Generalized Fuzzy q-open sets

[1]Rdhwan Mohammed Aqeel and [2]Rashed Alkhadher Nasser
[1]Department of Mathematics, Faculty of Science, Aden University, Aden, Yemen
[2]Department of Mathematics, Faculty of Education - Lawder, Abyan University, Lawder, Yemen
E-mail: rashednasser101@yahoo.com E-mail: raqeel1976@yahoo.com
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Abstract

In this paper, we introduce the concepts of μ-fuzzy q-open sets which is generalization of simply open sets defined by Neubrunnove [9]. We also introduce and investigate, with the help of this new concept, the concepts of qi_μ-Fuzzy open sets and qc_μ-Fuzzy closed sets. The relations between these concepts are investigated and several examples are presented.

Keywords: Generalized fuzzy topological spaces, μ-fuzzy q-open sets, qi_μ-Fuzzy open sets and qc_μ-Fuzzy closed sets.

1.Introduction and Preliminaries

The potential of the notion of fuzzy set studied by L. A. Zadeh [12] was realized by the researchers and has successfully been applied for new investigations in all the branches of science and technology for more than last five decades. Since Chang [3] defined the concept of a fuzzy topology, then many authors investigated different properties of fuzzy open sets which are weaker than the property of openness of a fuzzy set in a fuzzy topological space. For example, ([1],[10],[11],[13]) have considered such kind of properties of fuzzy sets and most of the collection forms a fuzzy supra topology therein. A significant contribution to the theory of generalized open sets has been reported by A. Csaszar ([6], [7], [8]) and extended by G. P. Chetty [5] in the context of fuzzy set theory with the name of generalized fuzzy topological space. Our aim is to study the parallel concept of topology in a given fuzzy space with an incomparable nature.

In the present paper, we introduce the concept of μ-fuzzy q-open sets and study some of their properties. Finally, we discuss about some fundamental properties of such structure and some related notions. In particular, we have shown that μ-fuzzy q-open sets is a weaker form of $\mu - fuzzy\ semi - open$ sets introduce by G. Palani Chettry [5] in 2008. Lastly, we define qi_μ-Fuzzy open set which is a weaker than μ-fuzzy q-open set.

We , now, state a few definitions and results that are required in our work.

Let X be a nonempty set and $\mathcal{F} = \{\lambda \mid \lambda : X \to [0,1]\}$ be the family of all fuzzy sets defined on X. A subfamily μ of $\mathcal{F}$ is called a *generalized fuzzy topology* (GFT) [16] if $0 \in \mu$ and $\vee\{\lambda_\alpha \mid \alpha \in \Delta\} \in \mu$ whenever $\lambda_\alpha \in \mu$ for every $\alpha \in \Delta$. For $\lambda \in \mathcal{F}$, the $\mu - interior$ of λ, denoted by $i_\mu(\lambda)$, is given by $i_\mu(\lambda) = \vee\{\nu \in \mu \mid \nu \leq \lambda\}$. Moreover, in [2], it is established that for all $\lambda \in \mathcal{F}$, $i_\mu(\lambda) \leq \lambda$, $i_\mu i_\mu(\lambda) = i_\mu(\lambda)$ and $\lambda \in \mu$ if and only if $\lambda = i_\mu(\lambda)$. A fuzzy set $\lambda \in \mathcal{F}$ is said to be a $\mu - fuzzy\ closed\ set$ if $1 - \lambda$ is a $\mu - fuzzy\ open\ set$. The intersection of all $\mu - fuzzy\ closed\ sets$ containing $\lambda \in \mathcal{F}$ is called the $\mu - closure$ of λ. It is denoted by $c_\mu(\lambda)$ and is given by $c_\mu(\lambda) = \wedge\{\gamma \mid 1 - \gamma \in \mu,\ \lambda \leq \gamma\}$. In [5], it is established that $c_\mu(\lambda) = 1 - i_\mu(1 - \lambda)$ for all $\lambda \in \mathcal{F}$. Finally, a fuzzy subset $\lambda \in \mathcal{F}$ is called $\mu - fuzzy\ regular\ open$ [5] (resp.$\mu - fuzzy\ semi - open$[5], $\mu - fuzzy\ pre\ open$ [4], $\mu - fuzzy\ \alpha - open$[4] and $\mu - fuzzy\ \beta - open$[4]) if $\lambda = i_\mu c_\mu(\lambda)$ (resp. $\lambda \leq c_\mu i_\mu(\lambda)$, $\lambda \leq i_\mu c_\mu(\lambda)$, $\lambda \leq i_\mu c_\mu i_\mu(\lambda)$ and $\lambda \leq c_\mu i_\mu c_\mu(\lambda)$). The complement of a $\mu - fuzzy\ regular\ open$ set is called $\mu - fuzzy\ regular\ closed$; similar as the case for the other types.

Lemma 1.1. For a GFTS (X, μ), then the following statements are equivalent :

(a) Every μ-fuzzy open set of X is μ-fuzzy regular open.

(b) Every fuzzy set of X is μ-fuzzy β-open.

(c) Every fuzzy set of X is μ-fuzzy β-closed.

(d) Every μ-fuzzy open set of X is μ-fuzzy semiclosed.

Proof . (a) ⇒ (b). Let λ be any fuzzy set of , then $1-\lambda$ is fuzzy set of X , by (a) $i_\mu c_\mu i_\mu(1-\lambda) = i_\mu(1-\lambda)$, or equivalently, $c_\mu i_\mu c_\mu \lambda = c_\mu \lambda$ and since $\lambda \le c_\mu \lambda$, then $\lambda \le c_\mu i_\mu c_\mu \lambda$ and hence λ is μ-fuzzy β-open.

