2}}\Gamma\left(\alpha+\frac{d+3}{2}\right)}\right)^{\frac{s}{2s+\alpha+(d+1)/2}}\left(\frac{2}{2\alpha+d+1}\right)^{\frac{\alpha+(d+1)/2}{2s+\alpha+(d+1)/2}}(2s+\alpha+(d+1)/2)\|f\|^{\frac{2s}{2s+\alpha+(d+1)/2}}_{L^2_\alpha(\mathbb{R}^d_+)}\||x|^{2s}f\|^{\frac{\alpha+(d+1)/2}{2s+\alpha+(d+1)/2}}_{L^1_\alpha(\mathbb{R}^d_+)}.$$

We conclude from proposition 2.1 and proposition 2.2 the following Theorem:

Theorem2.3. Let $s>0$. Then, there exists a constant $C(\alpha,s)$ such that for all $f\in L^1_\alpha(\mathbb{R}^d_+)\cap L^2_\alpha(\mathbb{R}^d_+)$, then

$$\||x|^{2s}f\|_{L^1_\alpha(\mathbb{R}^d_+)}\||\xi|^s\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_2)}\ge C(\alpha,s)\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\|f\|^2_{L^2_\alpha(\mathbb{R}^d_2)},$$

$C(\alpha,s)=\frac{1}{C_1(\alpha,s)C_2(\alpha,s)}$. where

For $s=1$, we have

$$\||x|^2f\|_{L^1_\alpha(\mathbb{R}^d_+)}\||\xi|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_2)}\ge C(\alpha,1)\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}\|f\|^2_{L^2_\alpha(\mathbb{R}^d_2)}.$$

which is a variation of the Heisenberg uncertainty inequality for Weinstein operator. Inequality (2.17) implies, in particular, that, if f is (ε,α)-concentrated at $x=0$, then $\mathcal{F}_W(f)$ cannot be (ε,α) is (ε,α)-concentrated at $\xi=0$.

A variation of Donoho-Stark's Uncertainty Principle for μ_α

We shall prove a variation of Donoho-Stark's uncertainty principle for Weinstein operator.

Proposition 2.4. Let $S,\Sigma\subset\mathbb{R}^d_+$ and $f\in L^1_\alpha(\mathbb{R}^d_+)\cap L^2_\alpha(\mathbb{R}^d_+)$. If f is (ε_1,α)-timelimited on S and $\hat{f}$ is (ε_2,α)-bandlimited on Σ, then

$$\mu_\alpha(S)\mu_\alpha(\Sigma)\ge(1-\varepsilon_1)^2(1-\varepsilon_2^2).$$

Proof. We know that

$$\|f\|_{L^1_\alpha(S)}\ge\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}-\|f\|_{L^1_\alpha(S^c)}.$$

Hence,

$$\|f\|_{L^1_\alpha(S)}\ge\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}(1-\varepsilon_1).$$

Moreover, by by using Cauchy-Schwarz inequality, we have

$$\|f\|^2_{L^1_\alpha(S)}\le\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}\mu_\alpha(S).$$

Thus,

$$\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}\mu_\alpha(S)\ge\|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)}(1-\varepsilon_1)^2. \quad (2.19)$$

On the other hand, by the orthogonality of the projection operator $g\longrightarrow g\chi_\Sigma$ we have

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma)} = \|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} - \|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma^c)}.$$

Since

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma^c)} \le \varepsilon_2^2\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma)} \ge \|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)}(1-\varepsilon_2^2).$$

By using (1.6), we have

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma)} \ge \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}(1-\varepsilon_2^2).$$

On the other hand, $\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\Sigma)} \le \|\mathcal{F}_W(f)\|^2_{L^\infty_\alpha(\mathbb{R}^d_+)}\mu_\alpha(\Sigma) \le \mu_\alpha(\Sigma)\|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)}$ (2.20)

By using (1.6), we get

$$\mu_\alpha(\Sigma)\|f\|^2_{L^1_\alpha(\mathbb{R}^d_+)} \ge \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}(1-\varepsilon_2^2). \qquad (2.21)$$

Now, multiply (2.19) by (2.21), this completes the proof.

A variation of the local uncertainty inequality

The first such inequalities for Fourier transform involving L^2-norms were obtained by Faris [9], they were subsequently sharpened and generalized by Price [17,18]. In this subsection, we will show the variations of local uncertainty inequality for Weinstein-Gabor transform. Our work is inspired from [10].

