On strong Semi* − I − Open sets in ideal topological spaces

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Abstract

In this paper, we first introduce the concept of a strong semi*-I-open set which is weaker than the concept of a Semi-I-open set and stronger than the concept of semi*-I-open set. Moreover, we will study its properties and discuss the relationships between this concept and relevant concepts intopological and ideal topological spaces. Finally, by using the new notion, we defined the strong semi*-I-interior and strong semi*-I-closure operators and establish their various properties.

Key words: local functions, ideal topological spaces, strong semi*-I-open sets and strong semi*-I-closed sets.

Introduction and Preliminaries

The notions of semi-open sets[11], semi-I-open sets [17] and semi*-I-open sets [8], and their properties have been introduced and studied in the literature. In the present paper, we have introduce and characterize the notionstrong semi*-I-open sets which is a generalization of the notion of semi-I-open sets. Many of its characterizations and properties have been studied. Moreover, the concept strong semi*-I-interior and strong semi*-I-closure operators were introduced and investigated.

Throughout the present paper, (X,τ) will denote topological spaces on which no separation property is assumed unless explicity stated. In topological space (X,τ), the closure and the interior of any subset A of X will be denoted by $Cl(A)$ and $Int(A)$, respectively. An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$, (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. Let (X,τ) be a topological space and I an ideal on X. An ideal topological space is a topological space (X,τ) with an ideal I on X and denoted by (X,τ,I). For a subset $A \subset X$, $A^*(I,\tau) = \{x \in X \mid U \cap A \notin I$ for each neighborhood U of $x\}$ is called the local function of A with respect to I and τ [15]. It is obvious that $(.)^*$: $(X) \rightarrow (X)$ is a set operator. Throughout this paper, we use A^* instead of $A^*(I,\tau)$. Besides, in [15], authors introduced a new Kuratowski closure operator $Cl^*(.)$ defined by $Cl^*(A) = A \cup A^*$ and obtained a new topology on X which is called $*$-topology. This topology is denoted by $\tau^*(I)$ which is finer than τ. We start with recalling some lemmas and definitions which are necessary for this study in the sequel.

Lemma 1.1[15]. Let (X,τ) be a topological space and I an ideal on X. For every subset A of X, $A^* \subset cl(A)$.

Definition 1.1. A subset A of an ideal topological space (X, τ, I) is called :

(1) semi-open, if $A \subset cl(int(A))$[17];

(2) semi-I-open, if $A \subset cl^*(int(A))$[11];

(3) $semi^*$-I-open, if $A \subset cl(int^*(A))$[8];

(4) α-I-open, if $A \subset int(cl^*(int(A)))$ [11];

(5) β^*-I-open, if $A \subset cl(int^*cl(A))$ [5];

(6) b-I-open, if $A \subset cl^*(int(A)) \cup int(cl^*(A))$ [9];

(7) regular closed, if $A = cl(int(A))$ [17];

(8) β-open, if $A \subset cl(int(cl(A)))$ [1];

(9) $*$- perfect, if $A = A^*$ [14];

(10) almost strong- I-open, if $A \subset cl^*(int(A^*))$ [13];
(11) I-open, if $A \subset int(A^*))$ [2];
(12) β^*-I-open, if $A \subset cl(int^*(cl(A)))$ [5];
(13) I-R-open, if $A = int^*(cl(A))$ [3];
(14) weakly semi-I-open, if $A \subset cl^*(int(cl(A)))$ [10];
(15) t-I-set, if $int(A) = int(cl^*(A))$ [11].
(16) strong β-I-open, if $A \subset cl^*(int(cl^*(A))$ [12]

Definition1.2[4]. Let (X, τ, I) be an ideal topological space, then I is said to be codense if $\tau \cap I = \{\emptyset\}$.

Lemma 1.2[17]. Let (X, τ, I) be an ideal space where I is codense, then the following hold:
(a) $Cl\,(G) = Cl^*\,(G)$ for every semi open set G.
(b) $Int(F) = Int^*\,(F)$ for every semi- closed set F.

Lemma 1.3[6]. For a subset A of an ideal topological space (X, τ, I), the followings hold:
(1) $pI\ Cl(A) = A \cup Cl(Int^*(A))$;
(2) $pI\ Int(A) = A \cap Int\,(Cl^*(A))$;
(3) $sIInt(A) = A \cap Cl^*(Int(A))$;
(4) $sICl(A) = A \cup Int^*\,(Cl(A))$.

Lemma 1.4 [15].For any two subsets, A and B of a space (X, τ, I) the following hold:
(1) If $A \subset B$, then $A^* \subset B^*$.
(2) If $U \in \tau$, then $U \cap A^* \subset (U \cap A)^*$.

Lemma 1.5 [13]. Let A be a subset of an ideal topological space (X, τ, I) and U be an open set, then, $U \cap cl^*(A) \subseteq cl^*(U \cap A)$.

Lemma 1.6 [19]. Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = Cl(A^*) = Cl(A) = Cl^*(A)$.

