Phase space localization of orthonormal sequences in $L_\alpha^2(\mathbb{R}_+^d)$

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Abstract

In this article, we prove Malinnikova's result for Weinstein operator as follows: Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L_\alpha^2(\mathbb{R}_+^d)$. If the sequences $\{e_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^d$ and $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^d$ are bounded, then

$$\sup_n \left(\||x - e_n|\phi_n\|_{L_\alpha^2(\mathbb{R}_+^d)} \||\xi - a_n|\mathcal{F}_W(\phi_n)\|_{L_\alpha^2(\mathbb{R}_+^d)} \right) < \infty.$$

Keywords: Weinstein operator; Uncertainty principle; Orthonormal bases; Time–frequency concentration.

Introduction

H.S. Shapiro proved in a number of uncertainty inequalities for orthonormal sequences that are stronger than corresponding inequalities for a single function. Quantitative versions of H.S. Shapiro's results appeared in a recent article by Ph. Jaming and A. Powell [7] where, in particular, the following sharp Mean Dispersion inequality was obtained. Let $\{e_k\}_{k\geq 0}$ be an orthonormal sequence in $L^2(\mathbb{R})$,then for all $N \geq 0$,

$$\sum_{k=0}^{N} \left(\left\| |t|^{\frac{1}{2}} e_n \right\|_{L^2(\mathbb{R})}^2 + \Delta^2(e_k) + \left\| |\xi|^{\frac{1}{2}} \mathcal{F}(e_n) \right\|_{L^2(\mathbb{R})}^2 + \Delta^2 \mathcal{F}(e_k) \right) \geq \frac{(N+1)(2N+1)}{4\pi}.$$

The equality is attained for the sequence of Hermite function. Here,

$$\left\| |t|^{\frac{1}{2}} e_n \right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} t|e_k|^2 dt, \Delta^2(e_k) = \int_{\mathbb{R}} \left(t - \left\| |t|^{\frac{1}{2}} e_n \right\|_{L^2(\mathbb{R})}^2 \right)^2 |e_k|^2 dx.$$

which are called the time mean of e_k, the variance of e_k respectively and $\mathcal{F}$ is the Fourier transform defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-ix\xi} dx,$$

and extended from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ in the usual way.

Next, E. Malinnikova in [8] proved the following Shapiro type inequality which is a generalization of the Mean-Dispersion principle :

Let $p > 0$ and $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbb{R}^d)$, then

$$\sum_{n=1}^{N} \left(\||x|^p \phi_n\|_{L^2(\mathbb{R}^d)}^2 + \left\| |\xi|^p \mathcal{F}(\phi_n) \right\|_{L^2(\mathbb{R}^d)}^2 \right) \geq C N^{1+p/2d},$$

where C depends only on d and p. Here

$$\||x|^p \phi_n\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |x|^{2p} |\phi_n|^2 dx.$$

In [4], J. Bourgain constructed an orthonormal basis for $L^2(\mathbb{R})$ consisting of functions $f_n \in L^2(\mathbb{R})$, such that

$$\sup_n \left(\inf_{a_n \in \mathbb{R}} \||x - a_n|^2 f_n\|_{L^2(\mathbb{R})} + \inf_{a_n \in \mathbb{R}} \||\xi - b_n|^2 \mathcal{F}(f_n)(\xi)\|_{L^2(\mathbb{R})} \right) < \infty.$$

J. Bourgain remarked that the exponent 2 of $|x-a_n|$ and $|\xi-b_n|$ is optimal. After that Karleinz Gröchenig and E. Malinnikova [6] proved astrong uncertainty principle for Reisz basis for $L^2(\mathbb{R}^d)$, such that

$$\sup_n\left(\inf_{a_n\in\mathbb{R}}\||x-a_n|^p f_n\|_{L^2(\mathbb{R})}+\inf_{a_n\in\mathbb{R}}\||\xi-b_n|^p\mathcal{F}(f_n)(\xi)\|_{L^2(\mathbb{R})}\right)<\infty.$$

This result therefore asserts that the J. Bourgain basis possesses the best possible phase space Localization.

E. Malinnikova [8] proved uncertainty principle inequality for an orthonormal basis in $L^2(\mathbb{R}^d)$ such for $p>d$,

$$\sup_n\left(\||x-a_n|^p\phi_n\|_{L^2(\mathbb{R}^d)}\||\xi-b_n|^p\mathcal{F}(\phi_n)(\xi)\|_{L^2(\mathbb{R}^d)}\right)=\infty.$$

After that, E. Malinnikova [8] constructed an orthonormal basis for uncertainty principle in $L^2(\mathbb{R}^d)$, , such that for $p\le d$,

$$\sup_n\left(\||x-a_n|^p\phi_n\|_{L^2(\mathbb{R}^d)}\||\xi-b_n|^p\mathcal{F}(\phi_n)(\xi)\|_{L^2(\mathbb{R}^d)}\right)<\infty.$$

The purpose of this article is to extend these type inequalities to the Weinstein transform.

