On certain a generalized $N_{im}$- Recurrent Finsler space

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Abstract

A Finsler space $F_n$ for which the normal projective curvature tensor $N^i_{jkh}$ satisfies $N^i_{jkh|m} = \lambda_m N^i_{jkh} + \mu_m (\delta^i_h g_{jk} - \delta^i_k g_{jh}), N^i_{jkh} \neq 0$, where $\lambda_m$ and $\mu_m$ are non-zero covariant vectors field, will be called a generalized $N_{im}-$recurrent space. The curvature vector $H_k$, the curvature scalar $H$ and Ricci tensor $N_{jk}$ are non-vanishing. When the generalized $N_{im}$- recurrent space is affinely connected space and under certain conditions, we obtain various results. Also, in generalized $N_{im}$- recurrent space, Weyl’s projective curvature tensors is a generalized recurrent tensor.

Keywords: Generalized $N_{im}$-Recurrent Space, Generalized Recurrent Tensor, Generalized $N_{im}$- Recurrent Affinely Connected Space, Weyl's projective curvature recurrent tensor.

1. Introduction

K. Yano [20] defined the normal projective connection $\Pi^i_{jk}$ by

$$\Pi^i_{jk} = G^i_{jk} - \frac{1}{n+1} y^i G^r_{kjr}.$$  

R.B. Misra and F.M. Meher [12] considered a space equipped with normal projective connection $\Pi^i_{jk}$ whose curvature tensor $N^i_{jkh}$ is recurrent with respect to normal projective connection $\Pi^i_{jk}$ and they called it $RNP-$Finsler space. P.N. Pandey and V.J. Diwedi [16] studied $RNP$-Finsler space and obtained many identities in $RNP$-Finsler space, most of these identities are also true in a recurrent Finsler space with respect to Berwald’s connection coefficients $G^i_{jk}$. F. Y. A. Qasem [17] obtained several results concerning the normal projective curvature tensor $N^i_{jkh}$ in such space.

Let us consider a set of quantities $g_{ij}$ defined by [18]

$$g_{ij}(x, y) = \frac{1}{2} \delta^i_l \delta^j_k F^2(x, y).$$

The tensor $g_{ij}(x, y)$ is positively homogeneous of degree zero in $y^i$ and symmetric in $i$ and $j$. According to Euler’s theorem on homogeneous functions, the vector $y_i$ and $y^i$ satisfy the following relations [18]

$$\delta y_i y^i = F^2, b) \ g_{ij} = \delta^i_l y_j = \delta^i_j y_l \ \text{and} \ c) \ g_{ij} y^i = y_j.$$

Cartan’s covariant derivative of the metric function $F$, vector $y^i$ and the metric tensor $g_{ij}$ vanish identically, i.e. [18]

$$a) \ F^i_{jk} = 0, b) \ y^i_{jk} = 0 \ \text{and} \ c) \ g_{ij,kl} = 0.$$  

A Finsler space whose connection parameter $G^i_{jk}$ is independent of $y^i$ is called an affinely connected space [1]. Thus, an affinely connected space is characterized by one of the equivalent equations

$$a) \ G^i_{jkh} = 0 \ \text{and} \ b) \ C^i_{jkh} = 0.$$  

The connection parameter $\Gamma^i_{jk}$ of Cartan and $G^i_{jk}$ of Berwald coincide in affinely connected space and they are independent of the direction argument, i.e. [18]

$$a) \partial^i_{jk} = 0 \ \text{and} \ b) \ \partial^i_{jk} = 0.$$  

Cartan’s connection parameter $\Gamma^i_{jk}$ coincides with Berwald’s connection parameter $G^i_{jk}$ for a Landsberg space, which is characterized by [18]

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(1.7) $\gamma_r G^i_{kh} = -2C^i_{khj} y^j = -2P^i_{kh} = 0.$

Various authors denote the tensor $G_{ij} y^i y^j$ by $P_{ijk}.$ F. Ikeda[2], H. Izumi ([4]-[7]), H. Izumiand M. Toshida [8], M. Matsumoto [10] and H. Wosoughi [20]. Since the equations (1.5a) and (1.6a) imply (1.7), an affinely connected space is necessarily Landsberg space[1]. However, a Landsberg space need not be an affinely connected space.

