On generalized for curvature Tensor P^i_{jkh} of second order in Finsler space

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Abstract

In this present paper, we introduced a Finsler space F_n which Cartan's second curvature tensor P^i_{jkh} satisfies the generalized birecurrence property with respect to Berwald's connection parameters G^i_{kh} which given by the condition

$\mathcal{B}_n\mathcal{B}_m P^i_{jkh} = a_{mn} P^i_{jkh} + b_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2\,\mu_m \mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})y^r \quad , P^i_{jkh} \neq 0$,

where $\mathcal{B}_n\mathcal{B}_m$ is Berwald' scovariant differential of second order with respect to x^m and x^n, successively, μ_m is non-zero covariant vector field, a_{mn} and b_{mn} are non-zero recurrence vectors field of second order, such space is called as *a generalized* $\mathcal{B}P$*-Birecurrent space* and denoted it briefly by $G\mathcal{B}P\text{-}BIRF_n$. We have obtained Berwald' scovariant derivative of second order for the h(v)-torsion tensor P^i_{kh}, P-Ricci tensor P_{jk} and the curvature vector P_k for Cartan's second curvature tensor P^i_{jkh}. Also, we find some theorems of the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of the v(hv)-torsion tensor P^i_{kh} in this space. We also obtained the necessary and sufficient condition for Cartan's fourth curvature tensor P^i_{jkh} to be generalized birecurrent and the necessary and sufficient condition of Berwald's covariant derivative of second order for the h(v)-torsion tensor H^i_{kh}, the R-Ricci tensor R_{jk} andthe deviation tensor H^i_h , also the necessary and sufficient condition for the curvature vector R_j and the deviation tensor H^i_j to be non-vanishing in this space.

Keywords: Generalized $\mathcal{B}P$-birecurrent space, Berwald's covariant derivative of second order, Cartan's second curvature tensor P^i_{jkh}.

1. Introduction

The generalized recurrent space is characterized by different curvature tensors and used the sense of Berwald discussed byZ. Ahsan and M. Ali[1], S.M.S. Baleedi [2] and P.N.Pandey, S. Saxena and A. Goswani [5], W^i_{jkh} generalizedbirecurrent Finsler space and the special birecurrent of first and second kind studied by F.Y.A. Qasem and A.A.M. Saleem[7] and others. The generalized birecurrent space characterized by different curvature tensors and used the sense of Berwald studied by W.H.A. Hadi [3], F.Y.A. Qasem [6],F.Y.A. Qasem and W.H.A. Hadi ([8], [9], [10]).

An n-dimensional Finsler space, equipped with the metric function F satisfies the requisite conditions [11]. Let consider the components of the corresponding metric tensor g_{ij}, Cartan's connection parameters Γ^{*i}_{jk} and Berwald's connection parameters G^i_{jk} *. These are symmetric in their lower indices.

The vectors y_i and y^i satisfy the following relations [11]

(1.1) a) $y_i = g_{ij}\, y^j$ and b) $y_i\, y^i = F^2$.

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by [11]

(1.2) $$g_{ij}g^{jk} = \delta^k_i = \begin{cases} 1 \;, & if \quad i = k \;, \\ 0 \;, & if \quad i \neq k \;. \end{cases}$$

The tensor C_{ijk} defined by

*The indices $i\,, j\,, k\,, ...$ assume positive integral values from 1 to n .

(1.3) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k F^2$

is known as *(h) hv - torsion tensor* [11].

The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by[11]

(1.4) a) $C^i_{jk}y^j = 0 = C^i_{kj}y^j$ and b) $C_{ijk}y^j = 0$.

Berwald's covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor filed T^i_j with respect to x^k is given by[11]

(1.5) $\mathcal{B}_k T^i_j := \partial_k T^i_j - (\dot{\partial}_r T^i_j)\, G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}$.

Berwald's covariant derivative of the metric function, Fig. (1.1) and the vector y^i vanish identically [11], i.e.

(1.6) a) $\mathcal{B}_k F = 0$ and b) $\mathcal{B}_k y^i = 0$.

