Applications of certain operational matrices of Dejdumrong polynomials

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Abstract

In this paper, we propose a numerical method based on Dejdumrong polynomials and their operational matrices for solving both linear and non-linear differential equations, calculus of variations, integral equations, optimal control and fraction differential equations. Several examples have been included to demonstrate the validity and applicability of the Dejdumrong operational matrices.

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1. Introduction:

Vast study areas in science have considered problems of the form (12) and (13) ranging from chemical to physical sciences in application to geophysics, reaction-diffusion processes, gas equilibrium amongst many others. As a result of the widespread areas of application of problems of the form under consideration, it is expedient to obtain the exact or an approximate solution for the problem and this has been explored by a good number of researchers. The wavelet analysis approach is adopted for the solution of linear and nonlinear initial (boundary) value problems was used by Nasab and Kilicman [12], Bataineh and Ishak Hashim applied Legendre Operational matrix to approximate the solution of two point boundary value problems [20], while Bhatti made use of the widely known Bernstein polynomial basis to obtain an approximate solution to the differential equation [10]. Similarity, Pandey and Kumar [14] and Isik and Sezer obtained an analytic solution to the Lane-Emden type equations. In a similar way,Yousefi gave an approximate solution to the Bessel differential equation. Moreover,Yuzbasi has attempted to solve the fractional riccati type differential equations [22]. a recent study conducted by Yiming Chen in which the researcher used Bernstein polynomials to obtainthe numerical solution for the variable order linear cable equation [23].A similar approach was used by Rostamy which still with respect to the Bernstein polynomials, but in a new operational matrix method solved the backward inverse heat conduction problems [5]. Likewise, the current researcher has adopted the use of the Dejdumrong operational matrix to obtain the solution of linear and nonlinear initial (boundary) value problems with the application of wavelet analysis method by [12]. From the numerical solutions obtained, it has been observed that there is commendable accuracy and less computational hassle as it is seen that only a few Dejdumrong polynomial basis functions is necessary for obtaining this approximate solution in direct comparison to the exact solution within the range of a maximum10 digits. Describing the structure this article follows, Section 2 describes the review of Dejdumrong polynomial and conventional derivation of Dejdumrong polynomials and its operational matrix differentiation, while Section 3 explains the applications of the operational matrix of derivative. Sections 4 shows the numerical findings, exact solution and, finally, justifying the validity, accuracy and applicability of the operational matrices. A brief summary and conclusion is given in Section5.

2. Dejdumrong polynomial Representation:

A polynomial of degree m can be explicitly formulated as [14]

$$\mathrm{D}_i^m(t) = \begin{cases} (3t)^i(1-t)^{i+3}, & \text{for } 0 \le i < \left\lceil \frac{m}{2} \right\rceil - 1, \\ (3t)^i(1-t)^{m-i}, & \text{if } i = \left\lceil \frac{m}{2} \right\rceil - 1, \\ 2\cdot 3^{i-1}(1-t)^i t^i, & \text{if } i \text{ is even and } i = \frac{m}{2}, \\ \mathrm{D}_{m-i}^m(1-t), & \text{for } \left\lfloor \frac{m}{2} \right\rfloor + 1 \le i \le m. \end{cases} \quad (1)$$

Definition:

The Dejdumrong monomial matrix is [4]

$$\mathrm{N} = \begin{bmatrix} n_{00} & n_{01} & \cdots & \cdots & n_{0m} \\ n_{10} & n_{11} & \cdots & \cdots & n_{1m} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ n_{m0} & n_{m1} & \cdots & \cdots & n_{mm} \end{bmatrix}_{(m+1)\times(m+1)} \quad (2)$$

where n_{kl} is given as

$$n_{kl} = \begin{cases} (-1)^{(l-k)} 3^k \binom{k+3}{l-k}, & \text{for } 0 \le k \le \left\lceil \frac{m}{2} \right\rceil - 1, \\ (-1)^{(l-k)} 3^k \binom{m-k}{l-k}, & \text{for } k = \left\lceil \frac{m}{2} \right\rceil - 1, \\ (-1)^{(l-k)} 2(3^{k-1}) \binom{k}{l-k}, & \text{for } k = \frac{m}{2} \text{ and } m \text{ is even}, \\ (-1)^{(l-k)} 3^{m-k} \binom{m-k}{l-k}, & \text{for } k = \left\lfloor \frac{m}{2} \right\rfloor + 1, \\ (-1)^{(l-m+k-1)} 3^{m-k} \binom{m-k}{l-m+k-3}, & \text{for } \left\lfloor \frac{m}{2} \right\rfloor + 1 \le k \le m. \end{cases} \quad (3)$$