(b) ⇒ (c). is clear .

(c) ⇒ (d). Let ν be any μ-fuzzy open set of , by (c) , $i_\mu c_\mu \nu = i_\mu c_\mu i_\mu \nu \le \nu$ and hence ν is μ-fuzzy semiclosed.

(d) ⇒ (a). Let ν be any μ-fuzzy open set of X , by (d) , $i_\mu c_\mu \nu \le \nu$ and since $\nu = i_\mu \nu \le i_\mu c_\mu \nu$. Thus $\nu = i_\mu c_\mu \nu$ and hence ν is μ-fuzzy regular open.

Lemma 1.2. For a GFTS (X, μ), then the following statements are equivalent :

(a) Every μ-fuzzy open set is μ-fuzzy closed.

(b) Every fuzzy set of X is μ-fuzzy preclosed.

(c) Every fuzzy set of X is μ-fuzzy preopen.

Proof . (a) ⇒ (b). Let λ be any fuzzy set of X, by (a) $c_\mu i_\mu \lambda = i_\mu \lambda$, and since $i_\mu \lambda \le \lambda$, then $c_\mu i_\mu \lambda \le \lambda$ and hence λ is μ-fuzzy preclosed.

(b) ⇔ (c). is clear .

(b) ⇒ (a). Let ν be any μ-fuzzy open set of , by (b) , $c_\mu \nu = c_\mu i_\mu \nu \le \nu$ and since $\nu \le c_\mu \nu$ then $c_\mu \nu = \nu$ and hence ν is μ-fuzzy closed.

2. μ-Fuzzy q-open sets.

Definition 2.1. A fuzzy sets λ of a GFTS (X,μ) is called a μ-fuzzy q-open if $i_\mu c_\mu \lambda \le c_\mu i_\mu \lambda$.

Theorem 2.1. Let λ be a fuzzy set of a GFTS (X,μ) , then the following statements are equivalent:

a) λ is a μ-fuzzy q-open.
b) $1-\lambda$ is a μ-fuzzy q-open.
c) $i_\mu c_\mu i_\mu \lambda = i_\mu c_\mu \lambda$.
d) $c_\mu i_\mu c_\mu \lambda = c_\mu i_\mu \lambda$
e) There exist μ-fuzzy open set ν contained in λ and a μ-fuzzy closed set η containing λ such that $i_\mu \eta \le c_\mu \nu$

Proof . (a) ⟹ (b). Let λ be a μ-fuzzy q-open set. Then $i_\mu c_\mu \lambda \le c_\mu i_\mu \lambda$, this implies that $(1 - c_\mu i_\mu \lambda) \le (1 - i_\mu c_\mu \lambda)$ and hence $i_\mu c_\mu (1-\lambda) \le c_\mu i_\mu (1-\lambda)$. Therefore $1-\lambda$ is a μ-fuzzy q-open set.

(b) ⟹ (c). Let $1-\lambda$ be a μ-fuzzy q-open set, then $i_\mu c_\mu (1-\lambda) \le c_\mu i_\mu (1-\lambda) \le i_\mu c_\mu i_\mu (1-\lambda)$. implies $i_\mu c_\mu \lambda \le i_\mu c_\mu i_\mu \lambda$. And since $i_\mu c_\mu i_\mu \lambda \le i_\mu c_\mu \lambda$ for each fuzzy set λ of X, $i_\mu c_\mu \lambda = i_\mu c_\mu i_\mu \lambda$.

(c) ⟹ (d). Obvios.

(d) ⟹ (e) : Since for each fuzzy set λ of X, $i_\mu \lambda \le \lambda \le c_\mu \lambda$. Put $i_\mu \lambda = \nu$ and $c_\mu \lambda = \eta$, i.e., ν ≤ λ ≤ η and by (d) $i_\mu \eta = i_\mu c_\mu \nu \le c_\mu \nu$.

(e) $\Longrightarrow$ (a). Since $\nu \leq \lambda \leq \eta$ where ν is μ-fuzzy open set and η is μ-fuzzy closed set. Then $i_\mu c_\mu \lambda \leq i_\mu \eta$ and $c_\mu \nu \leq c_\mu i_\mu \lambda$. And since $i_\mu \eta \leq c_\mu \nu$, this implies that $i_\mu c_\mu \leq c_\mu i_\mu \lambda$ and hence λ is μ-fuzzy q-open set .

Corollary 2.1. For a GFTS (X, μ) , if $i_\mu(1) \leq c_\mu(0)$, then every fuzzy set of X is μ-fuzzy q-open set.

Remark 2.1. If $\mu = \tau$ is the general topology the μ-fuzzy q-open set is reduced to the simply open set in [9].

Theorem 2.2. Let λ be a fuzzy set of a GFTS (X, μ)

(a) If λ is μ-fuzzy semiopen set, then λ is μ-fuzzy q-open set .

(b) If λ is μ-fuzzy semiclosed set, then λ is μ-fuzzy q-open set.

Proof : (a) Let λ be a μ-fuzzy semiopen set, then $\lambda \leq c_\mu i_\mu \lambda$ implies that $c_\mu \lambda \leq c_\mu i_\mu \lambda$, thus $i_\mu c_\mu \lambda \leq c_\mu \lambda \leq c_\mu i_\mu \lambda$ and hence λ is μ-fuzzy q-open set.

(b). Let λ be a μ-fuzzy semiclosed set, then $(1-\lambda)$ is μ-fuzzy semiopen, by (a) this implies that $(1-\lambda)$ is μ-fuzzy q-open and by Theorem 2.2 λ is μ-fuzzy q-open.

Remark 2.2. The converse of Theorem 2.2 is not true in general, this can be shown by the following example.