Strong SEMI*-I-Open Sets

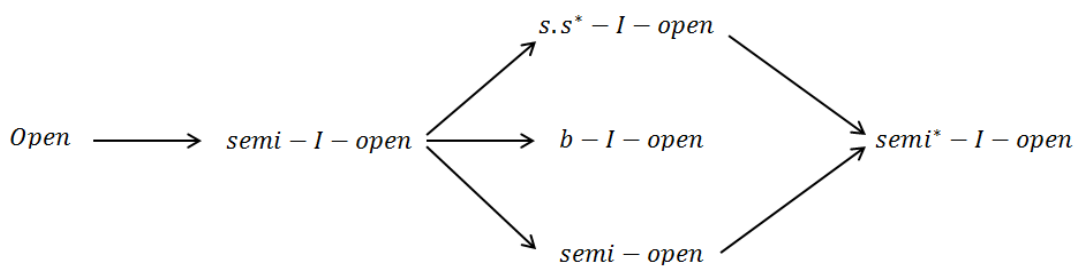
Definition 2 .1. A subset A of an ideal topological space (X, τ, I) is said to be strong semi*-I-open (briefly S.S^*-I-open) if $A \subseteq cl^*(int^*\,(A))$. We denote that all S.$S^*$-I-open by S$S^*$IO (X).

Proposition 2.1. Let (X, τ, I) be an ideal topological space. For A subset A of X the followings hold:
(1) Every semi-I-open set is an $s.s^*$-I-open.
(2) Every $s.s^*$-I-open set is a semi*-I-open.

Proof. The proofs come from Definitions 1.1 and 2.1.

The following diagram holds a subset A of an ideal topological space (X, τ, I) :

((Diagram I))



Remark 2.1 : The converses of these implications in Diagram I are not true, in general, as shown in the following examples:

Example 2.1. let $X = \{a, b, c, d\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$, then $A = \{b, c\}$ is semi*-I-open set, but it is not $s.s^*$-I-open.

Example 2.2. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{c\}, \{a, b, d\}, X\}$, $I = \{\emptyset, \{a\}\}$, then $B = \{b, d\}$is

an $s.s^*$-I-open set, but it is not semi-I-open.
Example2.3. Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{a\},\{b,c\},\{a,b,c\},X\}$, $I = \{\emptyset,\{a\},\{d\},\{a.d\}\}$, then $A = \{a,d\}$ is a semi-open set, but it is not $s.s^*$-I-open.
Example 2.4. let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{c\},\{a,b,d\},X\}$, $I = \{\emptyset,\{a\}\}$, then $A = \{b,c,d\}$ is an $s.s^*$-I-open set, but it is not semi-open.

From examples 2.3 and 2.4, we conclude that the concepts of semi - openness and $s.s^*$-I-openness are independent.
Remark 2.2. The$s.s^*$-I-open sets and b-I- open sets are independent notions.
From example 2.3, let $A = \{b\}$ is b - I - open but it is not a strong semi*-I-open set.
Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset,\{d\},\{a,c\},\{a,c,d\},X\}$, $I = \{\emptyset,\{c\},\{d\},\{c,d\}\}$, $A = \{a,b\}$ is not b-I-open set, but it is an $s.s^*$-I-open set.

Theorem 2.1. Let (X,τ,I) be an ideal topological space and $A \subset X$, then A is an $s.s^*$-I-open set if, and only if, $cl^*(A)= cl^*(int^*(A))$.
Proof : Let A be an $s.s^*$-I-open set in X, then we have $A \subset cl^*(int^*(A))$. We obtain $cl^*(\mathrm{A}) \subset cl^*(cl^*(int^*(\mathrm{A}))) = cl^*(int^*(\mathrm{A}))$. Hence, $cl^*(\mathrm{A}) \subset cl^*(int^*(\mathrm{A}))$.
Conversely, let $cl^*(\mathrm{A}) = cl^*(int^*(\mathrm{A}))$, since $cl^*(A)$ is a closure operator, we have $\mathrm{A} \subset cl^*(A)$, for every subset A of X. By using hypothesis, we have $\mathrm{A} \subset cl^*(int^*(\mathrm{A}))$.
This shows that A is an $s.s^*$-I- open set.

