A class of proper and improper partial bilateral generating Functions for some special Polynomials

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Abstract

In this paper, the group theoretic method is used to derive some classes of proper and improper partial bilateral generating functions for certain special polynomial. Some new and known results are obtained as special cases of the main results.

Keywords: Konhauser polynomials, Gegenbauer polynomials, Laguerre polynomials & proper and improper bilateral generating functions.

1. Introduction and Preliminaries:

Generating functions play a large role in the study of special functions. Generating functions which are available in the literature are almost bilateral in nature. There is a death of trilateral generating functions in the field of special functions. Group-theoretic method of obtaining generating functions for various special functions has been receiving much attention in recent years. Group Theoretic Method proposed by Louis Weisner in 1955, who employed this method to find generating relations for a large class of special functions. Weisner discussed the group-theoretic significance of generating functions for Hypergeometric, Hermite, and Bessel functions [10, 11 and 12] respectively. This technique was used by Khan et.al [2, 3, 4] for obtaining generating functions for various special functions. Miller, McBride, Srivastava and Manocha [6, 5 and 9], respectively , reported Group theoretic method for obtaining generating relations in their books.

In this paper, we have used the group-theoretic method to obtain new classes of generating functions from a given class of generating function. In this section, we give the brief introduction to Konhauser, Gegenbauer and Laguerre polynomials. Also, we define the recently used terms proper and improper partial bilateral generating functions.

The Konhauser biorthogonal polynomials are defined by [1]

$$\Upsilon_n^{(\alpha)}(x;k)=\frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n , \tag{1.1}$$

where $(\alpha)_n$ is the pochhammer symbol [9], $\alpha>-1$ and k is a positive integer.

In particular, we note [9, p.432]

$$\Upsilon_n^{(\alpha)}(x;1)=L_n^{(\alpha)}(x), \tag{1.2}$$

where $L_n^{(\alpha)}(x)$ denotes the modified Laguerre polynomials defined by [7]

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n}\,{}_1F_1(-n;1+\alpha;x), \tag{1.3}$$

which for $\alpha=0$ reduces to the classical Laguerre polynomials $L_n(x)$ given in [7].

The Gegenbauer polynomials, as introduced in [7], are defined by:

$$C_n^{(\nu)}(x)=\frac{(2\nu)_n}{n!}\,{}_2F_1\left(-n,n+2\nu;\nu+\frac{1}{2};\frac{1-x}{2}\right). \tag{1.4}$$

Note that

$$C_n^{(v)}(1) = \frac{(2v)_n}{n!}, \quad (1.5)$$

$$C_n^{(\frac{1}{2})}(x) = P_n(x), \quad (1.6)$$

$$C_n^{(1)}(x) = U_n(x), \quad (1.7)$$

where $P_n(x)$ is Legendre polynomials and $U_n(x)$ is Tchebycheff polynomials of second kind, respectively (see [7]).

It is also known that the Gegenbauer and Ultraspherical polynomials $P_n^{(\alpha,\beta)}(x)$ are linked by the following relation [7]:

$$C_n^{(v)}(x) = \frac{(2v)_n}{(v+\frac{1}{2})_n} P_n^{(v-\frac{1}{2},v-\frac{1}{2})}(x). \quad (1.8)$$

Now, we consider the following partial differential operators (cf. [1, 5]):

$$R_1 = xy^{-1}h\frac{\partial}{\partial x} + h\frac{\partial}{\partial y} + (k+1)y^{-1}h^2\frac{\partial}{\partial h} + y^{-1}h(1-h+km) \quad (1.9)$$

and

$$R_2 = (z^2-1)t\frac{\partial}{\partial z} + zt^2\frac{\partial}{\partial t} + (2\beta+s)zt. \quad (1.10)$$

So that

$$R_1\left[Y_{n+m}^{(\alpha)}(x;k)y^{\alpha}h^n\right] = k(1+n+m)Y_{n+m+1}^{(\alpha)}(x;k)y^{\alpha-1}h^{n+1}, \quad (1.11)$$

$$R_2\left[C_{s+n}^{(\beta)}(z)\ t^{s+n}\right] = (s+n+1)C_{s+n+1}^{(\beta)}(z)\ t^{s+n+1}. \quad (1.12)$$