Example 2.1 Let $X = [0,1]$ λ and ν be fuzzy sets of X defined as $(x) = 0.5$, $\forall\, x \in X$ and $\lambda(x) = x$. Cleary , μ = { 0 , ν } is a GFT on X and (X, μ) is a GFTS . We note that $i_\mu c_\mu \lambda = i_\mu(1) = \nu$ and $c_\mu i_\mu = c_\mu(0) = \nu$. Since $\leq \nu$, then λ is μ-fuzzy q-open. But $\lambda \nleq \nu = c_\mu i_\mu \lambda$ and $i_\mu c_\mu \lambda = \nu \nleq \lambda$. Thus λ is neither μ-fuzzy semiopen set nor μ-fuzzy semiclosed set.

The next theorem gives us under which condition the μ-fuzzy q-open set is μ-fuzzy semiopen (resp. μ-fuzzy semiclosed).

Theorem 2.3. A μ-fuzzy q-open set λ of a GFTS (X, μ) is μ-fuzzy semiopen (resp. μ-fuzzy semiclosed) if it is a μ-fuzzy β-open (resp. μ-fuzzy β-closed).

Proof. Let λ be a μ-fuzzy β-open set, *i.e.* $\lambda \leq c_\mu i_\mu c_\mu \lambda$. Since λ is μ-fuzzy q-open, then by Theorem 2.1 $c_\mu i_\mu c_\mu \lambda = c_\mu i_\mu \lambda$, this implies that $\lambda \leq c_\mu i_\mu \lambda$, and hence λ is μ-fuzzy semiopen set. The other case can be prove similarly.

Theorem 2.4. A μ-fuzzy preopen set λ (resp. μ-fuzzy preclosed set λ) of a GFTS (X, μ) is μ-fuzzy α-open (resp. μ-fuzzy α-closed) if it is a μ-fuzzy q-open set.

Proof . Let λ be a μ-fuzzy preopen set of X, *i.e.* $\lambda \leq i_\mu c_\mu \lambda$. Since λ is μ-fuzzy q-open, then $i_\mu c_\mu \lambda = i_\mu c_\mu i_\mu \lambda$, this implies that $\lambda \leq i_\mu c_\mu i_\mu \lambda$, and hence λ is μ-fuzzy α-open set. The other case is proved similary.

Remark 2.3. The finite union (resp. finite intersection) of μ-fuzzy q-open sets need not be a μ-fuzzy q-open. This can be shown by the following example.

Example 2.2. Let $X = [0,1]$ and , ν be fuzzy sets of X defined by

$$\lambda(x) = x \text{ and } \nu(x) = \begin{cases} 0, & 0 \leq x \leq \frac{1}{2} \\ 1, & otherwise. \end{cases}$$

Let $\mu = \{ 0, \lambda, \nu, \lambda \cup \nu \}$ be a GFT on X. Cleary $1-\lambda$ and $1-\nu$ are μ-fuzzy q-open sets. Since $i_\mu c_\mu[(1-\lambda) \cup (1-\nu)] = \lambda \cup \nu \nleq 1-(\lambda \cup \nu) = c_\mu i_\mu[(1-\lambda) \cup (1-\nu)]$, then $(1-\lambda) \cup (1-\nu)$ is not μ-fuzzy q-open. And since $(1-\lambda)\cup(1-\nu)=1-(\lambda\cap\nu)$ is not μ-fuzzy q-open. Thus $\lambda\cup\nu$ is not μ-fuzzy q-open although λ and ν are μ-fuzzy q-open sets.

Remark 2.4. The μ-fuzzy q-openness and μ-fuzzy β-openness (resp. μ-fuzzy β-closedness) are independent notions. These can be shown by referring to Example 2.1. and by the next example. From Example 2.1.the μ-fuzzy q-open set λ is neither μ-fuzzy β-open nor μ-fuzzy β-closed. The following example shows that the μ-fuzzy β-open (μ-fuzzy closed) set need not μ-fuzzy q-open.

Example 2.3. Let $X=[0,1]$ and ,η, ν be fuzzy sets of X defined as follows:
$\lambda(x)=x$, $\eta(x)=1-x$ *and* $\nu(x)=0.5$. Consider μ = {0, λ, η, λ ∪ η} is a GFT on X. Since $c_\mu i_\mu c_\mu \nu = 1 \geq \nu$ and $i_\mu c_\mu i_\mu \nu = 0 \leq \nu$.
Then ν is μ-fuzzy β-open (μ-fuzzy β-closed). But $i_\mu c_\mu \nu = i_\mu(1)=\lambda\cup\eta \nleq \lambda\cap\eta = c_\mu(0)=c_\mu i_\mu \nu$. Then ν is not μ-fuzzy q-open.

Theorem 2.5. Let λ be a fuzzy set of a GFTS (X,μ), then λ is μ-fuzzy q-open if and only if, there exist two μ-fuzzy q-open sets ν and η such that $\nu\leq\lambda\leq\eta$ and $i_\mu\eta\leq c_\mu\nu$.
Proof. If λ is μ-fuzzy q-open set, then the result is trivially true.
Conversely, since $\nu\leq\lambda\leq\eta$ where ν and η are μ-fuzzy q-open sets. Then $i_\mu c_\mu\lambda\leq i_\mu c_\mu\eta\leq c_\mu i_\mu\eta$ and $i_\mu c_\mu\nu\leq c_\mu i_\mu\nu\leq c_\mu i_\mu\lambda$. And since $i_\mu\eta\leq c_\mu\nu$, then $c_\mu i_\mu\eta\leq c_\mu\nu$. Therefore $i_\mu c_\mu\lambda\leq c_\mu i_\mu\eta\leq c_\mu\nu$, this implies that $i_\mu c_\mu\lambda\leq i_\mu c_\mu\nu$, thus $i_\mu c_\mu\lambda\leq i_\mu c_\mu\nu\leq c_\mu i_\mu\nu\leq c_\mu i_\mu\lambda$ and hence λ is μ-fuzzy q-open set.

