

On regular generalized N–Preopen sets

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DOI: <https://doi.org/10.47372/uajnas.2020.n1.a18>

Abstract

The purpose of this paper is to provide a new class of generalized N–preopen sets, namely, regular generalized N–preopen sets which is finer than the class of regular generalized preopen sets and the class of regular generalized open sets. Furthermore, we study the fundamental topological properties and introduce the notion of regular generalized N–precontinuous functions.

Keywords: Regular open; Preopen set; Generalized closed set; Decomposition of continuity; AMS classification: Primary 54A05, 54A10, 54C10.

1. Introduction

Through this paper, all topological spaces assumed no with separation axioms. Let A be a subset of a topological space (X, τ) . The closure and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. A is called preopen set [8] if $A \subseteq Int(Cl(A))$. The complement of preopen set is called preclosed set. The p –closure set of A is defined as the intersection of all preclosed subsets of X containing A and is denoted by $Clp(A)$. The p –interior set of A is defined as the union of all preopen subsets of X contained in A and is denoted by $Intp(A)$. A subset A is called regular open set (simply r –open) [12] if $A = Int(Cl(A))$. The complement of r –open set is called regular closed (simply r –closed).

A subset A of topological space (X, τ) is called a N –preopen set [1] if, for each $x \in A$, there exists a preopen set U_x containing x such that $U_x - A$ is a finite set. The complement of N –preopen set is called N –preclosed set. The N –closure set of A is defined as the intersection of all N –preclosed subsets of X containing A and is denoted by $Cl_N(A)$. The N –interior set of A is defined as the union of all N –preopen subsets of X contained in A and is denoted by $Int_N(A)$.

In 1970, Levine [7] introduced the notion of generalized closed sets. A subset A of a topological space (X, τ) is called generalized closed (simply g –closed) set if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of X . The complement of g –closed set is called generalized open (simply g –open) set. In [9] they introduced the notion of generalized preclosed sets. A subset A of a topological space (X, τ) is called generalized preclosed (simply g –preclosed) set if $Clp(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ) . The complement of g –preclosed set is called generalized preopen (simply g –preopen) set.

In 1993, Palaniappan and Rao [11] introduced the notion of regular generalized closed sets. A subset A of a topological space (X, τ) is called regular generalized closed (simply rg –closed) set, if $Cl(A) \subseteq U$, whenever $A \subseteq U$ and U is r –open subset of X . The complement of rg –closed set is called regular generalized open (simply rg –open) set. In [2], Al-Omari and Noorani introduced the notion of regular Generalized ω –closed sets, and [5] introduced the notion of regular generalized preclosed sets. A subset A of a topological space (X, τ) is called regular generalized preclosed set (simply rg –preclosed), if $Clp(A) \subseteq U$, whenever $A \subseteq U$ and U is r –open subset of (X, τ) . The complement of rg –preclosed set is called regular generalized preopen (simply rg –preopen).

This paper is organized as follow: Section 2 is devoted to some preliminaries. In Section 3, we give the concept of regular generalized N –preopen sets by utilizing the N –closure operator and we

On regular generalized N–Preopen sets.....Khaled M. A. Al-Hamadi, Ali Qassem, Amin Saif study its topological properties. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the notion of generalized N–precontinuous functions and approximately N–precontinuous.

2. Preliminaries

In this section, we provide some preliminary works that serve as background for the present study.

Theorem 2.1. [12] A subset A of a topological space (X, τ) is a r -closed set if and only if, $A = Cl(Int(A))$.

Theorem 2.2. [12] Let Y be an open subset of a topological space (X, τ) . If, A is a r -open set in $(Y, \tau|_Y)$, then $A = G \cap Y$ for some a r -open set G in X .

Theorem 2.3. [12] Let (X, τ) and (Y, ρ) be two topological spaces. If A is a r -open set (X, τ) and B is a r -open set (Y, ρ) , then $A \times B$ is a r -open set $(X \times Y, \tau \times \rho)$.

Theorem 2.4. [8] Let A and B be two subsets in a topological space (X, τ) . If A is a preopen set in X and B is an open set in X , then $A \cap B$ is a preopen set in X .

Theorem 2.5. [8] Let (X, τ) and (Y, ρ) be two topological spaces. If $A \times B$ is a preopen set in $(X \times Y, \tau \times \rho)$ if, and only if, A is a preopen set in (X, τ) and B is a preopen set in (Y, ρ) .