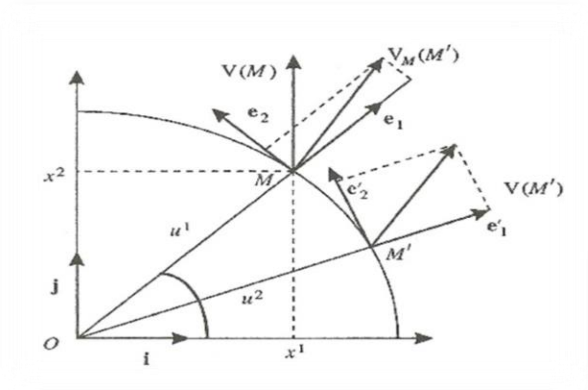

Fig. (1.1) Covariant Derivative Physics Forums

But Berwald's covariant derivative of the metric tensor g_{ij} does not vanish, i.e. $\mathcal{B}_k g_{ij} \neq 0$ and given by [11]

(1.7) $\mathcal{B}_k g_{ij} = -2\, C_{ijk|h} y^h = -2\, y^h \mathcal{B}_h C_{ijk}$.

The hv-curvature tensor (Cartan's second curvature tensor)is positively homogeneous of degree zero in the directional argument and is defined by [11]

$$P^i_{jkh} = \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jm} P^m_{kh} - C^i_{jh|k} \quad ,$$

or equivalent by

$$P^i_{jkh} = \dot{\partial}_h\, \Gamma^{*i}_{jk} + C^i_{jr}\, C^r_{kh|s}\, y^s - C^i_{jh|k} \quad ,$$

or

$$P^i_{jkh} = C^i_{kh|j} - g^{ir}\, C_{jkh|r} + C^r_{jk} P^i_{rh} - P^r_{jh} C^i_{rk} \quad .$$

The hv-curvature tensor P^i_{jkh} , itsassociate curvature tensor P_{ijkh} , the v(hv)–torsion tensor P^i_{kh} and the P-Ricci tensor P_{jk} satisfy [11]

(1.8) a) $P^i_{jkh}\, y^j = P^i_{kh}$, b) $g_{ir} P^r_{jkh} = P_{ijkh}$,

c) $g_{rp} P^r_{kh} = P_{kph}$, d) $P^i_{jki} = P_{jk}$ and e) $P^i_{ki} = P_k$.

Cartan's third curvature tensor R^i_{jkh} ,the R-Ricci tensor R_{jk} in sense of Cartan. The h(v)-torsion tensor H^i_{kh} , the deviation tensor H^i_k , the curvature vector H_k and the scalar curvature H in sense of Berwald, is given by [11]

(1.9) a) $R^i_{jkh} = \Gamma^{*i}_{hjk} + (\Gamma^{*i}_{ljk}) G^l_h + C^i_{jm}(G^m_{kh} - G^m_{kl} G^l_h) + \Gamma^{*i}_{mk}\, \Gamma^{*m}_{jh} - k/h\, **$,

b) $R^i_{jkh} y^j = H^i_{kh} = K^i_{jkh} y^j$, c) $R_{jk} y^j = H_k$,

d) $R_{jk} y^k = R_j$, e) $R^i_{jki} = R_{jk}$,

f) $H_k\, y^k = (n-1)H$ and g) $H^i_{jk}\, y^j = H^i_k$.

** $-k/h$ means the subtraction from the former term by interchanging the indices k and h .

2. Generalized $\mathcal{B}P$-Birecurrent Space

A Finsler space F_n whose Cartan's second curvature tensor P^i_{jkh} satisfies the condition[4]

(2.1) $\mathcal{B}_m P^i_{jkh} = \lambda_m P^i_{jkh} + \mu_m(\delta^i_h g_{jk} - \delta^i_k g_{jh})$, $P^i_{jkh} \neq 0$,

where $\mathcal{B}_m$ is covariant derivative of first order with respect to x^m, the quantities λ_m and μ_m are non-null covariant vectors field. He called such space as *a generalized $\mathcal{B}P$-recurrent space*, and denoted it briefly by $G\mathcal{B}P$-RF_n .

By taking the covariant derivative (in sense of Berwald) for (2.1), with respect to x^n, we get [12]

(2.2) $\mathcal{B}_n\mathcal{B}_m P^i_{jkh} = a_{mn} P^i_{jkh} + b_{mn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) - 2\,\mu_m \mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}) y^r$,

where $\mathcal{B}_n\mathcal{B}_m$ is covariant derivative of second order (Berwald's covariant differential operator) with respect to x^m and x^n, respectively, the quantities a_{mn} and b_{mn} are non-null covariant vectors field. She called such space as *a generalized $\mathcal{B}P$-birecurrent space,* and denoted it briefly by $G\mathcal{B}P$-$BIRF_n$.