Theorem.2.6. Let λ be a fuzzy set of a GFTS (X,μ), then the following statements are equivalent.
(a) λ is a μ-fuzzy q-open.
(b) There exists μ-fuzzy q-open set ν such that $\nu\leq\lambda\leq c_\mu\nu$.
(c) There exists μ-fuzzy q-open set η such that $i_\mu\eta\leq\lambda\leq\eta$.
Proof. It is immediate from Theorem. 2.5.

Corollary.2.2. Let λ be a fuzzy set of a GFTS (X,μ),
(a) if there exists μ-fuzzy semiclosed set η of X such that $i_\mu\eta\leq\lambda\leq\eta$, then λ is μ-fuzzy q-open.
(b) if there exists μ-fuzzy semiopen set ν of X such that $\nu\leq\lambda\leq c_\mu\nu$, then λ is μ-fuzzy q-open.

Theorem 2.7. Let λ be a fuzzy set of a GFTS (X,μ), then the following statements are equivalent :
(a) λ is μ-fuzzy q-open.
(b) There exist μ-fuzzy α-open set ν and μ-fuzzy α-closed set η of X such that ν ≤ λ ≤ η and $i_\mu\eta\leq c_\mu\nu$.
(c) There exist μ-fuzzy semiopen set δ and μ-fuzzy semiclosed set ξ of X such that δ ≤ λ ≤ ξ and $i_\mu\xi\leq c_\mu\delta$.
Proof . Follows from Theorem 2.1, Theorem 2.2 and Theorem 2.5.

Theorem 2.8. Let λ be a fuzzy set of a GFTS (X,μ), then the following statements are equivalent :
(a) λ is μ-fuzzy q-open.
(b) $i_\mu\alpha c_\mu\lambda\leq c_\mu\alpha i_\mu\lambda$.
(c) $i_\mu s c_\mu\lambda\leq c_\mu s i_\mu\lambda$.
Proof . Follows from Theorem 2.7.

Theorem 2.9. For a GFTS (X,μ), then the following statements are equivalent :
(a) Every μ-fuzzy open set is μ-fuzzy regular open.

(b) Every μ-fuzzy q-open set is μ-fuzzy semiopen.

(c) Every μ-fuzzy q-open set is μ-fuzzy semiclosed.

Proof . It follows from Lemma 1.1 and Theorem 2.3.

Theorem 2.10. For a GFTS (X,μ) , then the following the statements are equivalent

(a) Every μ-fuzzy open set is μ-fuzzy closed .

(b) Every μ-fuzzy q-open set is μ-fuzzy α-open.

(c) Every μ-fuzzy q-open set is μ-fuzzy α-closed.

(d) Every μ-fuzzy q-open set is μ-fuzzy open.

(e) Every μ-fuzzy q-open set is μ-fuzzy closed.

Proof . (a)⟹(b) Follows from Lemma 2.2. and Theorem 2.4.

(b)⟺(c) is clear .

(a)⟹(d) Let λ be a μ-fuzzy q-open set , *i.e.* $i_\mu c_\mu \lambda \leq c_\mu i_\mu \lambda$, by (a) . $i_\mu c_\mu \lambda \leq i_\mu$, by Lemma 2.2. $\lambda \leq i_\mu c_\mu \lambda \leq i_\mu \lambda$ and hence λ is μ-fuzzy open .

(d)⟺(e) is clear.

(e)⟹(a) Follows from observation that every μ-fuzzy open set is μ-fuzzy q-open.

3. μ-Fuzzy q-interior and μ-Fuzzy q-closure.

Definition 3.1. Let λ by a fuzzy set of a GFTS (X,μ) and defined the following sets:

$$qi_\mu\, \lambda = \cup \{\nu |\ \nu \leq \lambda, \nu\ is\ a\ \mu - fuzzy\ q - open\ set\ of\ X\};$$

$$qc_\mu\, \lambda = \cap \{\nu |\ \lambda \leq \nu, \nu\ is\ a\ \mu - fuzzy\ q - open\ set\ of\ X\}.$$

We call $qi_\mu\, \lambda$ the μ-fuzzy q-interior of λ and $qc_\mu\, \lambda$, the μ-fuzzy q-closure of λ.

Theorem 3.1. Let λ , ν and $\{\lambda_t | t \in T\ , T\ \ is\ an\ index\ set\}$ be fuzzy sets of a GFTS (X,μ), then the following properties hold :

(a) $i_\mu\, \lambda \leq qi_\mu\, \lambda \leq\ \lambda\ \leq qc_\mu \lambda\ \leq\ c_\mu\, \lambda$.

(b) If λ is μ-fuzzy q-open set , then $q\ i_\mu\ \lambda = q\ c_\mu\ \lambda = \lambda$.

(c) $1 - q\ i_\mu\ \lambda = q\ c_\mu (1 - \ \lambda)$.

(d) $1 - qc_\mu\ \ \lambda = q\ i_\mu (1 - \ \lambda)$.

(e) If $\lambda\ \leq\ \nu$, then $q\ i_\mu\ \lambda \leq q\ i_\mu\ \nu$.

(f) If $\lambda\ \leq\ \nu$, then $q\ c_\mu\ \lambda \leq q\ c_\mu\ \nu$.

(g) $q\ i_\mu\ q\ i_\mu \lambda = q\ i_\mu\ \lambda$.