Lemma 2.6. [3] For a topological space (X, τ) and $A \subseteq X$, the following hold:

1. $Intn(X - A) = X - Cln(A)$.
2. $Cln(X - A) = X - Intn(A)$.

Definition 2.7. [3] A subset A of a topological space (X, τ) is called generalized N–preclosed set (simply Ng–preclosed), if $Cln(A) \subseteq U$, whenever $A \subseteq U$ and U is open subset of (X, τ) . The complement of Ng–preclosed set is called generalized N–preopen set (simply Ng–preopen).

Lemma 2.8. [3] Let Y be an open subset of a topological space (X, τ) . Then the following hold:

1. If A is a N–preopen set in (X, τ) , then $A \cap Y$ is a N–preopen set in $(Y, \tau|_Y)$.
2. If A is a N–preclosed set in $(Y, \tau|_Y)$, then A is a N–preclosed set in (X, τ) .
3. If A is a N–preopen set in $(Y, \tau|_Y)$, then A is a N–preopen set in (X, τ) .
4. If $A \subseteq Y$ then $Cln|_Y(A) = Cln(A) \cap Y$.

Definition 2.9. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a topological space (Y, ρ) is called:

1. precontinuous function [8] if $f^{-1}(U)$ is a preopen set X for every open set U in Y
2. generalized precontinuous function (simply g–precontinuous function) [10] if $f^{-1}(U)$ is a g–preopen set in X for every open set U in Y .
3. N–precontinuous function [1] if $f^{-1}(U)$ is a N–preopen set in X for every open set U in Y .
4. generalized continuous function (simply g–continuous function) if $f^{-1}(U)$ is a g–open set in X for every open set U in Y .

5. Ng–precontinuous function [3] if $f^{-1}(U)$ is a Ng–preopen set in X for every open set U in Y .
6. regular generalized continuous function (simply rg–continuous function) [11] if $f^{-1}(U)$ is a rg–open set in X for every open set U in Y .
7. regular generalized precontinuous function (simply rg–precontinuous function) [5] if $f^{-1}(U)$ is a rg–preopen set in X for every open set U in Y .

3. Regular generalized N–preopen sets

Definition 3.1. A subset A of a topological space (X, τ) is called regular generalized N–preclosed set (simply RNg–preclosed), if $\text{Cl}_n(A) \subseteq U$, whenever $A \subseteq U$ and U is r–open subset of (X, τ) . The complement of RNg–preclosed set is called regular generalized N–preopen set (simply RNg–preopen).

Example 3.2. Let (\mathbb{R}, τ_u) be the real usual topological space on the set of real numbers \mathbb{R} and $a \in \mathbb{R}$. The set $(a, +\infty)$ is not RNg–preclosed set, since $(a, +\infty)$ is a r–open and it is not N–preclosed set. So, $\text{Cl}_n((a, +\infty)) \not\subseteq (a, +\infty)$. To show that $(a, +\infty)$ is not N–preclosed set, suppose that it is an N–preclosed set. Then, $(-\infty, a]$ is a N–preopen set. Hence there is a preopen set U_a in \mathbb{R} containing such that $U_a \cap (a, +\infty) = U_a - (-\infty, a]$ is a finite set. Since U_a is a preopen set and $(a, +\infty)$ is an open set in \mathbb{R} , then, by Theorem(2.4), the finite set $U_a \cap (a, +\infty)$ is a preopen in \mathbb{R} and

$$U_a \cap (a, +\infty) \subseteq \text{Int}(\text{Cl}[U_a \cap (a, +\infty)]) = \text{Int}([U_a \cap (a, +\infty)]) = \emptyset.$$

This implies $U_a \subseteq (-\infty, a]$, but this is a contradiction since

$$a \in U_a \subseteq \text{Int}(\text{Cl}(U_a)) \subseteq \text{Int}(\text{Cl}((-\infty, a])) = \text{Int}((-\infty, a]) = (-\infty, a).$$

It is clear that every Ng–preclosed set is RNg–preclosed set. The converse of this fact need not be true.