Theorem 2.2. Let(X,τ,I) be an ideal topological space, then B is a strong semi*-I-open set, if and only if, there exists an $s.s^*$-I-open set A such that $A \subset B \subset Cl^*(A)$.
Proof. Let B be an $s.s^*$-I-open, then $\mathrm{B} \subset cl^*(int^*(\mathrm{B}))$ .we put $A = int^*(B)$ be a$*$ −open set . i.e., A is $s.s^*$-I-open. And $A = int^*(B) \subset B \subset cl^*(int^*(B) = cl^*(A)$.
Conversely. If A is an $s.s^*$-I-open set such that $A \subset B \subset cl^*(\mathrm{A})$, then $cl^*(\mathrm{A}) = cl^*(\mathrm{B})$.On the other hand, $\mathrm{A} \subset cl^*(int^*(\mathrm{A}))$ and hence $B \subset cl^*(A) \subset cl^*(cl^*(int^*(\mathrm{A})))= cl^*(int^*(\mathrm{A}) = cl^*(int^*(\mathrm{B})$, which shows that B is an $s.s^*$-I-open set.
Corollary 2.1. Let(X,τ,I) be an ideal topological space, then B is an $s.s^*$-I-open set, if and only if, there exists an open set A such that $A \subset B \subset Cl^*(A)$.
Corollary 2.2. Let(X,τ,I) be an ideal topological space, and If A is an $s.s^*$-I-open set, then $cl^*(A)$ is$s.s^*$-I-open.
Proof. Let A be an $s.s^*$-I-open set, then $\mathrm{A} \subset cl^*(int^*(\mathrm{A}))$. This implies that $cl^*A \subset cl^*(cl^*(int^*(\mathrm{A}))) = cl^*(int^*(\mathrm{A})) \subset cl^*(int^* cl^*(\mathrm{A})))$.

Theorem 2. 3. Let (X,τ,I) be an idealtopological space and $\{U_\alpha : \alpha \in \Delta\}$a family ofsubsets of X, where Δ is an arbitrary index set.
(1) If $U \in \mathrm{SS}^*I\mathrm{O}(\mathrm{X},\tau)$ for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta}\{U_\alpha : \alpha \in \Delta\} \in \mathrm{SS}^*IO(X,\tau)$.
(2) If $\mathrm{A} \in \mathrm{SS}^*I\mathrm{O}(\mathrm{X},\tau)$ and $B \in \tau$, then $A \cap B \in \mathrm{SS}^*I\mathrm{O}(X,\tau)$.
Proof: (1) Since $U_\alpha \in SS^*IO(X,\tau)$, we have $U_\alpha \subset Cl^*(Int^*(U_\alpha))$ for each $\alpha \in \Delta$. we obtain $\bigcup_{\alpha\in\Delta} U_\alpha \subset \bigcup_{\alpha\in\Delta} Cl^*(Int^*(U_\alpha)) \subset \bigcup_{\alpha\in\Delta}\{(int^*(U_\alpha))^* \cup (int^*(U_\alpha))\} \subset (\bigcup_{\alpha\in\Delta}((int^*(U_\alpha))^* \cup int^*(\bigcup_{\alpha\in\Delta} U_\alpha) \subset (int^*(\bigcup_{\alpha\in\Delta} U_\alpha))^* \cup int^*(\bigcup_{\alpha\in\Delta} U_\alpha) = cl^*(int^*(\bigcup_{\alpha\in\Delta} U_\alpha))$. This shows that $\bigcup_{\alpha\in\Delta} U_\alpha \in SS^*IO(X,\tau)$.

(2) Let $A \in SS^*IO(X,\tau)$ and $B \in \tau$. Then $A \subset cl^*(int^*(\mathrm{A}))$ and by using Lemma 1.5 we obtain $A \cap B \subset cl^*(int^*(\mathrm{A}) \cap B) = ((int^*(\mathrm{A}))^* \cup int^*(A)) \cap B = (((int^*(\mathrm{A}))^* \cap B) \cup (int^*(\mathrm{A}) \cap B) \subset (int^*(\mathrm{A}) \cap B)^* \cup int^*(A \cap B) = (int^*(A \cap B))^* \cup int^*(A \cap B) = cl^*\left(int^*(A \cap B)\right)$.
This shows that $A \cap B \in \mathrm{SS}^*I\mathrm{O}(\mathrm{X},\tau)$.
Remark 2. 3. A finite intersection of $s.s^*$-I-open sets need notbe an$s.s^*$-I-open set, in general, as shown by the following example:
Example 2. 5.$Let\ X = \{a,b,c\}$, $\tau = \{\emptyset,\{a\},\{c\},\{a,c\},X\}$, $I = \{\emptyset,\{b\}\}$,

then $A = \{a, b\}$, $B = \{b, c\}$ are $s.s^*$-I-open sets, but $A \cap B = \{b\}$ is not $s.s^*$-I-open set.

Theorem 2.4. Let (X, τ, I) be an ideal topological space where I is condense. Then, for any set $A \subset X$, the followings hold:
(1) A is semi*-I-open set iff it is an $s.s^*$-I-open set.
(2) A is an $s.s^*$-I-open setif it is semi-open.
(3) A is an $s.s^*$-I-open set if it is regular closed.
(4) If A is an $s.s^*$-I-open set, then it is β-open set.
Proof. The proofs come directly from Definitions 1.1, 1.2 and 2.1.