(h) $q\ c_\mu\ q\ c_\mu \lambda = q\ c_\mu\ \lambda$.

(i) $\cup_{t\in T}\ [\ q\ i_\mu\ (\lambda t)] \leq q\ i_\mu\ [\cup_{t\in T}\ \lambda t]$.

(j) $q\ i_\mu [\ \cap_{t\in T}\ \ \lambda t] \leq \cap_{t\in T}\ [q\ i_\mu (\lambda t)]$.

(k) $\cup_{t\in T}\ [\ q\ c_\mu\ (\lambda t)] \leq q\ c_\mu\ [\cup_{t\in T}\ \lambda t]$.

(l) $q\ c_\mu [\ \cap_{t\in T}\ \ \lambda t] \leq \cap_{t\in T}\ [q\ c_\mu (\lambda t)]$.

Proof . (a) and (b) are clear from Definition 3.1.

(c) $qi_\mu\, \lambda = \cup \{\nu |\ \nu \leq \lambda, \nu\ is\ a\ \mu - fuzzy\ q - open\ set\ of\ X\}$ this implies that $1 - qi_\mu\, \lambda = 1 - (\cup \{\nu |\ \nu \leq \lambda, \nu\ is\ a\ \mu - fuzzy\ q - open\ set\ of\ X\})$

$$= \cap \{1 - \nu |\ 1 - \lambda \leq 1 - \nu, 1 - \nu\ is\ a\ \mu - fuzzy\ q - open\ set\ of\ X\} = qc_\mu (1 - \lambda)$$

(d) It is similar to the proof of (c) .

(e). Let ρ be a fuzzy point of X such that $\rho \in q\ i_\mu\ \lambda$. Then there exists a μ-fuzzy q-open set η such that $\eta \leq \lambda$ with $\rho \in \eta$. Since $\lambda \leq \nu$, then $\eta \leq \lambda$, this implies that $\eta \leq qi_\mu \nu$. Therefore $\rho \in qi_\mu \nu$. Hence $qi_\mu \lambda \leq qi_\mu \nu$.

(f). Since $\leq \nu$, then $1-\nu \leq 1-\lambda$, from (e), this implies that $qi_{\mu}\,(1-\nu) \leq q\; i_{\mu}$(1-λ) then , by (d), $1-qc_{\mu}\;\nu \leq 1-q\;c_{\mu}\lambda$, therefore $qc_{\mu}\;\lambda \leq q\;c_{\mu}\nu$.
(g). In view of (a) we need only to show that : $qi_{\mu}\lambda \leq q\;i_{\mu}\;q\;i_{\mu}\lambda$. Let ρ be a fuzzy point of X such that $\rho \in qi_{\mu}\lambda$. Then there exists a μ-fuzzy q-open set η such that $\rho \in \eta \leq$λ, then $\leq qi_{\mu}\lambda$. therefore $\eta \leq qi_{\mu}qi_{\mu}\lambda$. This implies that $\rho \in qi_{\mu}qi_{\mu}\lambda$. Thus the proof is complete.
(h). Since 1- λ is fuzzy set of X , then by (g) $qi_{\mu}\;qi_{\mu}(1-\lambda) = q\;i_{\mu}(1-\lambda)$. By (d), then implies that $qi_{\mu}\left(1-qc_{\mu}\lambda\right) = 1-q\;c_{\mu}\lambda$. Again by (d), this implies that $1-qc_{\mu}\;qc_{\mu}\lambda = 1-q\;c_{\mu}\lambda$ and hence $qc_{\mu}\;qc_{\mu}\lambda = q\;c_{\mu}\lambda$.
(i). Follows from (e).
(j). Follows from (e).
(k). Follows from (f), or from (j) and (c).
(l). Follows from (f), or from (i) and (c).
(later we can prove the properties (e) and (g) easily by using Theorem 4.6)

Corollary 3.1. Let λ be a fuzzy set of a GFTS (X,μ), then
(a) $i_{\mu}qi_{\mu}\lambda = qi_{\mu}\;i_{\mu}\lambda = qc_{\mu}\;i_{\mu}\lambda = i_{\mu}\lambda$,
(b) $c_{\mu}qc_{\mu}\lambda = qc_{\mu}\;c_{\mu}\lambda = qi_{\mu}c_{\mu}\lambda = c_{\mu}\lambda$.

Remark 3.1. The converse of (b) in Theorem 3.1 is not true, therefore qi_{μ} and $qc_{\mu}\lambda$ need not μ-fuzzy q-open sets. This can be shown by following example.

Example 3.1. let $X = [0,1]$ and λ a be fuzzy sets of X defined by :

$$\lambda_a(x) = \begin{cases} 1\,, & 0 < x < a \\ 0\,, & otherwise \end{cases}$$

Where $a \in (0,1]$ and $\lambda_0(x) = 0\,, \forall\, x \in X$.
The collection of all λ_a (where $a \in [0,1]$) form a GFT on X namely μ .
Consider

$$\nu(x) = \begin{cases} 1\,, & x \in Q \cap [0,1] \\ 0\,, & otherwise \end{cases}$$

is a fuzzy set of X. We find $qi_{\mu}\nu = \nu$ and $qc_{\mu}\nu = \nu$. But

$$i_{\mu}c_{\mu}\nu = i_{\mu}(1) = \lambda_1 \nleq 1-\lambda_1 = c_{\mu}(0) = c_{\mu}i_{\mu}\nu$$

Hence ν is not μ-fuzzy q-open set.

Remark 3.2. In (i) , (j) , (k) and (l) of Theorem 3.1. the equality does not necessarily hold as shown in the following examples.