Example 3.3. In topological space (N, T) , $N = \{1, 2, 3, 4, \dots\}$,
 $T = \{\emptyset\} \cup \{E_n : n \in \mathbb{N}\}$, $E_n = \{n, n + 1, n + 2, \dots\}$,

the set E_2 is RNg–preclosed set since N is the only r–open containing E_2 . The set E_2 is not Ng–preclosed set since E_2 is an open set and is not N–preclosed set, that is, $\text{Cl}_n(E_2) \not\subseteq E_2$. The set E_2 is not N–preclosed set since there is no a finite preopen subset of N containing $x = 1$. Let U_x be a preopen set in N containing $x = 1$ such that $U_x - \{1\}$ is a finite set. Then U_x will be a finite set in N. Since U_x is a preopen set in N, then

$$U_x - \text{Int}(\text{Cl}(U_x)) = \text{Int}[\{1, 2, 3, \dots, \text{Max}(U_x)\}] = \emptyset.$$

This is contradiction.

Theorem 3.4. Every rg–preclosed set is RNg–preclosed set.

Proof. Let A be rg–preclosed subset of a topological space (X, τ) and U be any r–open set such that $A \subseteq U$. Since A is a rg–preclosed set, $\text{Cl}_p(A) \in U$. Since $\text{Cl}_n(A) \in \text{Cl}_p(A)$, then $\text{Cl}_n(A) \in U$. Therefore, A is a RNg–preclosed set.

In Example (3.3), E_2 is RNg–preclosed set which is not Ng–preclosed set. That is, the set $E_2^c = \{1\}$ is RNg–preopen set which is not Ng–preopen set.

Remark 3.5. For any topological space (X, τ) , if X is a finite, then it's clear that every subset of X is a both RNg–preclosed and RNg–preopen set.

Example 3.6. Let (X, τ) be a topological space where $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$.

By the above remark, the set $\{b, c, d\}$ is RNg-preopen set, but is not rg-preopen set since $X - \{b, c, d\} = \{a\}$ is r-open set and $\text{Clp}(\{a\}) = \{a, c, d\} \not\subseteq \{a\}$.

We have the following implications for RNg-preopen set with theses sets:

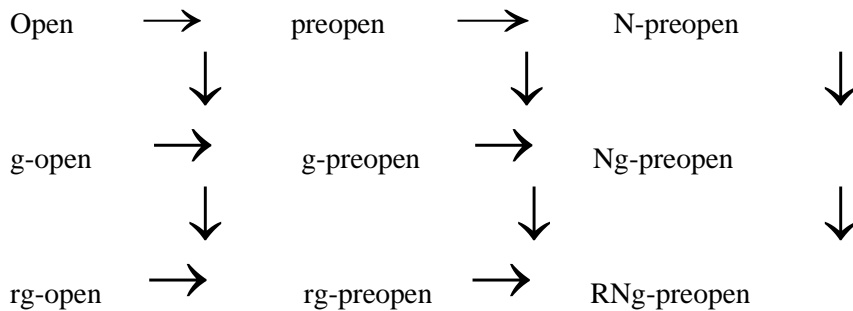


Figure 1

Theorem 3.7. A subset A of a topological space (X, τ) is a RNg-preopen set if, and only if, $F \subseteq \text{Intn}(A)$ whenever $F \subseteq A$ and F is a r-closed subset of (X, τ) .

Proof. Let A be a RNg-preopen subset of X and F be a r-closed subset of X such that $F \subseteq A$. Then $X - A$ is a RNg-preclosed, $X - A \subseteq X - F$ and $X - F$ is a r-open subset of X . Hence, by Lemma (2.6), $X - \text{Intn}(A) = \text{Cln}(X - A) \subseteq X - F$, that is, $F \subseteq \text{Intn}(A)$.

Conversely, suppose that $F \subseteq \text{Intn}(A)$ where F is a r-closed subset of X such that $F \subseteq A$. Then for any r-open subset U of X such that $X - A \subseteq U$, we have $X - U \subseteq A$ and $X - U \subseteq \text{Intn}(A)$. Then by, Lemma(2.6), $X - \text{Intn}(A) = \text{Cln}(X - A) \subseteq U$. Hence $X - A$ is a RNg-preclosed set. That is, A is a RNg-preopen set.

Theorem 3.8. If A is a RNg-preclosed subset of a topological space (X, τ) , then $\text{Cln}(A) - A$ does not contain any nonempty r-closed set.

Proof. Suppose that F be a r-closed subset of X such that $F \subseteq \text{Cln}(A) - A$. Then $F \subseteq X - A$ and hence $A \subseteq X - F$. Since A is a RNg-preclosed set and $X - F$ is a r-open subset of X , then $\text{Cln}(A) \subseteq X - F$ and so $F \subseteq X - \text{Cln}(A)$. Therefore, $F \subseteq \text{Cln}(A) \cap (X - \text{Cln}(A)) = \phi$. That is, $F = \phi$.