Theorem 2.5. Let(X, τ, I) be an ideal topological space, where A is $*$ - perfect set. Then, the following holds:
(1) If A is almost strong-I-open set, then it is ans.s*-I-open set.
(2) If A is I-open set, then it is an s.s*-I-open set.
Proof.
(1) Let A is almost strong-I-open set, then $A \subset cl^*(int(A^*)) = cl^*(int(A)) \subset cl^*(int^*(A))$. This implies A is $s.s^*$-I-open.
(2) suppose A is I-open set,then $A \subset int(A^*) = int(A) \subset cl^*(int^*(A))$.
This shows that A is s.s*-I-open.

Theorem 2.6. Let(X, τ, I) be an ideal topological space and $A \subset X$, If A is I- R-closed set, then it iss.s*-I-open.
Proof. Let A be I-R-closed set, then $A = cl^*(int(A) \subset cl^*(int^*(A))$. This implies A is s.s*-I-open.
Remark 2.4. The reverse of the above Theorem is not true, in general, as shown in the following example: Let $X = \{a, b, c, d\}$,$\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$, $I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$. Let $A = \{a\}$ *is* ans.s*-I-open set but it is not I-R-closed set.

Theorem2.7. Let(X, τ, I) be an ideal topological space where I is condense. If A is $*$ - perfect set, then the following are equivalent :
(1) A is s.s*-I-open.
(2) A is almost strong-I-open .
(3) A is β*-I-open.
(4) A is semi*-I-open.
Proof.
(1) ⟹ (2)$A \subset cl^*(int^*(A)) = cl^*(int(A^*))$.
(2) ⟹ (3) A $\subset cl^*(int(A^*)) = cl(int(A)) \subset cl\left(int^*(cl(A))\right)$.
(3) ⟹(4) $A \subset cl(int^*(cl(A)) = cl(int^*(A))$.
(4) ⟹(1) $A \subset cl(int^*(A)) = cl^*(int^*(A))$.

Theorem 2.8. Let(X, τ, I) be an ideal topological space and $A \subset X$ be a pre-open set. If A is either semiclosed or I-locally closed, then A is s.s*-I-open.
Proof.
Suppose$A$is I-locally closed. A is I-locally closed implies that $A = U \cap A^*$for some open set U. A is pre-open implies that $A \subset int(cl(A))$. Now $A = U \cap A^* \subset U \cap (int(cl(A)))^* \subset U \cap cl^*(int(cl(A)) \subset U \cap cl^*(int(cl(U \cap A^*))) \subset cl^*(U \cap int(cl(U \cap A^*))) = cl^*(int(U \cap cl(U \cap A^*)) \subset cl^*(int(U \cap cl(U) \cap cl(A^*))) = cl^*(int((U \cap A^*)) = cl^*(int(A)) \subset cl^*(int^*(A))$. Hence A is an s.s*-I-open.
Suppose A is semiclosed ,then $int(cl(A)) = int(A)$. Since A is pre-open, then $A \subset int(cl(A)) =$

$int(A) \subset cl^*(int^*(A))$.Hence it is s.s*-I-open.

Remark 2.5 : The reverse of the above Theorem is not true, in general, as shown in the following example: Let $X = \{a, b, c, \}, \tau = \{\emptyset, \{c\}, \{a, c\}, X\}$ and $I = \{\emptyset, \{b\}\}$.Tack$A = \{b, c\}$. Note that A is pre-open and s.s*-I-open. Moreover, A is neither I-locally closed nor semiclosed.
In general, the concepts of strong β-I-open sets and s.s*-I-open sets are independent notions, as shown in the following examples:
From example 2.3, let $A = \{b, d\}$, then it is strong β-I-open, but is not strong semi*-I-open.
Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}$, $I = \{\emptyset, \{a\}, \{d\}, \{a, d\}\}$.
Then $A = \{b\}$ is strong semi*-I-open, but it is not strong β-I-open.
The next theorem tells us under which condition that the concepts of the strong β-I-open setsandthe strong semi*-I-open sets are equivalent.

Theorem 2.9. Let(X, τ, I) be an ideal topological space, If A is $*$-closed set, where I is condense. Then A is s.s*-I-openif and only if it is strong β-I-open.
Proof.Let A be ans.s*-I-open. Since A is $*$-closed, then $A \subset cl^*(int^*(A)) = cl^*(int(cl^*(A)))$. Hence A is a strong β-I-open set.
Conversely, let Abea strong β-I-open set, then$A \subset cl^*(int(cl^*(A))) = cl^*(int^*(A))$. Hence A is s.s*-I-open.

Theorem 2.10. In the ideal topological space(X, τ, I), the union of a semi-open set with an s.s*-I-open set is a semi*- I-open set.
Proof.
Let A be ans.s*-I-open set and B is a semi-open set, then
$$A \cup B \subset cl^*(int^*(A)) \cup cl(int(B))$$
$$\subset cl(int^*(A)) \cup cl(int^*(B))$$
$$= cl(int^*(A) \cup int^*(B))$$
$$\subset cl(int^*(A \cup B)).$$
Hence $A \cup B$ is asemi*-I-open set.

Strong Semi*-I-Closed Sets
Definition 3.1. A subset A of a space (X, τ, I) is said to be a strong semi*-I-closed (briefly, s.s*-I-closed) if its complement is s.s*-I-open.