With $\left\lfloor \frac{t}{2} \right\rfloor$ represents $GI \le t$ and $\left\lceil \frac{t}{2} \right\rceil$ represents $LI \ge t$, where GI and LI are the greatest integer and least integer respectively. The Dejdumrong basis function satisfies the following properties:

i. The Dejdumrong basis function is non-negative, that is,

$$\mathrm{D}_i^m(t) \ge 0, \forall i = 0,1,\cdots,m. \quad (4)$$

ii. The partition of unity, that is,

$$\sum_{i=0}^{m} \mathrm{D}_i^m(t) = 1. \quad (5)$$

In general, we approximate any function $y(t)$ with the first $(m + 1)$ Dejdumrong polynomials as:

$$y(t) \approx \sum_{i=0}^{m} c_i \mathrm{D}_i^m(t) = C^T \phi(t) = C^T \mathrm{N} T(t) \tag{6}$$

where $C^T = [c_0, c_1, \ldots, c_m], T(t) = [1, t, \ldots, t^m]^T$ and N given in(2). The operational matrix of derivative of the Dejdumrong polynomials set $\phi(t) = \mathrm{N}T(t)$ is given by:

$\dfrac{d\phi(t)}{dt} = D^{(1)}\phi(t)$ is the $m + 1$ by $m + 1$ operational matrix of derivative define as

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}\mathrm{N}T(t)$$

$$= \mathrm{N}\frac{d}{dt}T(t)$$

$$= \mathrm{N}\frac{d}{dt}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix}$$

$$= \mathrm{N}\begin{bmatrix} 0 \\ 1 \\ 2t \\ \vdots \\ mt^{m-1} \end{bmatrix}$$

$$= \mathrm{N}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & m & 0 \end{bmatrix}\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{bmatrix}$$

$$= \left(\mathrm{N}V\mathrm{N}^{-1}\right)\mathrm{N}T(t)$$

$$= D^{(1)}\phi(t). \tag{9}$$

wher N are Dejdumrong monomial matrix form, and

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & m & 0 \end{bmatrix} \tag{10}$$

Thus,

$$D^{(1)} = \mathrm{N}V\mathrm{N}^{-1} \tag{11}$$

we can generalize Equation (11) as

$$\frac{d^n}{dt^n}\phi(t) = \frac{d^{n-1}}{dt^{n-1}}\left(\frac{d}{dt}\phi(t)\right) = \frac{d^{n-1}}{dt^{n-1}}\left(D^{(1)}\phi(t)\right) = \cdots = \left(D^{(1)}\right)^n\phi(t) = D^{(n)}\phi(t), n = 1, 2, \ldots$$

3. Applications of the Operational Matrix of Derivative:

In this section, the derivation of the method for solving differential equation of the form is presented.

$$p_0(t)y''(t)+p_1(t)y'(t)+p_2(t)\left(y(t)\right)^n=g(t), \qquad (12)$$

with initial (boundary) conditions

$$y(0)=1, \quad y'(0)=0, \quad \text{or } y(0)=\alpha_1, \quad y(1)=\alpha_2. \qquad (13)$$

where $p_j(t), \quad j=0,1,2$ and $g(t)$ are given, while $y(t)$ is unknown.

Approximating Equation (12) by Dejdumrong polynomials as follows:

$$p_0(t)C^T D^{(2)}\phi(t)+p_1(t)C^T D^{(1)}\phi(t)+p_2(t)\left(C^T\phi(t)\right)^n=G^T\phi(t) \quad (14)$$

Where $G^T=[g_0,g_1,\cdots,g_m]$, we can write the residual $\Re_n(t)$ for the Equation (14) as

$$\Re(t)=p_0(t)C^T D^{(2)}\phi(t)+p_1(t)C^T D^{(1)}\phi(t)+p_2(t)\left(C^T\phi(t)\right)^n-G^T\phi(t). \quad (15)$$

To find the solution of $y(t)$ that was given in (12), it can be split into two cases, linear and nonlinear.