Result 2.1. Every generalized $\mathcal{B}P$-recurrent space is generalized $\mathcal{B}P$-birecurrent space.

Transfecting the condition (2.2) by y^j, using (1.6b), (1.8a), (1.1a) and (1.4c), we get

(2.3) $\mathcal{B}_n\mathcal{B}_m P^i_{kh} = a_{mn} P^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h)$.

Contracting the indices I and h in (2.2) and (2.3), using(1.8d), (1.8e) and in view of (1.2), we get

(2.4) $\mathcal{B}_n\mathcal{B}_m P_{jk} = a_{mn} P_{jk} + b_{mn}(n-1) g_{jk} - 2\,\mu_m \mathcal{B}_r (n-1)\, C_{jkn} y^r$

and

(2.5) $\mathcal{B}_n\mathcal{B}_m P_k = a_{mn} P_k + b_{mn}(n-1) y_k$.

Therefore, we have

Theorem 2.1. *In $G\mathcal{B}P$-$BIRF_n$, the v(hv)-torsion tensor P^i_{kh} , the P-Ricci tensor P_{jk} and the curvature vector P_k (for Cartan's second curvature tensor P^i_{jkh}) are given by (2.3),(2.4) and (2.5), respectively.*

Trausvecting (2.2) and(2.3) by g_{ir}, using (1.8b), (1.8c), (1.7) and in view of (1.2), we get

(2.6) $$\mathcal{B}_n\mathcal{B}_m P_{jkrh} = a_{mn} P_{jkrh} + b_{mn}(g_{rh} g_{jk} - g_{rk} g_{jh}) - 2\,\mu_m \mathcal{B}_r(g_{rh} C_{jkn} - g_{rk} C_{jhn}) y^r - 2\left(\mathcal{B}_n C_{irm|s} y^s\right) P^i_{jkh} - 2\, C_{irm|s} y^s (\mathcal{B}_n P^i_{jkh}) - 2\left(C_{irn|p} y^p\right) \mathcal{B}_m P^i_{jkh}$$

and

(2.7) $$\mathcal{B}_n\mathcal{B}_m P_{krh} = a_{mn} P_{krh} + b_{mn}(g_{hr} y_k - g_{kr} y_h) - 2\left(\mathcal{B}_n C_{irm|s} y^s\right) P^i_{kh} - 2\, C_{irm|s} y^s (\mathcal{B}_n P^i_{kh}) - 2\left(C_{irn|p} y^p\right) \mathcal{B}_m P^i_{kh}.$$

Therefore, we have

Theorem 2.2. *In $G\mathcal{B}P$-$BIRF_n$, the associate curvature tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} and the associate tensor P_{jkh} of the v(hv)-torsion tensor P^i_{kh} (for Cartan's second curvature tensor P^i_{jkh}) is given by the equations (2.6) and (2.7), respectively.*

3. The Necessary and Sufficient Condition

In this section, we shall obtain the necessary and sufficient condition for some tensors to be generalized recurrent in $G\mathcal{B}R$-TRF_n.

For a Riemannian space V_4, the projective curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined by [1]

(3.1) $R^i_{jkh} = P^i_{jkh} + \frac{1}{3}(\delta^i_h R_{jk} - \delta^i_k R_{jh})$.

Taking covariant derivative of second order (Berwald's covariant derivative) of (3.1), with respect to x^m and x^n, successively, we get

(3.2) $\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = \mathcal{B}_n\mathcal{B}_m P^i_{jkh} + \frac{1}{3}(\delta^i_h \mathcal{B}_n\mathcal{B}_m R_{jk} - \delta^i_k \mathcal{B}_n\mathcal{B}_m R_{jh})$.

Using the condition (2.2) in (3.2), we get

$$\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{mn}\, P^i_{jkh} + b_{mn}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\,\mu_m\,\mathcal{B}_r\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)y^r + \tfrac{1}{3}\big(\delta^i_h\,\mathcal{B}_n\mathcal{B}_m R_{jk} - \delta^i_k\,\mathcal{B}_n\mathcal{B}_m R_{jh}\big).$$

By using (3.1), the above equation can be written as

(3.3) $$\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{mn}\, R^i_{jkh} + b_{mn}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\,\mu_m\,\mathcal{B}_r\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)y^r + \tfrac{1}{3}\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big) - \tfrac{1}{3}a_{mn}\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big).$$

This shows that

$$\mathcal{B}_n\mathcal{B}_m R^i_{jkh} = a_{mn}\, R^i_{jkh} + b_{mn}\big(\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\,\mu_m\,\mathcal{B}_r\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)y^r$$