Theorem 3.1.A subset A of a space (X, τ, I) is said to be s.s*-I-closed, if and only if, $int^*(cl^*(A)) \subset A$.
Proof: Let A be an s.s*-I-closedset of (X, τ, I), then $(X - A)$ is a s.s*-I-open and hence $(X - A) \subset cl^*(int^*(X - A)) = X - int^*(cl^*(A))$. Therefore, we obtain $int^*(cl^*(A)) \subset A$.
Conversely, let $int^*(cl^*(A)) \subset A$, then $(X - A) \subset cl^*(int^*(X - A))$ and hence $(X - A)$ is s.s*-I-open. Therefore A is s.s*-I-closed.

Theorem 3.2.A subset A of a space (X, τ, I) is said to be s.s*-I-closed if, and only if, there exists an s.s*-I-closed set B such that $Int^*(B) \subset A \subset B$.
Proof. Let A is a s.s*-I-closed set of a space (X, τ, I), then $Int^*(cl^*(A)) \subset A$. We put $B = cl^*(A)$ be a $*$-closed set. i.e, B is s.s*-I-closed. And $Int^*(B) = Int^*(cl^*(A) \subset A \subset cl^*(A) = B$.
Conversely, If B is a s.s*-I-closed set such that $Int^*(B) \subset A \subset B$, then $Int^*(A) = Int^*(B)$. On the other hand, $Int^*(cl^*(B) \subset B$ and hence $A \supset Int^*(B) \supset Int^*(Int^*(cl^*(B)) = Int^*(cl^*(B) = Int^*(cl^*(A))$, thus $A \supset Int^*(cl^*(A))$. Hence A is s.s*-I-closed.
Corollary 3.1. A subset A of a space (X, τ, I) is said to be s.s*-I-closed, if and only if, there

exists a $*$-closed set, B such that $Int^*(B) \subset A \subset B$.
Remark 3.1. The union of s.s*-I-closed sets need not be ans.s*-I-closed set. This can be shown by the following examples:
Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}, I = \{\emptyset, \{b\}\}$, then $A = \{a\}$ and $B = \{c\}$ ares.s*-I-closed sets but $A \cup B = \{a, c\}$ is not s.s*-I-closed set.
Remark 3.2. For a subset A of a space (X, τ, I), we have $X - Int(Cl^*(A)) \neq Cl^*(Int^*(X - A))$. as shown by the following example:
Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$, $I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$, If we take $A = \{a, b, c\}$, then $X - Int(Cl^*(A)) = \{b, d\}$, and $Cl^*(Int^*(X - A)) = \{d\}$.

Theorem 3.3. Let Abe a subset of a space (X, τ, I) such that $X - int(Cl^*(A)) = Cl^*(int^*(X - A))$. Then, A is ans.s*-I-closedset, if and only if, $int(Cl^*(A)) \subset A$.
Proof : This is an immediate consequence of Theorem 3.1.

Theorem 3.4. Let (X, τ, I)be an ideal topological space. If I is condense, then A is ans.s*-I-closed set, if and only if, $int(Cl^*(A)) \subset A$.
Proof. Let A beans.s*-I-closed set of X, then $A \supset int^*(Cl^*(A)) = int(Cl^*(A))$.
Conversely. Let A be any subset of X, such that $A \supset int(Cl^*(A))$. This implies that $A \supset int^*(Cl^*(A))$, i.e., A is s.s*-I-closed.

Theorem 3.5. Let (X, τ, I) be an ideal topological space and $A \subset X$, the following properties hold:
(1) If A is ans.s*-I-openset, then $\mathrm{PIcl(A)} = \mathrm{cl}(\mathrm{int}^*(\mathrm{A}))$.
(2) If A is ans.s*-I-closedset, then $\mathrm{PIint(A)} = \mathrm{int}(\mathrm{cl}^*(\mathrm{A}))$.
Proof.
(1) Let A be ans.s*-I-openset in X, then we have $\mathrm{A} \subset cl^*(int^*(A)) \subset cl(int^*(A))$**.** Thus we have $PIcl(A) = cl(int^*(A))$.
(2)Let A be a s.s*-I-closedset in X, then we have $A \supset int^*(cl^*(A)) \supset int(cl^*(A))$. Hence $PIint(A) = int(cl^*(A))$.
Remark 3.3. The reverse of the above theorem is not true, in general, as shown in the following examples:
From example 2.3, we show $PIcl(A) = cl(int^*(A))$ for the subset $A = \{a, d\}$, but it is not s.s*-I-open.
From Example 2.1, we show $PIint(A) = int(cl^*(A))$ for the subset $A = \{a, c\}$, but it is not s.s*-I-closed.
Lemma 3.1. The classes of t-I-set and s.s*-I-closed sets are not coincide.
Remark 3.4. From Example 2.1, we show the sets $\{a\}, \{a, c\}, and \{a, d\}$ are t -I- sets, but are not s.s*-I-closed.
Theorem 3.6. Let (X, τ, I) be an ideal topological space and $B \subset X$, If B is a semi -I-closed set, then B is both a s.s*-I-closedand t-I-set .
Proof. we obtain that every semi-I-closed set is strong semi*-I-closed by taking the complement of Diagram I.