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2.Preliminaries

We consider the Weinstein operator (also called Laplace-Bessel operator), (*see*[1,2]), defined on $\mathbb{R}^{d-1}\times(0,\infty)$ by

$$\Delta_W=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}+\frac{2\alpha+1}{x_d}+\frac{\partial}{\partial x_d};\qquad d\ge 2,\quad \alpha>-1/2.$$

For $d>3$, the operator $\Delta_W$is the Laplace-Beltrami operator on the Riemanian space on $\mathbb{R}^{d-1}\times(0,\infty)$ equipped with the metric (*see* [1])

$$ds^2=x_d^{4\alpha+2/d-2}\sum_{i=1}^{d}dx_i^2.$$

Weinstein operator has several applications in pure and applied mathematics especially in Fluid Mechanics (e. g.[5,9]).

2.1. Weinstein (or Laplace-Bessel) Transform

For $1\le p<\infty$, we denote by the Lebesgue space consisting of measurable functions f on $\mathbb{R}_+^d=\mathbb{R}^{d-1}\times\mathbb{R}_+$ equipped with the norm

$$\|f\|_{L_\alpha^p(\mathbb{R}_+^d)}=\left(\int_{\mathbb{R}_+^d}|f(x',x_d)|^p d\mu_\alpha(x',x_d)\right)^{1/p},$$

$$\|f\|_{L_\alpha^\infty(\mathbb{R}_+^d)}=\operatorname{ess\,sup}_{x\in\mathbb{R}_+^d}|f(x)|<\infty,$$

where for $x=(x_1,\cdots,x_{d-1},x_d)=(x',x_d)$ and

$$d\mu_\alpha(x)=\frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}}2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)}dx'\mathrm{d}x_d=\frac{x_d^{2\alpha+1}}{\pi^{\frac{d-1}{2}}2^{\alpha+\frac{d-1}{2}}\Gamma(\alpha+1)}dx_1,\cdots,dx_d.$$

For $f\in L_\alpha^1(\mathbb{R}_+^d)$, the Weinstein (or Laplace-Bessel) transform is defined by

$$\mathcal{F}_W(f)(\xi',\xi_d)=\int_{\mathbb{R}_+^d}f(x',x_d)\Psi(x,\xi)\,d\mu_\alpha(x',x_d),\qquad (2,1)$$

where $\Psi(x,\xi)=e^{-i\langle x',\xi'\rangle}j_\alpha(x_d\xi_d)$, $(x,\xi)\in\mathbb{R}_+^d\times\mathbb{R}_+^d$, the kernel of Weinstein and $j_\alpha$is the spherical Bessel function :

$$j_\alpha(z)=\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k},\quad z\in\mathbb{C}.$$

For radial function $f \in L^1_\alpha(\mathbb{R}^d_+)$, the function $\tilde{f}$ defined on $\mathbb{R}^d_+$ such that $f(x) = \tilde{f}(|x|)$ for all $x \in \mathbb{R}^d_+$, is integrable with respect to the measure $r^{2\alpha+d}dr$. Precisely, we have

$$\int_{\mathbb{R}^d_+} f(x)d\mu_\alpha(x) = a_{\alpha,d}\int_0^\infty \tilde{f}(\mathrm{r})\, r^{2\alpha+d}dr,$$

where

$$a_{\alpha,d} = \frac{W_{\alpha,d}}{\pi^{(d-1)/2}2^{\alpha+(d-1)/2}\Gamma(\alpha+1)} = \frac{1}{2^{\alpha+(d-1)/2}\Gamma(\alpha+(d+1)/2)}.$$

For $r > 0$, we denote by $B_r = \{x \in \mathbb{R}^d_+, |x| < r\}$ the ball in $\mathbb{R}^d_+$ of center 0 and radius r. The characteristic function of a set A is denoted by χ_A, so that

$$\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & otherwise \end{cases}.$$

The Weinstein transform satisfies the following properties

1- For all $f \in L^2_\alpha(\mathbb{R}^d_+)$, that is

$$\|\mathcal{F}_W(f)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \|f\|^2_{L^2_\alpha(\mathbb{R}^d_+)},$$

we have

$$\mathcal{F}_W{}^{-1}(f)(\xi) = \mathcal{F}_W(f)(-\xi',\xi_d), \qquad \xi = (\xi',\xi_d) \in \mathbb{R}^d_+.$$

2 - For $\beta > 0$, we define the fractional operator $(-\Delta_W)^{\frac{\beta}{2}}$

$$\mathcal{F}_W\left[(-\Delta_W)^{\frac{\beta}{2}}f\right](\xi) = |\xi|^\beta \mathcal{F}_W(f)(\xi).$$