Example 3.2. let $X = [0,1]$ and λ , ν , and η be fuzzy sets of X defined as follows:

$$\lambda(x) = x \quad , \quad \nu(x) = \begin{cases} x\,, & \leq x \leq \frac{1}{2} \\ 0\,, & otherwise \end{cases} \quad and \quad \eta(x) = \begin{cases} 0\,, & \leq x \leq \frac{1}{2} \\ x\,, & otherwise \end{cases}$$

Consider $\mu = \{0,\lambda\}$ be a GFTS on X.
Then $q\;i_{\mu}\;(\nu) = \nu \quad , \quad q\;i_{\mu}(\eta) = \begin{cases} 0\,, & \leq x \leq \frac{1}{2} \\ 1-x\,, & otherwise. \end{cases}$
This implies that $q\;i_{\mu}\;(\nu) \cup\; q\;i_{\mu}(\eta) = \lambda \cap (1-\lambda)$. But $q\;i_{\mu}(\nu \cup \eta) = \lambda$,
thus $q\;i_{\mu}(\nu \cup \eta) \;\neq\; q\;i_{\mu}(\nu) \cup\; q\;i_{\mu}(\eta)$. And by Theorem 2.2.1. (c), we have
$qc_{\mu}[\,(1-\nu) \cap (1-\eta)] \neq qc_{\mu}(1-\nu) \cap qc_{\mu}(1-\eta)$.

Example 3.3. Let $X = [0,1]$, λ and ν be fuzzy sets of X defined by :
$\lambda(x) = \begin{cases} 1 \ , \ 0 \le x \le \frac{1}{2} \\ x \ , otherwise \end{cases}$ and $\nu(x) = \begin{cases} x \ , \ 0 \le x \le \frac{1}{2} \\ 1 \ , otherwise \end{cases}$.
Consider $\mu = \{0, \lambda, \nu, 1\}$ be a GFTS on X. Then $qi_\mu\lambda = \lambda \ and \ qi_\mu(\nu) = \nu$ this implies that $qi_\mu\lambda \cap qi_\mu\nu = \lambda \cap \nu = (\lambda \cap \nu)(x) = x$.
But $qi_\mu(\lambda \cap \nu) = \nu \cap (\lambda - 1) = \nu \cap (1 - \lambda)(x) = \begin{cases} x \ , \ 0 \le x \le \frac{1}{2} \\ 1 - x \ , otherwis \end{cases}$
Thus $qi_\mu\lambda \cap qi_\mu\nu \ne qi_\mu(\lambda \cap \nu)$. And by Theorem 2.2.1.(c), we have
$qc_\mu[(1 - \nu) \cup (1 - \nu)] \ne qc_\mu(1 - \nu) \cup \ qc_\mu(1 - \lambda)$.

Theorem 3.2. Let λ be a fuzzy set of a GFTS (X , μ) , then the following statements are equivalent :
(a) $qi_\mu\lambda \ (resp. qc_\mu\lambda)$ is μ-fuzzy q-open.
(b) $i_\mu \, c_\mu qi_\mu\lambda \le c_\mu \, i_\mu \, \lambda \ (resp. i_\mu c_\mu\lambda \le c_\mu \, i_\mu \, qc_\mu\lambda)$.
(c) $qc_\mu(1 - \lambda) \ (resp. qi_\mu(1 - \lambda)$is μ-fuzzy q-open.

Proof. It is straight forward.

Theorem 3.3. For a GFTS (X , μ), if every μ-fuzzy open set is μ-fuzzy regular open , then $qi_\mu\lambda \ (qc_\mu\lambda)$ is μ-fuzzy q-open, for each fuzzy set λ of X.

Proof. Let λ be any fuzzy set of X, then by Theorem 2.9. $qi_\mu\lambda = si_\mu\lambda$ is μ-fuzzy semiopen set, therefore $qi_\mu\lambda$ is μ-fuzzy q-open.
The proof of other case is similarly

Theorem 3.4. Let λ be a fuzzy set of a GFTS (X , μ) , then λ is μ-fuzzy clopen iff $qi_\mu c_\mu\lambda \le qc_\mu i_\mu\lambda$.

Proof. It is immediate by Corollary 2.1.

4. qi_μ-Fuzzy set and qi_μ-Fuzzy.

Definition 4.1. A fuzzy set λ of a GFTS (X , μ) is called :
(i) qi_μ-Fuzzy set if $qi_\mu\lambda = \lambda$
(ii) qc_μ-Fuzzy sets if $qc_\mu\lambda = \lambda$ or equivalently, if $1 - \lambda \ is \ qi_\mu$-Fuzzy set.

Theorem 4.1. Let λ be a fuzzy set of GFTS (X , μ), then
(a) λ is $q \ i_\mu$-fuzzy set (resp. $q \ i_\mu$-fuzzy) set, iff $\lambda \le q \ i_\mu\lambda$ (resp. $q \ c_\mu\lambda \le \lambda$).
(b) $q \ i_\mu \ \lambda$ (resp. $q \ c_\mu \ \lambda$) is $q \ i_\mu$-fuzzy set (resp. $q \ c_\mu$-fuzzy set).

Proof . Follows from Theorem 3.1. and the above definition.

Theorem 4.2. Let λ be a fuzzy set of GFTS (X,μ), if λ is μ-fuzzy q-open , then λ is qi_μ-fuzzy set and qc_μ-fuzzy set .

Proof . Follows from Theorem 3.1. (b) and Definition 4.1.

Remark 4.1. It is clear , by Example 3.1, that the converse of Theorem 4.2 is not true .

Theorem 4.3. For a GFTS (X , μ), then every fuzzy set of X is $q \ i_\mu$-fuzzy set iff every fuzzy set of X is $q \ c_\mu$-fuzzy set.
Proof . It is clear from Definition 4.1.