Corollary 3.9. If A is a RNg-preclosed subset of a topological space (X, τ) , then $\text{Cln}(A) - A$ is a RNg-preopen set.

Proof. By Theorem(3.8), $\text{Cln}(A) - A$ does not contain any nonempty r-closed set. That is, the only r-closed set contained in $\text{Cln}(A) - A$ is ϕ . Hence $F = \phi \subseteq \text{Intn}(\text{Cln}(A) - A)$. By Theorem(3.7), $\text{Cln}(A) - A$ is a RNg-preopen set.

Theorem 3.10. If A is a RNg-preclosed subset of a topological space (X, τ) and $B \subseteq X$. If $A \subseteq B \subseteq \text{Cln}(A)$, then B is a RNg-preclosed set.

Proof. Let U be any r -open set in X such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since A is a RNg -preclosed set, then $Cl_n(A) \subseteq U$. Since $B \subseteq Cl_n(A)$ then

$$Cl_n(B) \subseteq Cl_n[Cl_n(A)] = Cl_n(A) \subseteq U.$$

That is, B is a RNg -preclosed set.

Theorem 3.11. Let A be a RNg -preclosed subset of a topological space (X, τ) . Then $A = Cl_n(Int_n(A))$ if and only if $Cl_n(Int_n(A)) - A$ is a r -closed set.

Proof. Let $Cl_n(Int_n(A)) - A$ be a r -closed set. Since $Int_n(A) \subseteq A$ and $A \subseteq Cl_n(A)$, then $Cl_n(Int_n(A)) - A \subseteq X - A \Rightarrow A \subseteq X - (Cl_n(Int_n(A)) - A)$. Since A is a RNg -preclosed set and $X - (Cl_n(Int_n(A)) - A)$ is a r -open set containing A , then $Cl_n(A) \subseteq X - (Cl_n(Int_n(A)) - A)$, this implies

$$Cl_n(Int_n(A)) - A \subseteq X - Cl_n(A).$$

Therefore,

$$Cl_n(Int_n(A)) - A \subseteq Cl_n(A) \cap (X - Cl_n(A)) = \phi.$$

Hence $Cl_n(Int_n(A)) - A = \phi$, that is, $Cl_n(Int_n(A)) = A$.

Conversely, if $A = Cl_n(Int_n(A))$ then $Cl_n(Int_n(A)) - A = \phi$ and, hence, $Cl_n(Int_n(A)) - A$ is a r -closed set.

Theorem 3.12. Let Y be an open subspace of a topological space (X, τ) and $A \subseteq Y$. If A is a RNg -preclosed subset of X , then A is a RNg -preclosed set in Y .

Proof. Let O be a r -open subset of Y such that $A \subseteq O$. Then, by Theorem (2.2), $O = U \cap Y$ for some r -open set U in X and so $A \subseteq U$. Since A is a RNg -preclosed subset of X , then $Cl_n(A) \subseteq U$. By Lemma(2.8), $Cl_n|_Y(A) = Cl_n(A) \cap Y \subseteq U \cap Y = O$. Hence A is a RNg -preclosed set in Y .

Lemma 3.13. Let (X, τ) and (Y, ρ) be two topological spaces. If $A \times B$ is a nonempty N -preopen set in $(X \times Y, \tau \times \rho)$, then A is a N -preopen set in (X, τ) and B is a N -preopen set in (Y, ρ) .

Proof. Let $x \in A$ be arbitrary point in A . Since $A \times B$ is a nonempty, take $y \in B$. Since $(x, y) \in A \times B$ and $A \times B$ is a N -preopen set in $(X \times Y, \tau \times \rho)$, then there is a preopen set $U \times G$ in $X \times Y$ containing (x, y) such that $(U \times G) - (A \times B)$ is a finite. Since $(U - A) \times (G - B) \subseteq (U \times G) - (A \times B)$, then $(U - A) \times (G - B)$ is a finite, that is, $U - A$ is also a finite. Since $U \times G$ is a preopen in $X \times Y$, then by, Theorem(2.5), U is a preopen set in (X, τ) and G is a preopen set in (Y, ρ) . Hence A is a N -preopen set in (X, τ) and similarly, B is a N -preopen set in (Y, ρ) .

Lemma 3.14. Let (X, τ) and (Y, ρ) be two topological spaces and $A \times B$ is a subset of $X \times Y$. Then $Int_n(A \times B) \subseteq Int_n(A) \times Int_n(B)$.