3.1. Linear Case:

For $n=1$ we generate $m-1$ linear equations as in a typical tau method [13] by applying

$$\int_0^1 \Re(t)\mathrm{D}_i^{m-1}(t)dt, \quad i=0,1,\cdots,m-1. \qquad (16)$$

Also, by substituting initial (boundary) conditions (13) into (12), we have

$$y(0)=C^T\phi(0)=\alpha_1, \quad y'(0)=C^T D^{(1)}\phi(0)=\alpha_2, \qquad (17)$$

Equations (16) and (17) generate $m+1$ set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C . Consequently, $y(t)$ that was given in (14) can be easily calculated.

3.2. Nonlinear Case:

For $n=2,3,\cdots$, we first collocate (15) at $(m-1)$ points. For suitable collection points, we use $t_i=\frac{1}{2}\left(cos\left(\frac{i\pi}{m}\right)+1\right), i=0,1,\cdots,m-1$ and $m\neq 0$. Theses equations together with (13) generate $(m+1)$ nonlinear equations which can be solved using Newton's iteration method. Consequently, $y(t)$ can be calculated.

4. Numerical examples

Example. 1:

At first, we consider the example given in [13]

$$y''(t)+\frac{1}{t}y'(t)+y(t)=4-9t+t^2-t^3, \qquad (18)$$

with boundary conditions

$y(0)=0 \quad y(1)=0.$ (19)

which has the exact solution is $y(t)=t^2-t^3$.

To solve (18) and (19), we use our purposed with $m = 3$, The approximate solution as

$$\begin{aligned} y(t) &\approx y_3(t)=C^T\phi(t) \\ &= [c_0,c_1,c_2,c_3][\mathrm{D}_0^3(t),\mathrm{D}_1^3(t),\mathrm{D}_2^3(t),\mathrm{D}_3^3(t)]^T \qquad (20) \\ &= c_0\mathrm{D}_0^3(t)+c_1\mathrm{D}_1^3(t)+c_2\mathrm{D}_2^3(t)+c_3\mathrm{D}_3^3(t), \end{aligned}$$

applying (11) we have

$$D^{(1)}=\begin{bmatrix}-3 & -1 & 0 & 0\\ 3 & -1 & -2 & 0\\ 0 & 2 & 1 & -3\\ 0 & 0 & 1 & 3\end{bmatrix},\quad D^{(2)}=\begin{bmatrix}6 & 4 & 2 & 0\\ -12 & -6 & 0 & 6\\ 6 & 0 & -6 & -12\\ 0 & 2 & 4 & 6\end{bmatrix}. \qquad (21)$$

Therefore, using (16) for (18)

$$<\Re(t),\mathrm{D}_i^2(t)>=\int_0^1 \Re(t)\mathrm{D}_i^2(t)dt=0,\quad i=0,1.,\qquad (22)$$

we obtain

$$-\frac{13}{60}c_0-\frac{1}{5}c_1-\frac{1}{5}c_2+\frac{47}{60}c_3+\frac{1}{15}=0,\qquad (23)$$

$$\frac{4}{15}c_0+\frac{3}{10}c_1-\frac{53}{20}c_2+\frac{29}{12}c_3+\frac{53}{60}=0,\qquad (24)$$

Now, by using the boundary conditions we have

$$c_0=0,\quad c_3=0.,\qquad (25)$$

Solving Equations (23), (24) and (25) we get $c_0=0,\quad c_1=0,\quad c_2=\frac{1}{3},\quad c_3=0.$ Thus

$$y(t)=c_0\mathrm{D}_0^3(t)+c_1\mathrm{D}_1^3(t)+c_2\mathrm{D}_2^3(t)+c_3\mathrm{D}_3^3(t)$$

$$=\begin{bmatrix}0 & 0 & \frac{1}{3} & 0\end{bmatrix}\begin{bmatrix}(1-1)^3\\ 3t(1-t)^2\\ 3t^2(1-t)\\ t^3\end{bmatrix}$$

$$=t^2-t^3.$$

which is the exact solution.