Theorem 4.4. In a GFTS (X ,μ)
(a) 0 and 1 are $q \ i_\mu$-fuzzy sets (0 and 1 are $q \ c_\mu$-fuzzy sets).

(b) The arbitrary union (resp. intersection) of qi_μ-fuzzy sets (resp. $q\ c_\mu$-fuzzy sets) is a $q\ i_\mu$-fuzzy set (resp. qc_μ-fuzzy set).

Proof . (a) is clear. (b). Let $\{\lambda_t |\ t \in T\}$ be a family of $q\ i_\mu$-fuzzy sets of X . From Theorem 3.1. (i) we have $\cup_{t\in T} \left(q\ i_\mu \lambda_t\right) \leq \cup_{t\in T} \left(q\ i_\mu t\right)$. We need only to show that $qi_\mu(\cup_{t\in T} \lambda_t) \leq \cup_{t\in T} \left(q\ i_\mu \lambda_t\right)$. By Theorem 3.1.(a) $qi_\mu(\cup_{t\in T} \lambda_t) \leq \cup_{t\in T} \lambda_t = \cup_{t\in T} \left(q\ i_\mu \lambda_t\right)$. Thus the proof is complete .
The proof of the other case follows from Definition 4.1 and the above.

Remark 4.2. the collection of all $q\ i_\mu$-fuzzy sets in a GFTS (X , μ) form a GFT on X , containing all μ-fuzzy q-open sets of λ. Thus (X , $q\ i_\mu$) is m-fuzzy space. Where if 1 is μ-fuzzy open set of X , X is called m-fuzzy space.
The following example shows that the finite intersection (resp. union) of $q\ i_\mu$-fuzzy set (resp. $q\ c_\mu$-fuzzy set) need not $q\ i_\mu$-fuzzy set (resp. $q\ c_\mu$-fuzzy set).

Example 4.1. Let λ and ν be fuzzy sets a GFTS (X , μ) as defined in Example 2.2
Then λ and ν are $q\ i_\mu$-fuzzy sets. But $\lambda \cap \nu$ is not $q\ i_\mu$-fuzzy set,
since $q\ i_\mu(\lambda \cap \nu) = \lambda \cap (1-\lambda) \neq \lambda \cap \nu$. Equivalently $(1-\lambda) \cup (1-\nu)$ is not $q\ c_\mu$-fuzzy set , althghou $(1-\lambda)$ and $(1-\nu)$ are $q\ c_\mu$-fuzzy sets.

Theorem 4.5. A fuzzy set λ of a GFTS (X , μ) is $q\ i_\mu$-fuzzy set iff for each fuzzy singleton p of λ , there exists a $q\ i_\mu$-fuzzy set λ_p of X such that $p \leq \lambda_p \leq \lambda$.
Proof . the sufficiency is clear.
Necessity . By hypothesis, we have $\lambda = \cup_{p\leq\lambda} \lambda_p$, then, by Theorem 4.4.(b), λ is $q\ i_\mu$-fuzzy set of X
Theorem 4.6. Let λ be a fuzzy set of a GFTS (X , μ), then
$q\ i_\mu\lambda = \cup \{\nu |\ \nu \leq \lambda, \nu\ is\ a\ q\ i_\mu$-fuzzy set of X $\}$. Therefore , $q\ i_\mu\lambda$ is the largest $q\ i_\mu$-fuzzy set contained in λ and $q\ c_\mu\lambda = \cap \{\ \eta |\ \lambda \leq \eta, \eta\ is\ a\ q\ c_\mu$-fuzzy set of X$\}$. Therefore $q\ c_\mu\lambda$ is the smallest $q\ i_\mu$-fuzzy set containing λ.
Proof . By definition of $q\ i_\mu\lambda$ and since every μ-fuzzy q-open set is $q\ i_\mu$-fuzzy set , then $q\ i_\mu\lambda \leq \cup \{\ \nu |\ \nu \leq \lambda, \nu\ is\ a\ q\ i_\mu$-fuzzy set of X$\}$. And since every $q\ i_\mu$-fuzzy set is the union of μ-fuzzy q-open sets ,then for each $q\ i_\mu$-fuzzy set $\nu \leq \lambda\ \ \nu$ is the union of μ-fuzzy q-open sets contained in λ, and from definition of $q\ i_\mu\lambda$, then
$\cup \{\nu |\ \nu\ \leq \lambda, \nu\ is\ a\ q\ i_\mu$-fuzzy set of X $\} \leq q\ i_\mu\lambda$.
Thus $q\ i_\mu\lambda = \cup \{\nu |\ \nu \leq \lambda\ , \nu\ is\ a\ q\ i_\mu$-fuzzy set of X$\}$. Therefore by Theorem 4.4. (b) $q\ i_\mu\lambda$ is the largest $q\ i_\mu$-fuzzy set contained in λ.
The proof of the other case is similar

Theorem 4.7. For a GFTS (X , μ), if every fuzzy point of X is μ-fuzzy q-open , then every fuzzy set of X is $q\ i_\mu$-fuzzy set and $q\ c_\mu$-fuzzy set.
Proof . Let λ be any fuzzy set of X . Since $\lambda = \cup \{\ p | p\ \in \lambda\ , p$ is μ-fuzzy q-open set$\}$. Then λ is the union of μ-fuzzy q-open sets. Thus $q\ i_\mu\lambda = \lambda$and hence λ is $q\ i_\mu$-fuzzy set. Also from Theorem 4.3. λ is $q\ c_\mu$-fuzzy set.