Proof. Let $(x, y) \in Int_n(A \times B)$. Then, by the definition of $Int_n(A \times B)$ there is at least one a N -preopen set $U \times G$ in $X \times Y$ such that $(x, y) \in U \times G \subseteq A \times B$. This implies $x \in U \subseteq A$ and y

$\in G \subseteq B$. Since $U \times G$ is a N–preopen set in $X \times Y$, then by Lemma, (3.13), U is a N–preopen set in X and G is a N–preopen set in Y . Then $x \in \text{Intn}(A)$ and $y \in \text{Intn}(B)$. This implies $(x, y) \in \text{Intn}(A) \times \text{Intn}(B)$. Therefore $\text{Intn}(A \times B) \subseteq \text{Intn}(A) \times \text{Intn}(B)$.

Theorem 3.15. Let (X, τ) and (Y, ρ) be two topological spaces. If $A \times B$ is a nonempty RNg–preopen set in $(X \times Y, \tau \times \rho)$, then A is a RNg–preopen set in (X, τ) and B is a RNg–preopen set in (Y, ρ) .

Proof. Let F_1 be a r –closed set in X and F_2 be a r –closed set in Y such that $F_1 \subseteq A$ and $F_2 \subseteq B$. By Theorem(2.3), $F_1 \times F_2$ is a r –closed set in $X \times Y$. Since $A \times B$ is a nonempty RNg–preopen set in $(X \times Y, \tau \times \rho)$ and by Lemma (3.14), then

$$F_1 \times F_2 \subseteq \text{Intn}(A \times B) \subseteq \text{Intn}(A) \times \text{Intn}(B).$$

This implies $F_1 \subseteq \text{Intn}(A)$ and $F_2 \subseteq \text{Intn}(B)$. Hence A is a RNg–preopen set in (X, τ) and B is a RNg–preopen set in (Y, ρ) .

4. Regular generalized N –precontinuous functions:

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is called regular generalized N–precontinuous function (simply RNg–precontinuous) if $f^{-1}(U)$ is a RNg–preopen set in X for every open set U in Y .

Theorem 4.2. If $f : (X, \tau) \rightarrow (Y, \rho)$ is a RNg–precontinuous function, then for each $x \in X$ and each open set U in Y with $f(x) \in U$, there exists a RNg–preopen set V in X such that $x \in V$ and $f(V) \subseteq U$.

Proof. Let $x \in X$ and U be any open set in Y containing $f(x)$. Put $V = f^{-1}(U)$. Since f is a RNg–precontinuous, then V is a RNg–preopen set in X such that $x \in V$ and $f(V) \subseteq U$.

The converse of the last theorem need not be true.

Example 4.3. Let $f : (N, T) \rightarrow (Y, \rho)$ be a function defined by $f(n) = a$ if $n \in D$ and $f(n) = b$ if $n \notin D$, where

$$T = \{\emptyset, N, \{1\}, E_n, \{1\} \cup E_n : E_n = \{n, n + 1, n + 2, \dots\}, n \in N \text{ and } n \geq 6\},$$

$$Y = \{a, b\}, \rho = \{\emptyset, Y, \{a\}\} \text{ and } D = \{1, 2, 3, 4, 5\}.$$

The function f is not a RNg–precontinuous, see Example (3.8), $f^{-1}(\{a\}) = D$ is not RNg–preopen set in N . On the other hand, for each $n \in N$ and each open set U in Y containing $f(n)$, the set $V = \{n\}$ is a RNg–preopen set in N containing n and $f(V) \subseteq U$.

It is clear that every Ng–precontinuous function is RNg–precontinuous and the converse of this fact need not be true.

Example 4.4. Let $f : (N, T) \rightarrow (Y, \rho)$ be a function defined by $f(1) = a$ and $f(n) = b$ if $n \neq 1$, where

$$N = \{1, 2, 3, 4, \dots\}, T = \{\emptyset\} \cup \{E_n : n \in N\}, E_n = \{n, n + 1, n + 2, \dots\},$$

$$Y = \{a, b\} \text{ and } \rho = \{\emptyset, Y, \{a\}\}.$$

The function f is a RNg–precontinuous, since $f^{-1}(\{a\}) = \{1\}$ and $f^{-1}(Y) = N$ are RNg–preopen sets in N . The function f is not Ng–precontinuous, see Example (3.3), $f^{-1}(\{a\}) = \{1\}$ is not Ng–preopen sets in N .