Exampl. 2

Consider the differential equation [6]

$$y''(t)+\frac{2}{t}y'(t)+y(t)=0,\qquad y(0)=1,y'(0)=0.\qquad (26)$$

The exact solution is given by $y(t)=\frac{sin(t)}{t}$, we solve the above equation when m = 7 and m = 8. Figure.(1) shows the absolute error. From this figure, one can conclude that our method has obtained ,highly accurate solutions even in large computational intervals.

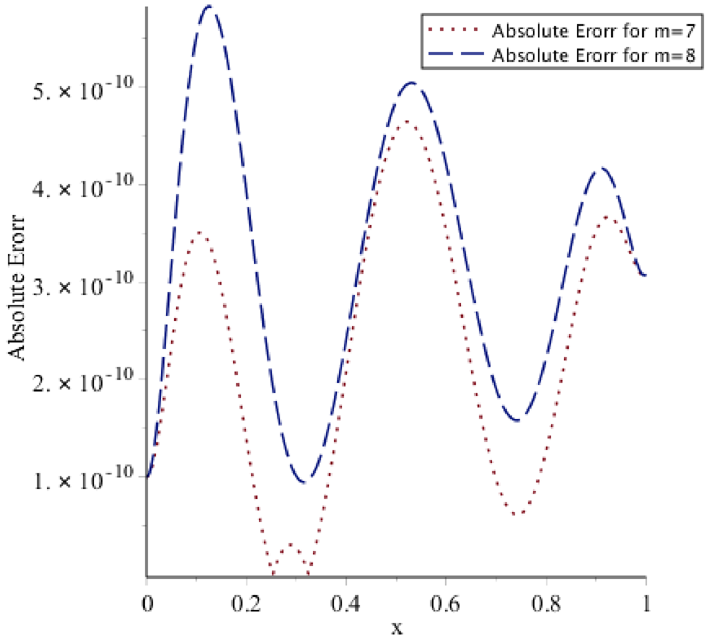

Figure 1: Absolute errors at m = 7 and m = 8 for Example.2 using Dejdumrong

Example. 3
Consider the Bessel differential equation of order zero given in [24, 15, 18, 7]

$$ty''(t)+y(t)'+ty(t)=0, \quad (27)$$

with the initial conditions

$$y(0)=1, \quad y'(0)=0.$$

The exact solution of this example is

$$J_0(t)=\sum_{q=0}^{\infty}\frac{(-1)^q}{(q!)^2}\left(\frac{t}{2}\right)^{2q}.$$

It has been noticed that $g(t) = 0$, The numerical results of our scheme, together with three other, [16, 17, 18],are provided in Table (1).

Table 1: Comparison of the absolute error functions for $m = 8$ of the example. 3

t	PM m = 8	Method of [17] for m = 3, k = 2	Method of [18] for m = 3, k = 2	Method of [16]
0.0	1e-10	9.36e-05	6.01e-05	4.1506e-07
0.1	9e-10	2.78e-05	6.15e-05	1.6138e-07
0.2	1.1e-9	3.60e-05	5.99e-05	7.5736e-08
0.5	1.0e-9	2.695e-04	1.695e-04	1.3032e-07
1.0	9e-10	2.689e-04	1.636e-04	4.1524e-07

Example. 4
We consider the isothermal gas spheres equation as follows [6, 16, 17]

$$y''(t)+\frac{2}{t}y'(t)+e^{y(t)}=0, \quad t\geq 0 \quad (30)$$

with the initial conditions $y(0) = y'(0) = 0$. In this case, we have $g(t) = 0$. We approximate $e^{y(t)}$by using the five terms of its Maclaurin expansion as follows:

$$e^{y(t)} \approx 1 + y(t) + \frac{y^2(t)}{2} + \frac{y^3(t)}{6} + \frac{y^4(t)}{24}.$$

We apply our method for solving this problem using value of m = 10: The solution of this problem was given in [21]. We provide the numerical solutions at the points $t = [0, 0, 0, 1, 0.2, 0.5, 1.0]$ in the case of $m = 10$ in Table 3.