Consequently, we have

$$\exp(wR_1)f(x,y,h) = (1-kwy^{-1}h)^{-(\frac{km+1}{k})}\exp\left(x - \frac{x}{(1-kwy^{-1}h)^{\frac{1}{k}}}\right)$$

$$\times f\left(\frac{x}{(1-kwy^{-1}h)^{\frac{1}{k}}}, \frac{y}{(1-kwy^{-1}h)^{\frac{1}{k}}}, \frac{h}{(1-kwy^{-1}h)^{\frac{1}{k}}}\right) \quad (1.13)$$

and

$$\exp(vR_2)f(z,t) = (1-2vzt+v^2t^2)^{-\beta}\ f\left(\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}}, \frac{t}{\sqrt{1-2vzt+v^2t^2}}\right), \quad (1.14)$$

where $\left|2vzt - v^2t^2\right| < 1$

Next, we define the recently used terms proper partial bilateral generating relation and improper partial bilateral generating relation for two classical polynomials introduced by Sarkar (cf. [8]).

Definition 1.1 Proper partial bilateral generating function for two classical polynomials is the relation

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n w^n\, p_{m+n}^{(\alpha)}(x)\, q_{m+n}^{(\beta)}(z), \quad (1.15)$$

where the coefficients a_n's are quite arbitrary, $p_{m+n}^{(\alpha)}(x)$ *and* $q_{m+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n)$ with the parameters α and β, respectively. where α *and* β are complex numbers , n is a positive integer or zero and m is integer number.

Definition 1.2 Improper partial bilateral generating relation for two classical polynomials, is the relation

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n\, p_{m+n}^{(\alpha)}(x)\, q_{k+n}^{(\beta)}(z) \quad , \tag{1.16}$$

where the coefficients a_n's are quite arbitrary and $p_{m+n}^{(\alpha)}(x), q_{k+n}^{(\beta)}(z)$ are any two classical polynomials of order $(m+n)$ *and* $(k+n)$ with the parameters α *and* β respectively.

In the following section, we have obtained new classes of improper bilateral generating functions for some orthogonal polynomials.

2. Improper partial bilateral generating functions:

We derive the main result in the form of the following theorem:

Theorem (1) : If there exist the following class of improper partial bilateral generating functions for the Konhauser and Gegenbauer polynomials by means of the relation

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n\, Y_{n+m}^{(\alpha)}(x;k)\; C_{n+s}^{(\beta)}(z) \quad , \tag{2.1}$$

where a_n is arbitrary , then the following generating function holds

$$(1-wk)^{\frac{-(1+\alpha+km)}{k}}\left(1-2vz+v^2\right)^{-(\beta+\frac{s}{2})}\exp\left(x-\frac{x}{(1-wk)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p}v^{n+q}}{p!q!}k^p(m+n+1)_p(s+n+1)_q\; Y_{n+m+p}^{(\alpha)}(x;k)C_{s+n+q}^{(\beta)}(z), \tag{2.2}$$

where $\left|2vz-v^2\right|<1$. Equation (2.2) represents a new class of improper partial bilateral generating function for Konhauser and Gegenbauer polynomials.

Proof of the Theorem 1

Let

$$G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n\, Y_{n+m}^{(\alpha)}(x;k)\; C_{n+s}^{(\beta)}(z) \tag{2.3}$$

on multiplying both sides of above equation by $y^{\alpha}t^{s}$, which gives us

$$y^{\alpha}t^{s}G(x,z,w)=\sum_{n=0}^{\infty} a_n w^n\, Y_{n+m}^{(\alpha)}(x;k)\; C_{n+s}^{(\beta)}(z)y^{\alpha}t^{s} \tag{2.4}$$

Now, replacing w *by* $whtv$ in (2.4), we get

$$y^{\alpha}t^{s}G(x,z,whtv)=\sum_{n=0}^{\infty} a_n (wv)^n Y_{n+m}^{(\alpha)}(x;k)y^{\alpha}h^{n}C_{n+s}^{(\beta)}(z)t^{s+n} \tag{2.5}$$