It is clear that every rg-precontinuous function is RNg-precontinuous function and the converse of this fact need not be true.

Example 4.5. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a function defined by $f(1) = f(2) = c$, $f(3) = a$, and $f(4) = b$ where $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c\}$, $\tau = \{\emptyset, X, \{1, 2\}, \{4\}, \{1, 2, 4\}\}$ and $\rho = \{\emptyset, Y, \{a, b\}\}$. The function f is a RNg-precontinuous, see Example (??) $f^{-1}(\{a, b\}) = \{3, 4\}$ and $f^{-1}(Y) = X$ are RNg-preopen sets in X . The set $f^{-1}(\{a, b\}) = \{3, 4\}$ is not rg-preopen set in X , since $\{1, 2\}$ is a r-open set in X and $X - \{3, 4\} = \{1, 2\} \subseteq \{1, 2\}$ but $\text{Clp}(X - \{3, 4\}) = \text{Clp}(\{1, 2\}) = \{1, 2, 3\} \not\subseteq \{1, 2\}$,

that is, the function f is not rg-precontinuous.

We have the following implications for RNg-precontinuous function with the other known functions:

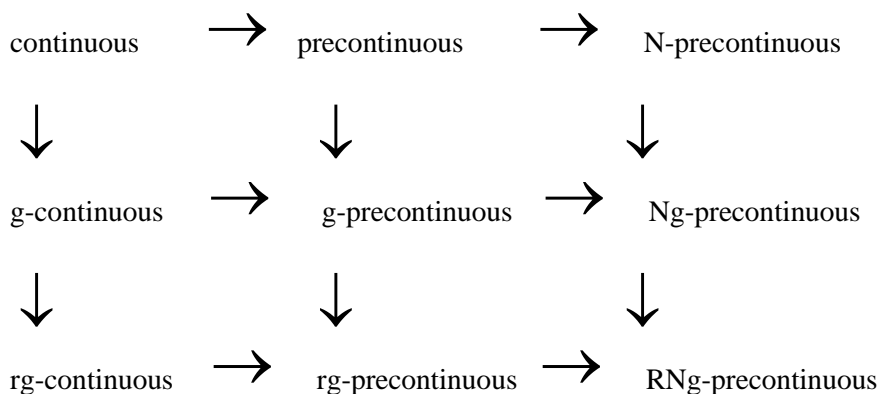


Figure 2

Theorem 4.6. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is RNg-precontinuous if, and only if, $f^{-1}(F)$ is a RNg-preclosed set in X for every closed set F in Y .

Proof. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a N-precontinuous and F be any closed set in Y . Then $f^{-1}(Y - F) = X - f^{-1}(F)$ is a RN-preopen set in X , that is, $f^{-1}(F)$ is RNg-preclosed set in X . Conversely, suppose that $f^{-1}(F)$ is a RNg-preclosed set in X for every closed set F in Y . Let U be any open set in Y . Then, by the hypothesis, $f^{-1}(Y - U) = X - f^{-1}(U)$ is a RNg-preclosed set in X , that is, $f^{-1}(U)$ is a RNg-preopen set in X . Hence f is a RN-precontinuous.

Definition 4.7. A topological space (X, τ) is called regular a generalized N- $T_{1/2}$ -space (simply $N_g^r T_{1/2}$ -space) if every RNg-preclosed set in X is N-preclosed.

Theorem 4.8. A topological space (X, τ) is $N_g^r T_{1/2}$ -space if, and only if, every RNg–preopen set in X is N–preopen.

Lemma 4.9. A topological space (X, τ) is $N_g^r T_{1/2}$ -space if, and only if, every singleton set in X is either r -closed or N–preopen.

Proof. Suppose that (X, τ) is $N_g^r T_{1/2}$ -space and $\{x\}$ is not r -closed subset of X for some $x \in X$. Then $X - \{x\}$ is not r -open set in X . Hence X is the only r -open set containing $X - \{x\}$. That is, $X - \{x\}$ is a RNg–preclosed set in X . Since (X, τ) is $N_g^r T_{1/2}$ -space, then $X - \{x\}$ is a N–preclosed set in X . That is, $\{x\}$ is a N–preopen set in X .