Example. 5

we consider the 5th example as follows [16]

$$y''(t) + \frac{2}{t} y'(t) + sin(y(t)) = 0, \quad t \geq 0 \qquad (31)$$

with the initial conditions $y(0) = 1,\ y'(0) = 0$ In this case, we have $g(t) = 0$ We approximate sin $(y(t))$using the five terms of its Maclaurin expansion as follows:

$$sin(y(t)) = y(t) - \frac{y^3(t)}{6} + \frac{y^5(t)}{120}.$$

The solution of this example was reported in [21].Table. 4 contains the comparison of absolute error of our method with [19] at the values of $m = 10$ at the points $t = 0.0, 0.1, 0.2, 0.5$ and $t = 1.0$.

Table 2. Sin table The exact and approximated solution of Example 5 together with the error comparisons for $m = 10$.

Table 3. The error comparisons for Example. 4 isothermal gas

t	Solution of PM	Exact Solution	Error of PM	Error of [19]
0.0	1.000000000	1.000000000	0.00e+00	0.00e+00
0.1	0.9985976024	0.9985979274	3.25e-07	7.21e-06
0.2	0.9943949771	0.9943962649	1.29e-06	1.00e-05
0.5	0.9651702490	0.9651777802	7.53e-06	1.04e-05
1.0	0.8636571814	0.8636811042	2.39e-05	7.03e-06

Table 4: The exact and approximated solution of Example 5 together with the error comparisons for $m = 10$

t	Solution of PM	Exact Solution	Error of PM	Error of [17]
0.0	1.000000000	1.000000000	0.00e+00	0.00e+00
0.1	0.9985976024	0.9985979274	3.25e-07	7.21e-06
0.2	0.9943949771	0.9943962649	1.29e-06	1.00e-05
0.5	0.9651702490	0.9651777802	7.53e-06	1.04e-05
1.0	0.8636571814	0.8636811042	2.39e-05	7.03e-06

Example. 6

We now consider the Lane-Emden equation [6, 16, 3]

$$y''(t) + \frac{2}{t} y'(t) + y^3(t) = 0, \qquad (32)$$

with the initial conditions $y(0) = 1, y'(0) = 0$. The exact solution of this equation was reportedin [19].

The numerical results of our scheme, together with three other [6, 16, 3], are provided in Table 5. Not only does our method need to lower values of Dejdumrong polynomials.

Table 4. The absolute error comparisons for Example. 6,

t	Error of PM	Error of [15], m=10	Error of [19]	Error of [22], m=25
0.0	0.e+00	0.0e+00	0.0e+00	0.0e+00
0.1	9.80e-12	2.95e-08	1.40e-06	2.95e-08
0.5	2.00e-09	3.00e-08	2.99e-06	3.00e-08
1.0	1.11e-05	3.14e-08	1.99e-06	3.14e-08

5. Conclusions

A general procedure of forming these matrices are given. These matrices can be used to solve problems, such as the calculus of variations, differential equations, optimal control and integral equations, like that of the other basis. The method is general, easy to implement, and yields very accurate results. Moreover, onlya small number of bases are needed to obtain a satisfactory result. Numerical treatment is included to demonstrate the validity and applicability of the operational matrices.

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تطبيقات لبعض المؤثرات المصفوفية لكثيرات حدود ديمورجن

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الملخص

في هذا البحث، نقترح طريقة عددية تعتمد على ديديمرج كثيرات الحدود والمصفوفات التشغيلية لحل المعادلات التفاضلية الخطية وغير الخطية، وحساب التفاضل والتكامل، والمعادلات المتكاملة، والتحكم الأمثل، والمعادلات التفاضلية للكسور. يتم تضمين العديد من الأمثلة لإثبات صحة وإمكانية تطبيق المصفوفات التشغيلية ديديمرج.

الكلمات المفتاحية: كثيرات حدود ديمورجن، المؤثرات المصفوفية، المعادلات التفاضلية.