Remark 4.3. In (X , μ) that is in Example 3.1. we note that every fuzzy point of X is μ-fuzzy q-open set, since either $|p| \neq 0$, then $i_\mu c_\mu(p) = i_\mu\left(1-\lambda_{|p|}\right) = 0$, or $|p| = 0$, then $i_\mu c_\mu(p) = i_\mu(1-\lambda_1) = 0$. Thus by the theorem above $qi_\mu\lambda = qc_\mu = \lambda$, for each fuzzy set λ of X.

Theorem 4.8. For a DFTS (X , μ), if every μ-fuzzy open set of X is μ-fuzzy regular open , then every qi_μ-fuzzy set (qc_μ-fuzzy set) is μ-fuzzy q-open.
Proof. Let λ be any qi_μ-fuzzy set of X , *i.e.* $\lambda = qi_\mu\lambda$. From Theorem 2.9 , then $\lambda = qi_\mu\lambda = si_\mu\lambda$ and since $si_\mu\lambda$ is μ-fuzzy semiopen set ,then λ is μ-fuzzy semiopen , therefore λ is μ-fuzzy q-open .
The proof of the other case is similar.

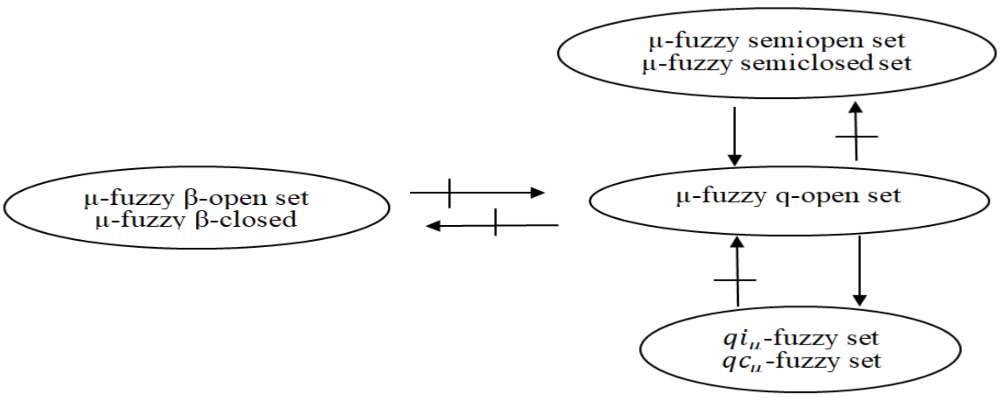
Theorem 4.9. Let λ be a qi_μ-fuzzy set (resp. qc_μ-fuzzy set), then. $\lambda \le qi_\mu qc_\mu qi_\mu\lambda$ (resp. $qc_\mu qi_\mu qc_\mu\lambda \le \lambda$).
Proof . Since λ is qi_μ-fuzzy set ,then $\lambda \le qi_\mu\lambda$, then
$\lambda \le qi_\mu\lambda \le qc_\mu qi_\mu\lambda \Rightarrow \lambda \le qc_\mu qi_\mu\lambda$. Thus $\lambda \le qi_\mu\lambda \le qi_\mu qc_\mu qi_\mu\lambda$.
The proof of the other case is similar.

Corollary 4.1. Let λ be a qi_μ-fuzzy set (resp. qc_μ-fuzzy set), then
(a) $\lambda \le qc_\mu qi_\mu\lambda$ (resp. $qi_\mu qc_\mu\lambda \le \lambda$).
(b) $\lambda \le qi_\mu qc_\mu\lambda$ (resp. $qc_\mu qi_\mu\lambda \le \lambda$).
(c) $\lambda \le qc_\mu qi_\mu qc_\mu\lambda$ (resp. $qi_\mu qc_\mu qi_\mu\lambda \le \lambda$).
(d) $qi_\mu qc_\mu\lambda \le qc_\mu qi_\mu\lambda$

Theorem 4.10. Let λ be a fuzzy set of a GFTS (X , μ), then λ is μ-fuzzy q-open iff λ is qi_μ-fuzzy set (resp. qc_μ-fuzzy set) and $qi_\mu\lambda$ (resp. $qc_\mu\lambda$) is μ-fuzzy q-open.
Proof . It is obvious.

Theorem 4.11. For a GFTS (X , μ) , if every fuzzy point of X is μ-fuzzy q-open set, then for each fuzzy set λ such that $qi_\mu\lambda$ ($qc_\mu\lambda$) is μ-fuzzy q-open, λ is μ-fuzzy q-open.
Proof . It follows Theorem 4.7. and Theorem 4.10.

In the end of this paper, the following diagram give summary of the last results.



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المجموعات الضبابية المعممة المفتوحة من النوع q.

رضوان محمد عقيل[1] وراشد الخضر ناصر[2]
[1]قسم الرياضيات، كلية العلوم، جامعة عدن
[2]قسم الرياضيات، كلية التربية لودر، جامعة أبين
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الملخص

في هذا البحث قدمنا مفهوم المجموعات الضبابية المعممة المفتوحة من النوع q والتي تُعد تعميمًا للمجموعات بسيطة الفتح التي قدمها بايسويس. ومن خلال هذا المفهوم عرفنا نوع جديد من المجموعات الضبابية المعممة الأضعف من المجموعات الضبابية المفتوحة المعممة وأسميناه المجموعات الضبابية المعممة المفتوحة من النوع q_i ودرسنا الكثير من خواصه وخصائصه، كما قمنا بدراسة العلاقة بين هذه المفاهيم والمفاهيم المعروفة سابقًا.

الكلمات المفتاحية: الفضاءات التوبولوجية الضبابية المعممة، المجموعات الضبابية المعممة المفتوحة من النوع q، المجموعات الضبابية المعممة المفتوحة من النوع q_i ، والمجموعات الضبابية المعممة المغلقة من النوع q_c.