Applying the results in (1.5), (1.6), (1.7) and (1.8) to equation (2.5), we obtain

$$\left(1-wky^{-1}h\right)^{\frac{-(1+\alpha+km)}{k}}\left(1-2vzt+v^2t^2\right)^{-(\beta)}\exp\left(x-\frac{x}{(1-wky^{-1}h)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wky^{-1}h)^{\frac{1}{k}}},\frac{z-vt}{\sqrt{1-2vzt+v^2t^2}},\frac{wvyt}{(1-wky^{-1}h)^{\frac{1}{k}}\sqrt{1-2vzt+v^2t^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}(ky^{-1}h)^p y^{\alpha}\left(m+n+1\right)_p(s+n+1)_q\ \mathrm{Y}_{n+m+p}^{(\alpha)}(x;k)C_{s+n+q}^{(\beta)}\left(z\right)t^{s+n+q} \quad (2.6)$$

Finally, when putting $y=t=h=1$ in the above equation (2.6), we arrive at the result (2.2).

It may be of special interest to point out that, for s = m, the above theorem has become a nice class of generating functions forms proper partial bilateral generating functions. We state these results in the form of following corollary:

Corollary If there exist the following class of (proper) partial bilateral generating function for the Konhauser and Gegenbauer polynomials by means of the relation

$$G(x,z,w)=\sum_{n=0}^{\infty}a_n w^n Y_{n+m}^{(\alpha)}(x;k)\ C_{n+m}^{(\beta)}(z)$$

where a_n is arbitrary , then the following general class of generating function is hold

$$\left(1-wk\right)^{\frac{-(1+\alpha+km)}{k}}\left(1-2vz+v^2\right)^{-(\beta+\frac{m}{2})}\exp\left(x-\frac{x}{(1-wk)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}k^p\left(m+n+1\right)_p(m+n+1)_q\ \mathrm{Y}_{n+m+p}^{(\alpha)}(x;k)C_{m+n+q}^{(\beta)}\left(z\right), \quad (2.7)$$

where $\left|2vz-v^2\right|<1$.

Remark : using relation (1.8) in results (2.2) and (2.7), we get the following relations:

$$\left(1-wk\right)^{\frac{-(1+\alpha+km)}{k}}\left(1-2vz+v^2\right)^{-(\beta+\frac{s}{2})}\exp\left(x-\frac{x}{(1-wk)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}k^p\left(m+n+1\right)_p(s+n+1)_q\frac{(2\beta)_{s+n+q}}{(\beta+\frac{1}{2})_{s+n+q}}\mathrm{Y}_{n+m+p}^{(\alpha)}(x;k)P_{s+n+q}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}\left(z\right) \quad (2.8)$$

and

$$\left(1-wk\right)^{\frac{-(1+\alpha+km)}{k}}\left(1-2vz+v^2\right)^{-(\beta+\frac{m}{2})}\exp\left(x-\frac{x}{(1-wk)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}k^p(m+n+1)_p(m+n+1)_q\frac{(2\beta)_{m+n+q}}{(\beta+\frac{1}{2})_{m+n+q}}\Upsilon_{n+m+p}^{(\alpha)}(x;k)P_{m+n+q}^{(\beta-\frac{1}{2},\beta-\frac{1}{2})}(z), \quad (2.9)$$

respectively.

In the following section, we have obtained new and known classes of proper partial bilateral generating functions as particular cases of the above main results.

Particular Cases:

I. On putting $k=1$ in Theorem 1, we get the following known result due to Sarkar [8]:

$$(1-w)^{-(1+\alpha+m)}(1-2vz+v^2)^{-(\beta+\frac{s}{2})}\exp\left(-\frac{wx}{(1-w)}\right)$$

$$\times G\left(\frac{x}{(1-w)},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}(m+n+1)_p(s+n+1)_q\, L_{n+m+p}^{(\alpha)}(x)C_{s+n+q}^{(\beta)}(z), \quad (2.10)$$

where $\left|2vz-v^2\right|<1$ and $G(x,z,w)=\sum_{n=0}^{\infty}a_nw^nL_{n+m}^{(\alpha)}(x)C_{n+m}^{(\beta)}(z)$