Theorem 2.5.(Local uncertainty inequality for $\mathcal{F}_W$)

Let $T, \mathcal{W} \subset \mathbb{R}^d_+$ and $f \in L^1_\alpha(\mathbb{R}^d_+) \cap L^2_\alpha(\mathbb{R}^d_+)$. Then, there exists a constant $C(\alpha, s)$ such that $\|\mathcal{F}_W(f)\|_{L^2_\alpha(\mathcal{W})} \le C(\alpha,s)\sqrt{\mu_\alpha(\mathcal{W})}\, \|f\|^{\frac{2s}{2s+\alpha+(d+1)/2}}_{L^2_\alpha(\mathbb{R}^d_+)} \||x|^{2s}f\|^{\frac{\alpha+(d+1)/2}{2s+\alpha+(d+1)/2}}_{L^1_\alpha(\mathbb{R}^d_+)}$.

Proof . From (2.20), we have

$$\|\mathcal{F}_W(f)\|_{L^2_\alpha(\mathcal{W})} \le \|\mathcal{F}_W(f)\|_{L^\infty_\alpha(\mathbb{R}^d_+)}\sqrt{\mu_\alpha(\mathcal{W})} \le \sqrt{\mu_\alpha(\mathcal{W})}\|f\|_{L^1_\alpha(\mathbb{R}^d_+)}.$$

Applying Proposition 2.2 Thus, the theorem is proved.

Theorem2.6 (Local uncertainty inequality for $\mathcal{G}_g f$)

Let $s > 0$ and $X \subset \mathbb{R}^d_+ \times \mathbb{R}^d_+$ such that $0 < \omega_\alpha(X) < \infty$. Then, for all functions $f, g \in L^2_\alpha(\mathbb{R}^d_+)$

$$\|\mathcal{G}_g f\|^2_{L^2_{\omega_\alpha}(X)} \le C^{-1}_{(\alpha,s)}\omega_\alpha(X)\||x|^s\mathcal{G}_g f\|_{L^2_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)}\, \||\xi|^s\mathcal{G}_g f\|_{L^2_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)}. \qquad (2.22)$$

Proof. By using (1.2.12), we get

$$\|\mathcal{G}_g f\|^2_{L^2_{\omega_\alpha}(X)} \le \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|^2_{L^2_\alpha(\mathbb{R}^d_+)}\omega_\alpha(X).$$

Then, from [3, Theorem 5.2], for all functions $f, g \in L^2_\alpha(\mathbb{R}^d_+)$ there exists a constant $C_{\alpha,s}$ such that

$$C_{\alpha,s}\|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)}\|g\|^2_{L^2_\alpha(R^d_+)} \le \||x|^s\mathcal{G}_g f\|_{L^2_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)}\, \||\xi|^s\mathcal{G}_g f\|_{L^2_{\omega_\alpha}(\mathbb{R}^d_+\times\mathbb{R}^d_+)}.$$

Consequently, we obtain the desired result.

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تباينات على متراجحات مبدأ عدم اليقين تحت تأثير معامل وينشتاين

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الملخص

الهدف في هذا المقال إثبات تباينات جديدة من مبدأ عدم اليقين تحت تأثير معامل وينشتاين . أول هذه النتائج إثبات تباين متراجحة هزينبرغ تحت تأثير محول وينشتاين, حيث بينا أن لكل $s > 0$ يوجد ثابت $C(\alpha, s)$ بحيث انه لكل $f \in L_{\alpha}^{1}(\mathbb{R}_{+}^{d}) \cap L_{\alpha}^{2}(\mathbb{R}_{+}^{d})$ فإن.

$$\||x|^{2s}f\|_{L_{\alpha}^{1}(\mathbb{R}_{+}^{d})}\||\xi|^{s}\mathcal{F}_{W}(f)\|_{L_{\alpha}^{2}(\mathbb{R}_{+}^{d})}^{2} \geq C(\alpha, s)\|f\|_{L_{\alpha}^{1}(\mathbb{R}_{+}^{d})}\|f\|_{L_{\alpha}^{2}(\mathbb{R}_{+}^{d})}^{2}.$$

ثاني هذه النتائج تحقيق تباين عدم اليقين لدنهنو- ستراك تحت تأثير وينشتاين، نفرض ان $S, \Sigma \in \mathbb{R}_{+}^{d}$ and $f \in L_{\alpha}^{1}(\mathbb{R}_{+}^{d}) \cap L_{\alpha}^{2}(\mathbb{R}_{+}^{d})$ إذا f $(\mathcal{E}_{1}, \alpha)$–timelimited على S banlimited $S(\mathcal{E}_{1}, \alpha)$ على Σ فإن $\mu_{\alpha}(S)\mu_{\alpha}(\Sigma) \geq (1-\mathcal{E}_{1})^{2}(1-\mathcal{E}_{2}{}^{2})$. والنتيجة الثالثة برهنة تباين متراجحة مبدأ عدم اليقين المحلية تحت تأثير محولي وينشتاين و وينشتاين-جابور.

الكلمات المفتاحية : معامل ويشتاين, مبدأ عدم اليقين لهيزنبرج.