Cartan’s covariant derivative of an arbitrary tensor $T^i_h$ with respect to $x^k$ is given by

(1.8a) $\delta_j (T^i_h y^j) = (\delta_j T^i_h) y^j = T^i_h (\delta_j y^j) - T^i_h (\delta_j y^j)$

where

and

$$ b) \quad P^r_{kj} = (\delta_j T^i_{kh}) y^j = T^r_{jk} = \Gamma^r_{jk} y^h$$

$$ c) \quad P^r_{kj} = g^{ir} P_{kh}.$$  

The tensor $H^i_{jk}$ is called Berwald curvature tensor, it is positively homogeneous of degree zero in $y^i$ and skew-symmetric in its last two lower indices which defined by [18]

$$ H^i_{jk} := \partial_h G^i_{jk} + G^r_{jk} G^i_{rh} + G^i_{rk} G^r_{jh} - h/k.$$  

In view of Euler’s theorem on homogeneous functions, we have the following relations [18]

(1.9) a) $\delta_j H^i_{k} = H^i_{jk}$, b) $H^i_{jk} y^j = H^i_{kh}$, c) $H^i_{jkh} := g_{jr} H^i_{kr}$

d) $H^i_{k} y^k = H^i_{h}$, e) $H^i_{k} = \delta_k H^i_{kh}$, f) $H^i_{jkr} = H^i_{jkr}$

g) $H_k = H^i_{k}$, which defined by

The tensor $H^i_{jkh}$ is given by

(1.10) $H^i_{jkh} = g_{lk} H^l_{jh}.$

2. Normal Projective Curvature Tensor

P.N. Pandey ([13] - [15]) obtained a relation between the normal projective curvature tensor $N^i_{jkh}$ and Berwald curvature tensor $H^i_{jkh}$ as follows:

(2.1) $N^i_{jkh} = H^i_{jk} - \frac{1}{n+1} y^i \delta_j H^i_{kh}.$

The normal projective curvature tensor $N^i_{jkh}$ is homogeneous of degree zero in $y^i.$

Contracting the indices i and j in (2.1) and using the fact that the tensor $H^r_{kh}$ is positively homogeneous of degree zero in $y^i$, we get

(2.2) $N^r_{jkh} = H^r_{jkh}.$

Transvecting (2.1) by $y^j$ and using (1.9a), we get

(2.3) $N^i_{jkh} y^j = H^i_{kh}.$

The projective curvature tensor $W^i_{jkh}$ and the normal projective curvature tensor $N^i_{jkh}$ are connected [9] by

(2.4a) $W^i_{jkh} = N^i_{jkh} + (\delta^i_k M_{hj} - M_{kh} \delta^i_j - k[h])$

where

$$ b) \quad M_{kh} := -\frac{1}{n^2 - 1} (n N_{kh} + N_{kh})$$

and

$$ c) \quad N_{jk} := N^r_{jkr}.$$  

The projective curvature tensor $W^i_{jkh}$ satisfies the following [18]:

(2.5) a) $W^i_{jkh} y^j = W^i_{kh}$ and b) $W^i_{kh} y^k = W^i_{h}.$

A Finsler space is called a recurrent Finsler space if it’s normal projective curvature tensor $N^i_{jkh}$ satisfies ([11], [14], [19])

(2.6) $N^i_{jkh|m} = \lambda_m N^i_{jkh}, \quad N^i_{jkh} \neq 0,$

where $\lambda_m$ is non-zero covariant vector field.

3. Generalized Recurrent Space

Let us consider a Finsler space \( F_n \) for which the normal projective curvature tensor \( \nabla_{ijk} \) satisfies the condition
\[
\nabla_{ijk} = \lambda_m \nabla_{ijk} + \mu_n \left( \delta^i_{jk} g_{ij} - \delta^j_{ik} g_{ij} \right), \nabla_{ijk} \neq 0,
\]
where \( \lambda_m \) and \( \mu_n \) are non-zero covariant vector fields, such space will be called a generalized \( N_{im} \) recurrent space and the tensor will be called generalized \( N_{im} \) recurrent tensor.