For $s=m$ in (2.10) gives the following class of (proper) partial bilateral generating function for the Laguerre and Gegenbauer polynomials due to Sarkar [8]:

$$(1-w)^{-(1+\alpha+m)}(1-2vz+v^2)^{-(\beta+\frac{m}{2})}\exp\left(-\frac{wx}{(1-w)}\right)$$

$$\times G\left(\frac{x}{(1-w)},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-w)\sqrt{1-2vz+v^2}}\right)$$

$$=\sum_{n,p,q=0}^{\infty}a_n\frac{w^{n+p}v^{n+q}}{p!q!}(m+n+1)_p(m+n+1)_q\, L_{n+m+p}^{(\alpha)}(x)C_{m+n+q}^{(\beta)}(z), \quad (2.11)$$

where $\left|2vz-v^2\right|<1$.

II. On putting $\beta=\frac{1}{2}$ in Theorem (1) and using relation (1.6), we get the following new improper partial bilateral generating functions for Konhauser and Legendre polynomials:

$$(1-wk)^{\frac{-(1+\alpha+km)}{k}}(1-2vz+v^2)^{-(\frac{s+1}{2})}\exp\left(x-\frac{x}{(1-wk)^{\frac{1}{k}}}\right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}},\frac{z-v}{\sqrt{1-2vz+v^2}},\frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}}\right)$$

$$= \sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p} v^{n+q}}{p!q!} k^p (m+n+1)_p (s+n+1)_q \ \mathrm{Y}_{n+m+p}^{(\alpha)}(x;k) P_{s+n+q}(z), \qquad (2.12)$$

where $\left|2vz - v^2\right| < 1$.

III. On putting $\beta = 1$ in Theorem (1) and using relation (1.7), we get the following improper partial bilateral generating functions for Konhauser and Tchebycheff polynomials:

$$(1-wk)^{\frac{-(1+\alpha+km)}{k}} \left(1-2vz+v^2\right)^{-(1+\frac{s}{2})} \exp\left(x - \frac{x}{(1-wk)^{\frac{1}{k}}} \right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}}, \frac{z-v}{\sqrt{1-2vz+v^2}}, \frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2vz+v^2}} \right)$$

$$= \sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p} v^{n+q}}{p!q!} k^p (m+n+1)_p (s+n+1)_q \ \mathrm{Y}_{n+m+p}^{(\alpha)}(x;k) U_{s+n+q}(z), \qquad (2.13)$$

where $\left|2vz - v^2\right| < 1$.

IV. On putting $z = 1$ in Theorem (1) and using relation (1.5), we get

$$(1-wk)^{\frac{-(1+\alpha+km)}{k}} \left(1-2v+v^2\right)^{-(\beta+\frac{s}{2})} \exp\left(x - \frac{x}{(1-wk)^{\frac{1}{k}}} \right)$$

$$\times G\left(\frac{x}{(1-wk)^{\frac{1}{k}}}, \frac{1-v}{\sqrt{1-2v+v^2}}, \frac{wv}{(1-wk)^{\frac{1}{k}}\sqrt{1-2v+v^2}} \right)$$

$$= \sum_{n,p,q=0}^{\infty} a_n \frac{w^{n+p} v^{n+q}}{p!q!(s+n+1)!} k^p (m+n+1)_p (s+n+1)_q \ \mathrm{Y}_{n+m+p}^{(\alpha)}(x;k) \ , \qquad (2.14)$$

where $\left|2vz - v^2\right| < 1$.

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فئة دوال مولدة ثنائية جزئية معتلة وصحيحة لبعض كثيرات الحدود الخاصة

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الملخص

في هذا البحث استعملنا طريقة الزمر النظري للحصول على دوال مولدة ثنائية معتلة وصحيحة لأنواع محددة من كثيرات الحدود. بعض النتائج الجديدة والمعروفة أوجدت كحالات خاصة من النتائج الرئيسية.

الكلمات المفتاحية: كثيرات حدود كونهوسر، كثيرات حدود جيجن باور، كثيرات حدود لاجير، الدوال المولدة الثنائية المعتلة والصحيحة.