If, and only if ,

$$\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big) = a_{mn}\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big).$$

Thus, we conclude:

Theorem 3.1. *In $G\mathcal{B}P$-$BIRF_n$, Cartan's fourth curvature tensor R^i_{jkh} is generalized birecurrent if, and only if, the tensor $\big(\delta^i_h R_{jk} - \delta^i_k R_{jh}\big)$ is birecurrent.*

Transvecting (3.3) by y^j, using (1.6b), (1.9b), (1.1a), (1.4b) and (1.9c), we get

(3.4) $$\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{mn}\, H^i_{kh} + b_{mn}\big(\delta^i_h y_k - \delta^i_k y_h\big) + \tfrac{1}{3}\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h H_k - \delta^i_k H_h\big) - \tfrac{1}{3}a_{mn}\big(\delta^i_h H_k - \delta^i_k H_h\big).$$

This shows that

(3.5) $$\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{mn}\, H^i_{kh} + b_{mn}\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

If, and only if,

$$\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h H_k - \delta^i_k H_h\big) = a_{mn}\big(\delta^i_h H_k - \delta^i_k H_h\big)\ .$$

Therefore, using the above assumptions and mathematical analysis, the following theorem have been derived.

Theorem 3.2. *In $G\mathcal{B}P$-$BIRF_n$, Berwald's covariant derivative of the second order for the h(v) – torsion tensor H^i_{kh} is given by the condition (3.5) if, and only if, the tensor $\big(\delta^i_h H_k - \delta^i_k H_h\big)$ is birecurrent.*

Contracting the indices i and h in (3.3), using (1.9e) and in view of (1.2), we get

(3.6) $$\mathcal{B}_n\mathcal{B}_m R_{jk} = a_{mn}\, R_{jk} + (n-1)\, b_{mn}\, g_{jk} - 2\,(n-1)\mu_m\,\mathcal{B}_r C_{jkn} y^r + \tfrac{1}{3}(n-1)\mathcal{B}_n\mathcal{B}_m R_{jk} - \tfrac{1}{3}(n-1)a_{mn}\, R_{jk}.$$

This shows that

(3.7) $$\mathcal{B}_n\mathcal{B}_m R_{jk} = a_{mn}\, R_{jk} + (n-1)\, b_{mn}\, g_{jk} - 2\,(n-1)\mu_m\,\mathcal{B}_r C_{jkn} y^r$$

If, and only if,

$$\mathcal{B}_n\mathcal{B}_m R_{jk} = a_{mn}\, R_{jk}\ .$$

Therefore, we conclud the following:

Theorem 3.3. *In $G\mathcal{B}P$-$BIRF_n$,Berwald's covariant derivative of the second order for the R-Ricci tensor R_{jk} is given by the equation (3.7) if, and only if, the R-Ricci tensor R_{jk} is birecurrent.*

Transfecting the equation (3.6) by y^k, using (1.4a), (1.9d), (1.1a) and (1.4b), yields

$$\mathcal{B}_n\mathcal{B}_m R_j = a_{mn}\, R_j + (n-1)\, b_{mn}\, y_j + \tfrac{1}{3}(n-1)\mathcal{B}_n\mathcal{B}_m R_j - \tfrac{1}{3}(n-1)a_{mn}\, R_j.$$

This shows that

(3.8) $$\mathcal{B}_n\mathcal{B}_m R_j = a_{mn}\, R_j + (n-1)\, b_{mn}\, y_j$$

If, and only if ,

$$\mathcal{B}_n\mathcal{B}_m R_j = a_{mn}\, R_j\ .$$

Further, transvecting the equation (3.6) by y^j, using (1.4a), (1.9c), (1.1a) and (1.4b), yields

$$\mathcal{B}_n\mathcal{B}_m H_j = a_{mn}\, H_j + (n-1)\, b_{mn}\, y_j + \tfrac{1}{3}(n-1)\mathcal{B}_n\mathcal{B}_m H_j - \tfrac{1}{3}(n-1)a_{mn}\, H_j.$$

This shows that

(3.9) $$\mathcal{B}_n\mathcal{B}_m H_j = a_{mn}\, H_j + (n-1)\, b_{mn}\, y_j$$

if, and only if,

$$\mathcal{B}_n\mathcal{B}_m H_j = a_{mn} H_j \ .$$

The equations (3.8) and (3.9) show that the curvature vector R_j and the curvature vector H_j cannot vanish if, and only if, the curvature vector R_j and the curvature vector H_j are birecurrent, because the vanishing of them would imply the vanishing of the covariant vector field b_{mn}, i.e. $b_{mn} = 0$, a contradiction.