**Remark 3.1.** Any curvature tensor which satisfies similar to the condition (3.1) will be called generalized recurrent tensor.

Contracting the indices \( i \) and \( j \) in (3.1) and using (2.2), we get
\[
(3.2) \quad \nabla_{ijk} = \lambda_m \nabla_{ijk} + \mu_n \left( \delta^i_{jk} g_{ij} - \delta^j_{ik} g_{ij} \right).
\]

Thus, the following theorem

**Theorem 3.1.** In a generalized \( N_{im} \) recurrent space, Cartan's covariant derivative of the tensor \( \nabla_{ijk} \) behaves as recurrent.

Transvecting (3.1) by \( y^l \), using (1.4b), (2.3) and (1.3c), we get
\[
(3.3) \quad \nabla_{klm} = \lambda_m \nabla_{klm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right).
\]

Transvecting (3.3) by \( y^k \), using (1.4b), (1.9d) and (1.3a), we get
\[
(3.4) \quad \nabla_{lm} = \lambda_m \nabla_{lm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right).
\]

Thus, the following theorem

**Theorem 3.2.** In a generalized \( N_{im} \) recurrent space, Cartan's covariant derivative of the \( h(v) \)-torsion tensor \( \nabla_{klm} \) and the deviation tensor \( \nabla_{lm} \) are given by (3.3) and (3.4), respectively.

Contracting the indices \( i \) and \( j \) in (3.3) and using (1.9g), we get
\[
(3.5) \quad \nabla_{klm} = \lambda_m \nabla_{klm} + (\nu - 1) \mu_n g_{km}.
\]

Contracting the indices \( i \) and \( j \) in (3.4) and using (1.9h), we get
\[
(3.6) \quad \nabla_{lm} = \lambda_m \nabla_{lm} + \mu_n F^2.
\]

Contracting the indices \( i \) and \( j \) in (3.1) and using (2.4c), we get
\[
(3.7) \quad \nabla_{lm} = \lambda_m \nabla_{lm} + (\nu - 1) \mu_n g_{km}.
\]

Thus, the following theorem

**Theorem 3.3.** The curvature vector \( \nabla_{klm} \) of the curvature scalar \( H \) and the Ricci tensor \( \nabla_{lm} \) of generalized \( N_{im} \) recurrent space are non-vanishing.

Differentiating (3.2) partially with respect to \( y^l \), we get
\[
(3.8) \quad \nabla_{klm} = \lambda_m \nabla_{klm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right).
\]

Differentiating (2.1) covariantly with respect to \( x^m \) in the sense of Cartan and using (1.4b), we get
\[
(3.9) \quad \nabla_{klm} = \lambda_m \nabla_{klm} + \frac{1}{n+1} \gamma^l \left( \nabla_{jk} \nabla_{klm} \right).
\]

Using commutation formula exhibited by (1.8a) for \( \nabla_{jk} \) in (3.9), using (3.1) and (3.8), we get
\[
(3.10) \quad \lambda_m \nabla_{klm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right) = \lambda_m \nabla_{klm} + \frac{1}{n+1} \gamma^l \left( \nabla_{jk} \nabla_{klm} \right).
\]

Using (2.1) in (3.10), we get
\[
(3.11) \quad \lambda_m \nabla_{klm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right) = \lambda_m \nabla_{klm} + \frac{1}{n+1} \gamma^l \left( \nabla_{jk} \nabla_{klm} \right).
\]

This shows that
\[
(3.12) \quad \nabla_{jk} \nabla_{klm} = \lambda_m \nabla_{jk} \nabla_{klm} + \mu_n \left( \delta^l_{km} g_{lm} - \delta^k_{lm} g_{lm} \right).
\]
Theorem 3.4. In generalized N\textsubscript{m}–recurrent space, Berwald curvature tensor H\textsubscript{jk} is generalized recurrent if and only if (3.12) holds.