Conversely, suppose that every singleton set in X is either r -closed or N–preopen. Let A be any RNg–preclosed set in X and $x \in \text{Cln}(A)$. We show that $x \in A$. By the Hypothesis, $\{x\}$ is either r -closed or N–preopen set in X . If $\{x\}$ is r -closed and $x \notin A$, then $x \in \text{Cln}(A) - A \subseteq X - A$. Then $\{x\} \subseteq X - A$ and so $A \subseteq X - \{x\}$. Since A is a RNg–preclosed set in X contained in r -open set $X - \{x\}$, then $\text{Cln}(A) \subseteq X - \{x\}$ and so $\{x\} \subseteq X - \text{Cln}(A)$. Therefore $\{x\} \subseteq \text{Cln}(A) \cap [X - \text{Cln}(A)] = \emptyset$, and this is a contradiction. Hence $x \in A$, that is, $\text{Cln}(A) = A$ and so A is a N–preclosed. If $\{x\}$ is N–preopen and $x \in \text{Cln}(A)$, then we have $\{x\} \cap A \neq \emptyset$; Hence $x \in A$, that is, $\text{Cln}(A) = A$ and so A is a N–preclosed.

Lemma 4.10. Let (X, τ) be a topological space. If (X, τ) is a $N_g^r T_{1/2}$ -space, then every RNg–preclosed set in X is N–preclosed.

Proof. Let A be a RNg–preclosed set in X . Suppose that A is not N–preclosed set. Then there is at least $x \in \text{Cln}(A)$ such that $x \notin A$. Since (X, τ) is a $N_g^r T_{1/2}$ -space then by Lemma(4.9), $\{x\}$ is either r -closed or N–preopen in X . If $\{x\}$ is a r -closed set in X then $X - \{x\}$ is a r -open set in X . Since $x \notin A$, then $A \subseteq X - \{x\}$. Since A is a RNg–preclosed set and $X - \{x\}$ is a r -open subset of X containing A , then $\text{Cln}(A) \subseteq X - \{x\}$. Hence $x \in X - \text{Cln}(A)$ and this a contradiction, since $x \in \text{Cln}(A)$. If $\{x\}$ is a N–preopen and since $x \in \text{Cln}(A)$, then we have $\{x\} \cap A \neq \emptyset$. That is, $x \in A$ and this a contradiction. Hence A is a N–preclosed set in X .

Lemma 4.11. Let (X, τ) be a topological space. If (X, τ) is a $N_g^r T_{1/2}$ -space, then every RNg–preclosed set in X is Ng–preclosed.

Proof. From Lemma(4.10).

Theorem 4.12. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a function of a $N_g^r T_{1/2}$ -space (X, τ) into a space (Y, ρ) . If f is a RNg–precontinuous, then it is a Ng–precontinuous.

Proof. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a RNg–precontinuous function and U be any open set in Y . Then $f^{-1}(U)$ is a RNg–preopen set in X . Since X is a $N_g^r T_{1/2}$ -space, then by Lemma, (4.11), $f^{-1}(U)$ is a Ng–preopen set in X . That is, f is a RN–precontinuous function.

Theorem 4.13. Let $f : (X, \tau) \rightarrow (Y, \rho)$ be a function of a N_g^r $T_{1/2}$ -space (X, τ) onto a space (Y, ρ) . Then the following are equivalent:

1. f is g -precontinuous.
2. f is N_g -precontinuous.
3. f is RN_g -precontinuous.

Proof. 1 \Rightarrow 2: Trivial.

2 \Rightarrow 3: Trivial.

3 \Rightarrow 1: By Theorem (4.12).

Lemma 4.14. Let Y be an open subspace of a topological space (X, τ) and $A \subseteq Y$. If A is a RN_g -preclosed subset in X , then A is a RN_g -preclosed set in Y .

Proof. Let O be any r -open subset in Y such that $A \subseteq O$. Then $O = U \cap Y$ for some r -open set U in X and so $A \subseteq U$. Since A is a RN_g -preclosed subset of X , then $Cl_X(A) \subseteq U$. By Lemma(2.8),

$$Cl_Y(A) = Cl_X(A) \cap Y \subseteq U \cap Y = O.$$

Hence A is a RN_g -preclosed set in Y .

Theorem 4.15. If $f : (X, \tau) \rightarrow (Y, \rho)$ is a RN_g -precontinuous function and A is an open subspace of a topological space (X, τ) , then the restriction function $f|_A : (A, \tau|_A) \rightarrow (Y, \rho)$ of f on A is a RN_g -precontinuous.

Proof. Let U be an open set in Y . since f is a RN_g -precontinuous then $f^{-1}(U)$ is a RN -preopen set in X . Since A is an open in X , then $A \cap f^{-1}(U)$ is a RN -preopen set in X .