Theorem 3.7. Let (X, τ, I) be an ideal topological space$A, B \in X$, then
(1) If A is ans.s*-I-closed set and B is a semi-closed set, then A ∩ B is a semi*-I-closed set.
(2) If A is ans.s*-I-closed set and B is t-I- set , then A ∩ B is a semi*-I-closed set.
Proof.
(1) Proved similarly (2) Theorem 2.10.

(2) Let A is strong semi*-I-closed, then $A \supset int^*(cl^*(A))$, and B is t-I-set

Then, $int(B) = int(cl^*(B))$.

Now, $A \cap B \supset int^*\big(cl^*(A)\big) \cap B$

$\supset int^*\big(cl^*(A)\big) \cap int(B)$

$\supset int\big(cl^*(A)\big) \cap int(cl^*(B))$

$= int(cl^*(A) \cap cl^*(B))$

$\supset int\big(cl^*(A \cap B)\big)$.

Hence $A \cap B$ is semi*-I-closed.

Theorem 3.8. Let (X, τ, I) be an ideal topological space and $A, B \in X$, then, if A is ans.s*-I-closed set and B is a weakly semi-I-closed set, then $A \cap B$ is a β^*-I-closed set.

Proof. Let A is s.s*-I-closed, then $A \supset int^*(cl^*(A))$ and since B is weakly semi- I-closed, then $B \supset int^*(cl(int(B)))$.

Now; $A \cap B \supset int^*\big(cl^*(A)\big) \cap int^*(cl\big(int(B)\big))$

$\supset int(cl^*(int(A))) \cap int(cl^*\big(int(B)\big)$

$= int\,(cl^*\big(int(A)\big) \cap cl^*\big(int(B)\big))$

$\supset int\,(cl^*\big(int\,(A \cap B)\big))$.

Hence $A \cap B$ is β^*-I-closed .

Corollary 3.2. Let(X, τ, I) be an ideal topological space, If A is $*$-open set, where I is codence. Then A is s.s*-I-closed, if and only if, it is strong β-I-closed.

Strong Semi*-I-Interior and Strong Semi*-I- Closure Operators

Definition 4.1. The strong semi*-I-interior of a subset A of an ideal topological space (X, τ, I), denoted by $s.s^* - I - Int(A)$, is defined by the union of all s.s*-I-opensets of X contained in A, i.e.,

$s.s^* - I - Int(A) = \{\cup B : B \subset A$, B is ans.s*-I-openset$\}$

Theorem4.1. For a subset A of an ideal topological space (X, τ, I), $s.s^* - I - Int(A) = A \cap Cl^*(Int^*(A))$.

Proof. If A is any subset of X, then $A \cap Cl^*(Int^*(A)) \subset Cl^*(Int^*(A)) =$

$Cl^*(Int^*(Int^*(A))) = Cl^*(Int^*(A \cap Int^*(A))) \subset Cl^*(Int^*(A \cap Cl^*(Int^*(A))))$.

Thus $A \cap Cl^*(Int^*(A)) \subset Cl^*(Int^*(A \cap Cl^*(Int^*(A))))$ and so $A \cap Cl^*(Int^*(A))$ is ans.s*-I-open set contained in A. Therefore, $A \cap Cl^*(Int^*(A)) \subset s.s^* - I - Int(A)$.

Conversely, since $s.s^* - I - Int(A)$ is s.s*-I-open, then $s.s^* - I - Int(A) \subset Cl^*(Int^*(s.s^* - I - Int(A)) \subset Cl^*(Int^*(A))$. So, $s.s^* - I - Int(A) \subset A \cap Cl^*(Int^*(A))$. Therefore, $s.s^* - I - Int(A) = A \cap Cl^*(Int^*(A))$.

Corollary 4.1 : Let (X, τ, I) be an ideal topological space and $A \in X$, then :

A is s.s*-I-open if and only if $s.s^* - I - Int(A) = A$.

Proof. Let A is s.s*-I-open, then $A \subset Cl^*(Int^*(A))$. Hence $s.s^* - I - Int(A) = A \cap Cl^*(Int^*(A)) = A$.

Conversely, since $s.s^* - I - Int(A) = A \cap Cl^*(Int^*(A))$ and by hypothesis $s.s^* - I - Int(A) = A$, we get $A \subset Cl^*(Int^*(A))$. This implies A is s.s*-I-open.

Definition 4.2. The strong semi* -I-closure of a subset A of an ideal topological space (X, τ, I), denoted by $s.s^*ICl(A)$, is defined by the intersection of all s.s*-I-closedsets of X containing A.