Contracting the indices i and h in (3.11) and using (1.9f), we get
\begin{equation}
\lambda_m H_{jk} + \mu_m (n-1) \varphi_{jk} = H_{jk|m} - \frac{1}{n+1} y^i (\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s.
\end{equation}

This shows that
\begin{equation}
H_{jk|m} = \lambda_m H_{jk} + \mu_m (n-1) \varphi_{jk}.
\end{equation}

if and only if
\begin{equation}
y^i (\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s = 0.
\end{equation}

Thus, the following theorem

Theorem 3.5. In generalized N\textsubscript{m}–recurrent space, Ricci tensor H\textsubscript{jk} is non–vanishing if and only if (3.14) holds.

Also, (3.11) can be written as
\begin{equation}
H_{jk|m}^b = \lambda_m H_{jk}^b - \mu_m (\delta_h g_{jk} - \delta_k g_{jh}) = \frac{1}{n+1} \left( (\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s \right).
\end{equation}

Transvecting (3.15) by y\textsubscript{jk} and using (1.3a), we get
\begin{equation}
y^i \left( H_{jk|m}^b - \lambda_m H_{jk}^b - \mu_m (\delta_h g_{jk} - \delta_k g_{jh}) \right) = \frac{1}{n+1} \left( (\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s \right).
\end{equation}

From (3.15) and (3.16), we get
\begin{equation}
H_{jk|m}^b - \lambda_m H_{jk}^b - \mu_m (\delta_h g_{jk} - \delta_k g_{jh}) = \frac{y^i y^j}{n+1} \left[ H_{jk|m}^b - \lambda_m H_{jk}^b - \mu_m (\delta_h g_{jk} - \delta_k g_{jh}) \right].
\end{equation}

Thus, the following theorem

Theorem 3.6. In generalized N\textsubscript{m}–recurrent space, the curvature tensor H\textsubscript{jk} is generalized recurrent if and only if y\textsubscript{jk} \{ H_{jk|m}^b - \lambda_m H_{jk}^b - \mu_m (\delta_h g_{jk} - \delta_k g_{jh}) = 0 \} holds.

Transvecting (3.3) by y\textsubscript{jk}, using (1.10) and (1.4c), we get
\begin{equation}
H_{ks|h}^s = \lambda_m H_{ks|h} + \mu_m (g_{sh} Y_k - g_{sk} Y_h).
\end{equation}

Thus, the following theorem

Theorem 3.7. In generalized N\textsubscript{m}–recurrent space, Cartan’s covariant derivative of the associate tensor H\textsubscript{ks|h} of the h–torsion tensor H\textsubscript{jk} is given by (3.18).

Transvecting (3.11) by g\textsubscript{tij}, using (1.9c), (1.3c) and (1.4c), we get
\begin{equation}
\lambda_m H_{ftkh} + \mu_m (g_{tj} g_{jk} - g_{tk} g_{jh}) = H_{ftkh|m} - \frac{1}{n+1} y^i (\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s.
\end{equation}

This shows that
\begin{equation}
H_{ftkh|m} = \lambda_m H_{ftkh} + \mu_m (g_{tj} g_{jk} - g_{tk} g_{jh})
\end{equation}

if and only if
\begin{equation}
(\hat{\partial}_i \lambda_m) H_{rkt}^r + H_{rst}^r (\hat{\partial}_j \Gamma_{km}^s) + H_{rks}^r (\hat{\partial}_t \Gamma_{jm}^s) + \hat{\delta}_s H_{rkt}^r P_{mj}^s = 0.
\end{equation}

Thus, the following theorem

Theorem 3.8. In generalized N\textsubscript{m}–recurrent space, the associate tensor H\textsubscript{jk} of the curvature tensor H\textsubscript{jk} is given by (3.20) if and only if (3.21) holds.

Remark 3.2. If the generalized N\textsubscript{m}–recurrent space is affinely connected space, so the new space will be called generalized N\textsubscript{m}–recurrent space affinely connected space. It will be sufficient to call the curvature tensor which satisfies this space by generalized recurrent.
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Let us consider generalized $N_{|m}$-recurrent affinely connected space.