Then $f^{-1}(U) \cap A = (f|_A)^{-1}(U)$ is a RN_g -preopen set in X . Hence, by Lemma(4.14), $(f|_A)^{-1}(U) \subseteq A$ is a RN_g -preopen set in A . That is, $f|_A$ is a RN_g -precontinuous.

Definition 4.16. A function $f : (X, \tau) \rightarrow (Y, \rho)$ of a topological space (X, τ) into a space (Y, ρ) is called an approximately N -precontinuous function if $Cl_X(A) \subseteq f^{-1}(V)$ whenever V is a r -open subset of Y , A is a RN_g -preclosed subset of X and $A \subseteq f^{-1}(V)$.

Theorem 4.17. A topological space (X, τ) is N_g^r $T_{1/2}$ -space if, and only if, every function $f : (X, \tau) \rightarrow (Y, \rho)$ is an approximately N -precontinuous for every topological space (Y, ρ) .

Proof. Suppose that (X, τ) is N_g^r $T_{1/2}$ -space, (Y, ρ) be any topological space and $f : (X, \tau) \rightarrow (Y, \rho)$ be any function from (X, τ) into (Y, ρ) . We show that f is an approximately N -precontinuous. Let V be a r -open subset of Y and A is a RN_g -preclosed subset of X such that $A \subseteq f^{-1}(V)$. Since (X, τ) is N_g^r $T_{1/2}$ -space, then A is a N -preclosed, that is, $Cl_X(A) = A$. Then $Cl_X(A) = A \subseteq f^{-1}(V)$. Hence f is an approximately N -precontinuous.

Conversely, Let A be any RN_g -preclosed subset of X . If $A = \emptyset$; then A is a N -preclosed. If $A \neq \emptyset$; then take $Y = X$, $\rho = \{\emptyset, Y, A\}$ and $f : (X, \tau) \rightarrow (Y, \rho)$ is the identity function. By the hypothesis, f is an approximately N -precontinuous. Since A is a RN_g -preclosed subset of X and

open in Y such that $A \subseteq f^{-1}(A)$, then $\text{Cln}(A) \subseteq f^{-1}(A) = A$. Then $\text{Cln} = A$, that is, A is a N -preclosed. Hence (X, τ) is $N_g^r T_{1/2^-}$ space.

Theorem 4.18. If the r -open and r -closed subset of X coincide, then a function $f : (X, \tau) \rightarrow (Y, \rho)$ is an approximately N -precontinuous if, and only if, $f^{-1}(V)$ is a N -preclosed set in X for every r -open subset V of Y .

Proof. Suppose that $f : (X, \tau) \rightarrow (Y, \rho)$ is an approximately N -precontinuous and V is a r -open subset of Y . Firstly, we show that $f^{-1}(V)$ is a RNg -preclosed set in X . Let U be any r -open set in X containing $f^{-1}(V)$. By the hypothesis, U is r -closed and so closed in X . Then

$$\text{Cln}(f^{-1}(V)) \subseteq \text{Cln}(U) \subseteq \text{Cl}(U) = U.$$

That is, $f^{-1}(V)$ is a RNg -preclosed set in X . Since V is a r -open subset of Y , $f^{-1}(V) \subseteq f^{-1}(V)$ and f is an approximately N -precontinuous, then $\text{Cln}[f^{-1}(V)] \subseteq f^{-1}(V)$. That is, $f^{-1}(V)$ is a N -preclosed set in X .

Conversely, Let V be a r -open subset of Y and A is a RNg -preclosed subset of X such that $A \subseteq f^{-1}(V)$. Then by the hypothesis, $f^{-1}(V)$ is a N -preclosed set in X . Hence $\text{Cln}(A) \subseteq \text{Cln}[f^{-1}(V)] = f^{-1}(V)$, that is, f is an approximately N -precontinuous.

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حول مجموعات N – ما قبل المفتوحة المعممة المنتظمة

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DOI: <https://doi.org/10.47372/uajnas.2020.n1.a18>

الملخص

في هذا البحث تم دراسة نوع جديد من مجموعات N – ما قبل المفتوحة المعممة، تسمى مجموعات N – ما قبل المفتوحة المعممة المنتظمة، وتقديم مفهوم دوال N – ما قبل المتصلة المعممة المنتظمة.

الكلمات المفتاحية: المجموعة المنتظمة المفتوحة، المجموعة ما قبل المفتوحة، المجموعة المعممة المغلقة، الاتصال.