$s.s^*ICl(A) = \{\cap B : B \supset A$, $B$is ans.s*-I-closedset$\}$

Theorem 4.2. For a subset A of an ideal topological space (X, τ, I),

$$s.s^* - I - Cl(A) = A \cup Int^*\big(Cl^*(A)\big).$$

Proof. Since $A \cup Int^*\big(Cl^*(A)\big) \supset Int^*\big(Cl^*(A)\big) = Int^*\big(Cl^*(Cl^*(A))\big) =$

$Int^*\big(Cl^*(A \cup Cl^*(A))\big) \supset Int^*\big(Cl^*(A \cup Int^*(Cl^*(A)))\big)$.

Thus $A \cup Int^*(Cl^*(A)) \supset Int^*(Cl^*(A \cup Int^*(Cl^*(A))))$. Hence $A \cup Int^*(Cl^*(A))$ is ans.s*-I-closed set containing A and so $s.s^*-I-Cl(A) \subset A \cup Int^*(Cl^*(A))$.

Conversely, since $s.s^*-I-Cl(A)$ is s.s*-I-closed, we have $s.s^*-I-Cl(A) \supset Int^*(Cl^*(s.s^*-I-Cl(A)) \supset Int^*(Cl^*(A))$. Therefore, $s.s^*-I-Cl(A) \supset A \cup Int^*(Cl^*(A))$. Hence $s.s^*-I-Cl(A) = A \cup Int^*(Cl^*(A))$.

Corollary 4.2. Let (X, τ, I) be an ideal topological space and $A \in X$, then :
A is s.s*-I-closed, if and only if, $s.s^*-I-Cl(A) = A$.

Proof. Let$A$ is s.s*-I-closed, then $A \supset Int^*(Cl^*(A))$. Hence $s.s^*-I-Cl(A) = A \cup Int^*(Cl^*(A)) = A$.

Conversely, since $s.s^*-I-Cl(A) = A \cup Int^*(Cl^*(A))$ and by hypothesis $s.s^*-I-Cl(A) = A$, we get $A \supset Int^*(Cl^*(A))$. This implies A is s.s*-I-closed.

Remark 4.1. Let (X, τ, I) be an ideal topological space and $A \subset X$. Then,

$$s-I-Int(A) \subset s.s^*-I-Int(A) \subset A \subset s.s^*-I-Cl(A) \subset s-I-Cl(A)$$

Theorem 4.3. Let (X, τ, I) be an ideal topological space and $A \subset X$**,** then the following properties are hold:

(1) If A is pre-I-open set, then $s.s^*-I-Cl(A) = int^*(cl^*(A))$.

(2) If A is pre-I-closed set, then $s.s^*-I-Int(A) = cl^*(int^*(A))$.

Proof.(1) suppose A is a pre-I-open set in X, then we have $A \subset int(Cl^*(A)) \subset int^*(cl^*(A))$.This implies that $s.s^*ICl(A) = A \cup Int^*(Cl^*(A)) = Int^*(Cl^*(A))$.

(2) Let A be a pre-I-closed set in X, it follows that $cl^*(int^*(A)) \subset cl(int^*(A)) \subset A$. This implies that $s.s^*IInt(A) = A \cap cl^*(int^*(A)) = cl^*(int^*(A))$.

Remark 4.2. The reverse implications of the above Theorem are not true, in general, as shown in the following example : Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}, I = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}$, then $s.s^*-I-Cl(A) = int^*(cl^*(A))$, for the subset $A = \{a, b, c\}$, but A is not pre-I-open set. Moreover, $s.s^*-I-Int(A) = cl^*(int^*(A))$, for the subset $B = \{c, d\}$, but B is not pre-I-closed.

Theorem 4.4. Let (X, τ, I) be an ideal topological space and $A \subset X$**,**the following properties hold:

(1) If A is I-R-closed set, then $s.s^*-I-Int(A) = cl^*(int(A))$.

(2) If A is I-R-open set, then $s.s^*-I-Cl(A) = int^*(cl(A))$.

Proof. (1) Let A be I-R-closed set in X, it follows that $A = cl^*(int(A))$. This implies that
$s.s^*-I-Int(A) = A \cap cl^*(int^*(A)) = cl^*(int(A)) \cap cl^*(int^*(A)) = cl^*(int(A))$.

(2) : suppose A is I-R-open set in X, then we have $A = Int^*(cl(A))$. This implies that
$s.s^*-I-Cl(A) = A \cup Int^*(Cl^*(A)) = Int^*(cl(A)) \cup Int^*(cl(A)) = Int^*(cl(A))$.

Remark 4.3. The reverse implications of the above Theorem are not true, in general, as shown in the following example: Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}$. Then, $s.s^*-I-Int(A) = cl^*(int(A))$ for the subset $A = \{a\}$ but A is not $I-R$- closed set . Moreover, $s.s^*-I-cl(A) = int^*(cl(A))$, for the subset $B = \{b, d\}$, but B is not I-R-open set.