In view of (1.8c), (1.7) and if $\delta p_{\lambda m} = 0$, (3.11) becomes

$$(\text{3.22}) H^{i}_{jkh|m} = \lambda_{m} H^{i}_{jkh} + \mu_{m}(\delta^{i}_{k} g_{jk} - \delta^{i}_{j} g_{jh}).$$

Thus, the following theorem

**Theorem 3.9.** In the generalized $N_{|m}$- recurrent affinely connected space, if the directional derivative of covariant vector field vanish, then the curvature tensor $H^{i}_{jkh}$ is generalized recurrent.

In view of (1.8c), (1.7) and if $\delta p_{\lambda m} = 0$, (3.19) becomes

$$(\text{3.23}) H^{i}_{jkh|m} = \lambda_{m} H^{i}_{jkh} + \mu_{m}(g_{th} g_{jk} - g_{tk} g_{jh}).$$

Thus, the following theorem

**Theorem 3.10.** In the generalized $N_{|m}$- recurrent affinely connected space, if the directional derivative of covariant vector field vanish, then H - Ricci tensor $H^{i}_{hik}$ is non-vanishing.

**Remark 3.** A affinely connected space is necessarily Landsberg space. However, Landsberg space need not be an affinely connected space. Hence, any result obtained in affinely connected space carries Landsberg space.

4. Weyl’s Projective Curvature Generalized $N_{|m}$ - Recurrent Space

Let us consider a Finsler space $F_{n}$ for which the normal projective curvature tensor $N^{i}_{jkh}$ satisfies the condition (3.1).

Differentiating (2.4b) covariantly with respect to $x^{m}$ in the sense of Cartan, we get

$$(\text{4.1}) M_{kh|m} = - \frac{1}{n^{2} - 1} \left(n N_{kh|m} + N_{hk|m}\right).$$

Using (3.7) in (4.1), we get

$$(\text{4.2}) M_{kh|m} = \lambda_{m} \left[- \frac{1}{n^{2} - 1} (n N_{kh} + N_{hk})\right] - \frac{2}{n+1} \mu_{m} g_{kh}.$$}

Using (2.4b) in (4.2), we get

$$(\text{4.3}) M_{kh|m} = \lambda_{m} M_{kh} - \frac{2}{n+1} \mu_{m} g_{kh}.$$}

Thus, the following theorem

**Theorem 4.1.** In generalized $N_{|m}$- recurrent space, Cartan derivative of the tensor $M_{kh|m}$ is given by (4.3).

Differentiating (2.4a) covariantly with respect to $x^{m}$ in the sense of Cartan, we get

$$(\text{4.4}) W^{i}_{jkh|m} = N^{i}_{jkh|m} + \left(\delta^{i}_{k} M_{hj|m} - M_{kh|m} \delta^{i}_{j} - k/h\right).$$

Using (3.1) and (4.3) in (4.4), we get

$$(\text{4.5}) W^{i}_{jkh|m} = \lambda_{m} (N^{i}_{jkh} + \left(\delta^{i}_{k} M_{hj} - M_{kh} \delta^{i}_{j} - k/h\right) + \mu_{m}(\delta^{i}_{h} g_{jk} - \delta^{i}_{j} g_{kh}).$$

Using (2.4a) in (4.5), we get

$$(\text{4.6}) W^{i}_{jkh|m} = \lambda_{m} W^{i}_{jkh} + \mu_{m}(\delta^{i}_{h} g_{jk} - \delta^{i}_{j} g_{kh}).$$

Thus, the following theorem

**Theorem 4.2.** In generalized $N_{|m}$- recurrent space, the projective curvature tensor $W^{i}_{jkh}$ is generalized recurrent.