Therefore, we conclude the following:

Theorem 3.4. *In $G\mathcal{B}P$-$BIRF_n$, the curvature vector R_j and the curvature vector H_j are non-vanishing if, and only if, the curvature vector R_j and the curvature vector H_j are birecurrent.*

Transvecting (3.4) by y^k, using (1.6b), (1.9g), (1.1a),(1.1b), (1.4b) and (1.9f), we get

$$\mathcal{B}_n\mathcal{B}_m H^i_h = a_{mn} H^i_h + b_{mn}(F^2 - 1)\,\delta^i_h$$
$$+\frac{1}{3}\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h\,(n-1)H - H_h y^i\big) - \frac{1}{3}a_{mn}\big(\delta^i_h\,(n-1)H - H_h y^i\big).$$

This shows that

(3.10) $\quad \mathcal{B}_n\mathcal{B}_m H^i_h = a_{mn} H^i_h + b_{mn}(F^2 - 1)\,\delta^i_h$

If, and only if,

$$\mathcal{B}_n\mathcal{B}_m\big(\delta^i_h\,(n-1)H - H_h y^i\big) = a_{mn}\big(\delta^i_h\,(n-1)H - H_h y^i\big) \ .$$

Therefore, using the above assumptions and mathematical analysis, the following theorem have been derived.

Theorem 3.5. *In $G\mathcal{B}P$-$BIRF_n$, Berwald's covariant derivative of the second order for the deviation tensor H^i_h is is given by the equation (3.10) if, and only if, the tensor $\big(\delta^i_h\,(n-1)H - H_h y^i\big)$ is birecurrent.*

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حول تعميم الموتر التقوسي P^i_{jkh} ثُنائي المُعاودة في فضاء فنسلر

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الملخص

في هذه الورقة، قدمنا فضاء فنسلر الذي يكون فيه الموتر التقوسي الثاني لكارتان P^i_{jkh} يحقق خاصية ثنائي المعاودة المعمم بالنسبة لروابط بروالد والمعطى بالحالة
$\mathcal{B}_n\mathcal{B}_m P^i_{jkh} = a_{mn}\, P^i_{jkh} + b_{mn}\big(\,\delta^i_h g_{jk} - \delta^i_k g_{jh}\big) - 2\,\mu_m\,\mathcal{B}_r\big(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\big)y^r\ ,\ P^i_{jkh} \neq 0\ ,$
حيث $\mathcal{B}_n\mathcal{B}_m$ هي مشتقة بروالد ثنائية المعاودة بالنسبة إلى المسقط الوضعي $x^m x^n$ وعلى التعاقب حيث a_{mn} و b_{mn} هي حقول غير صفرية لمتجهات متحدة الاختلاف من الرتبة الثانية وأطلقنا عليه تعميم فضاء $\mathcal{B}P$ - ثنائي المعاودة ورمزنا إليه $BIRF_n G\mathcal{B}P$-. أوجدنا المشتقة المتحدة الاختلاف ثنائية المعاودة في مفهوم بروالد للموتر الألتوائي P^i_{kh}، تقوسات رتشي P_{jk}، المتجه التقوسي P_k، كذلك تم الحصول على بعض المبرهنات للموتر التقوسي المرافق P_{ijkh} للموتر التقوسي P^i_{jkh} والموتر المرافق P_{jk} للموتر الاتوائي P^i_{kh} في هذا الفضاء، كذلك أوجدنا الشرط اللازم والكافي للموتر التقوسي الثالث لكارتان R^i_{jkh} ليكون معمماً ثنائي المعاودة، كذلك الشرط اللازم والكافي لإيجاد المشتقة المتحدة الاختلاف بمفهوم بروالد للموتر الالتوائي H^i_{kh}، تقوس رتشي R_{jk} والموتر الانحرافي H^i_h وأيضا أوجدنا الشرط اللازم والكافيلكي تكون المتجهات التقوسية R_j و H_j غير منتهية في هذا الفضاء.

الكلمات المفتاحية: تعميم فضاء $\mathcal{B}P$-ثُنائي المُعاودة، المشتقة المتحدة الاختلاف من الدرجة الثانية بمفهوم بروالد، الموتر التقوسي الثاني لكارتان P^i_{jkh}.