Theorem 4.5. For a subset A of an ideal topological space(X, τ, I), If I is condense, the following properties hold :

(1) $S\beta-I-cl(A) \subset s.s^*-I-Cl(A)$,

(2) $s.s^*-I-Int(A) \subset wsIint(A)$,

(3) $wsIcl(A) \subset s.s^*-I-Cl(A)$.

Proof. (1) by Lemma 3.14[8] $S\beta-I-cl(A) = A \cup Int^*(cl(Int^*(A))) = A \cup int^*(cl^*(int^*(A))) \subset A \cup Int^*(Cl^*(A))$.

(2) From Theorem 3.1[18] $wsIint(A) = A \cap cl^*(int(cl(A)))$

Since $s.s^*IInt(A) = A \cap cl^*(int^*(A)) = A \cap cl^*(int(A)) \subset cl^*(int(cl(A)))$.

Hence $s.s^*IInt(A) \subset wsIint(A)$.

(3) From Theorem 3.1[18]

$$wsIcl(A) = A \cup Int^*cl(int(A)))$$
$$= A \cup int^*cl^*\big(int^*(A)\big)$$
$$\subset A \cup Int^*(cl^*(A))).$$

Hence $wsIcl(A) \subset s.s^*ICl(A)$.

Theorem 4.6. For a subset A of an ideal topological space (X, τ, I), the following properties hold :

(1) $int^*(s.s^* - I - Cl(A)) = int^*(cl^*(A))$,

(2) $cl^*(s.s^* - I - Int(A)) = cl^*\big(int^*(A)\big)$.

Proof:(1) We have :

$$int^*\big(s.s^* - I - Cl(A)\big) = int^*(A \cup int^*\big(cl^*(A)\big))$$
$$\supset int^*(A) \cup int^*(int^*\big(cl^*(A)\big))$$
$$= int^*(A) \cup int^*\big(cl^*(A)\big) = int^*\big(cl^*(A)\big).$$

Conversely,

$$int^*\big(s.s^* - I - Cl(A)\big) = int^*\Big(A \cup int^*\big(cl^*(A)\big)\Big)$$
$$\subset int^*(cl^*\big(A\big) \cup int^*\Big(cl^*(A)\Big)$$
$$= int^*(cl^*\big(A\big)).$$

This implies $int^*(s.s^* - I - Cl(A)) = int^*(cl^*(A))$.

(2) we have$cl^*(s.s^* - I - Int(A)) = cl^*(A \cap cl^*int^*(A)))$

$$\subset cl^*(cl^*(A) \cap cl^*(int^*\big(A\big)))$$
$$\subset cl^*(cl^*int^*\big(A\big))$$
$$= \mathrm{cl}^*(\mathrm{int}^*\big(\mathrm{A}\big)).$$

Conversely ,

$$cl^*(s.s^* - I - Int(A)) = cl^*(A \cap cl^*int^*(A))$$
$$\supset cl^*(int^*(A) \cap cl^*(int^*\big(A\big)))$$
$$= cl^*(int^*\big(A\big)).$$

This implies $cl^*(s.s^* - I - Int(A)) = cl^*\big(int^*(A)\big)$.

Theorem 4.7. For a subset A of an ideal topological space (X, τ, I), the following properties hold:

(1) $s.s^* - I - Cl(s.s^*IInt(A)) = s.s^* - I - Int(A) \cup int^*(cl^*\big(int^*(A)\big)$.

(2) $s.s^* - I - Int(s.s^* - I - Cl(A) = s.s^* - I - Cl(A) \cap cl^*(int^*(cl^*(A))$.

Proof. The proof comes directly By Theorem 4.4.

Conclusion

A definition of a strong semi*-I-open set was provided. In this paper, it has been shown that the concept of a strong semi*-I-open set is weaker than the concept of a Semi-I-open set and stronger than the concept of semi*-I-open set. A discussion illustrating the relationships between strong semi*-I-open sets and some known concepts in ideal topological spaces. Moreover, many counter examples were investigated. Finally, the strong semi*-I-interior and strong semi*-I-closure operators were introduced and their various properties were established.

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حول المجموعات شبه المفتوحة المثالية القوية من النوع * في الفضاءات التوبولوجية المثالية

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الملخص

في بداية هذا البحث عرفنا المجموعات شبه المفتوحة المثالية القوية من النوع* والتي هي أضعف من المجموعات شبه المفتوحة المثالية وأقوى من المجموعات شبه المفتوحة المثالية من النوع*، بالإضافة ألى ذلك قمنا بدراسة خواص هذا المفهوم وعلاقتة بالمفاهيم المعروفة سابقا في الفضاءات التوبولوجية والتوبولوجية المثالية, وأخيرا قدمنا مفاهيم الداخلية والخارجية شبه المفتوحة المثالية القوية من النوع* ودرسنا العديد من خواصهما.

الكلمات المفتاحية: الرواسم المحلية, الفضاءات التوبولوجية المحلية, المجموعات شبه المفتوحة المثالية القوية من النوع * و المجموعات شبه المغلقة المثالية القوية من النوع *.