Transvecting (4.6) by $y^{i}$, using (2.5a), (1.4b) and (1.3c), we get

$$(\text{4.7}) W^{i}_{kh|m} = \lambda_{m} W^{i}_{kh} + \mu_{m}(\delta^{i}_{h} y_{k} - y^{i} g_{kh}).$$

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Transvecting (4.7)by $yy^k$, using (2.5b), (1.4b) and (1.3a), we get
\[(4.8) W_{iklm}^j = \lambda_m W_{ij}^m + \mu_m (\delta^i_k F^2 - \gamma^i_{xy}).\]

Thus, the following theorem

**Theorem 4.2.** In generalized $N_{lm}$ - recurrentspace, Cartan derivative of the projective
torsion tensor $W_{ik}^{jl}$ and the projective deviation tensor $W_{ik}^{jl}$ given by (4.7) and (4.8), respectively.

Now, we know that Finsler space $F_n$, in general, is not generalized $N_{lm}$-recurrent spacethesetensor
$M_{kh}$ of Finsler space $F_n$ is given by (4.3). But if the projective curvature tensor $W_{ik}^{jl}$ is generalized
recurrent tensor, our space is necessarily generalized $N_{lm}$ - recurrent space and this may be seen as follows:

Let us consider a Finsler space $F_n$ in which the projective curvature tensor $W_{ik}^{jl}$ and the tensor
$M_{kh}$ are generalized recurrent tensors.

Differentiating (2.4a) covariantly with respect to $x^m$ in the sense of Cartan, we get
\[(4.9) N_{iklm}^{jl} = W_{ik}^{jl}m - (\delta^i_k M_{jl}^{m} - M_{klm}^{jl} - k[h]).\]

Using (4.3), (4.6) and the properties $\delta^i_k$ in (4.9), we get
\[(4.10) N_{iklm}^{jl} = \lambda_m W_{ik}^{jl}m - (\delta^i_k M_{jl}^{m} - M_{klm}^{jl} - k[h]) + \mu_m (\delta^i_k \gamma_{jk} - \gamma^i_{gkh}).\]

Using (2.4a) in (4.10), we get
\[(4.11) N_{iklm}^{jl} = \lambda_m N_{iklm}^{jl} + \mu_m (\delta^i_k \gamma_{jk} - \gamma^i_{gkh}).\]

Thus, the following theorem

**Theorem 4.3.** In Finsler space $F_n$, if the projective curvature tensor $W_{ik}^{jl}$ and the tensor $M_{kh}$ are
generalized recurrent tensors, then the space considered is necessarily generalized $N_{lm}$-recurrent space.

Reference

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فضاء فنسلر الذي يحقق فيه الموتر التقوسي الإسقاطي العادي $\mathcal{N}_{jk}^{|m}$ الشرط الأتية:

$$\mathcal{N}_{jk|m} = \lambda_m \mathcal{N}_{jk} + \mu_m (\delta_h^j g_{jk} - \delta_h^k g_{jh}), \quad \mathcal{N}_{jk|m} \neq 0,$$

حيث $\lambda_m$ و $\mu_m$ هي متجهات متحدة الاختلاف لا تساوي الصفر، وتم تسمية هذا الفضاء الذي يحقق الشرط أعلاه بعموم فضاء $\mathcal{N}_{jk|m}$ - أحادي المعودة. كما أثبت أن المتجه التقوسي $\mathcal{N}_{jk}$، الثابت التقوسي $H_k$, وموتر ريشي $\mathcal{N}_{jk|m}$ - أحادي المعودة. وكذلك لكي يكون موتر ريشي معتمد كرتان $H_{jk}$ غير منتظرية عندما يكون عموم $\mathcal{N}_{jk|m}$ - أحادي المعودة هو فضاء أفينلي وتمتلك المشتقة الاتجاهية بالنسبة للإحداثي الإقليمي لتتجهات $\mathcal{W}_{jk|m}$ - أحادي المعودة أثبت أن موثر ويلي التقوسي الإسقاطي $\mathcal{N}_{jk|m}$ وهو عموم أحادي المعودة. ملف المفتاحية: تعميم فضاء $\mathcal{N}_{jk|m}$ - أحادي المعودة، تعميم موثر حادي المعودة، تعميم فضاء أفينلي $\mathcal{N}_{jk|m}$ - أحادي المعودة، تعميم موثر ويلي التقوسي الإسقاطي.