3 - If $f \in L^1_\alpha(\mathbb{R}^d_+)$, then

$$\|\mathcal{F}_W(f)\|_{L^\infty_\alpha(\mathbb{R}^d_+)} \leq \|f\|_{L^1_\alpha(\mathbb{R}^d_+)}.$$

4 - If f and $\mathcal{F}_W(f)$ are both in $L^1_\alpha(\mathbb{R}^d_+)$, then inverse Weinstein transform is defined for almost every $x \in \mathbb{R}^d_+$by

$$f(x) = \int_{\mathbb{R}^d_+} \mathcal{F}_W(f)(\xi)\overline{\Psi(x,\xi)}d\mu_\alpha(\xi),$$

where $\overline{\Psi(x,\xi)} = \Psi(-x,\xi)$.

The generalized translation operator τ_x, $x \in \mathbb{R}^d_+$ associated with the Weinstein operator Δ_W is defined for a continue function f on $\mathbb{R}^d_+$even with respect to the last variable by

$$\tau_x f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}\int_0^\pi f\left(x'+y'; \sqrt{x_d^2+y_d^2+2x_d^2y_d^2cos\theta}\right)(sin\theta)^{2\alpha}d\theta, \tag{2.2}$$

$y \in \mathbb{R}^d_+$, where $(x'+y' = x_1+y_1,\cdots,x_{d-1}+y_{d-1})$.

In particular for all $x, y \in \mathbb{R}^d_+$we have $\tau_x f(y) = \tau_y f(x)$ and $\tau_0 f = f$. Moreover for all $1 \leq p < \infty$, $f \in L^p_\alpha(\mathbb{R}^d_+)$, the function $x \to \tau_x f$ belongs to $L^p_\alpha(\mathbb{R}^d_+)$ and we have

$$\|\tau_x f\|_{L^p_\alpha(\mathbb{R}^d_+)} \leq \|f\|_{L^p_\alpha(\mathbb{R}^d_+)}.$$

It is also well known that

$$\tau_x\Psi(\cdot,\lambda)(\mathrm{y}) = \Psi(x,\lambda)\Psi(\mathrm{y},\lambda), \qquad x,y,\lambda \in \mathbb{R}^d_+.$$

Therefore, for $f \in L^p_\alpha(\mathbb{R}^d_+)$, $p = 1$ or 2

$$\mathcal{F}_W(\tau_x f)(y) = \Psi(x,\mathrm{y})\mathcal{F}_W(f)(y), \qquad x,y \in \mathbb{R}^d_+.$$

By using the generalized translation, we define the generalized convolution prod- uct $f *_W g$ of functions $f, g \in L^1_\alpha(\mathbb{R}^d_+)$ as follows:

$$f *_W g(x) = \int_{\mathbb{R}_+^d} \tau_x f(-y', y_d)(g)(y) d\mu_\alpha(y), \qquad x \in \mathbb{R}_+^d.$$

This convolution is commutative and associative. Then (see [1]), if $1 \leq p, q, r \leq \infty$ are such that $1/p + 1/q - 1 = 1/r$, $f *_W g \in L_\alpha^r(\mathbb{R}_+^d)$ and we have the following Young inequality:

$$\|f *_W g\|_{L_\alpha^r(\mathbb{R}_+^d)} \leq \|f\|_{L_\alpha^p(\mathbb{R}_+^d)} \|g\|_{L_\alpha^q(\mathbb{R}_+^d)}.$$

This allows us to define $f *_W g$ for $f \in L_\alpha^p(\mathbb{R}_+^d)$ and $g \in L_\alpha^q(\mathbb{R}_+^d)$. Moreover, for $f \in L_\alpha^1(\mathbb{R}_+^d)$ and $g \in L_\alpha^q(\mathbb{R}_+^d)$, $q = 1$ or 2, we have

$$\mathcal{F}_W(f *_W g) = \mathcal{F}_W(f)\mathcal{F}_W(g).$$

3. Existence of Some Orthonormal Bases for $L_\alpha^2(\mathbb{R}_+^d)$:

In this article utilizing Weinstein transform $L_\alpha^2(\mathbb{R}_+^d)$, we prove uncertainty principle inequality for orthonormal basis. The proof of the following theorem is based on the idea of bourgain [4] and Malinnikova [8].

Remark 3.1 : There is an orthonormal sequence $\{\psi_n\}_{n=1}^{\infty}$ in $L_\alpha^2(\mathbb{R}_+^d)$ with bounded product of dispersion. Indeed for $\psi_n : \mathbb{R}_+^d \to \mathbb{R}$ a radial real-valued Schwartz function supported in $B_+(0,2)/B_+(0,1)$ with $\|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 = 1$, consider $\psi_n(x) = 2^{\frac{-(2\alpha s + ds + s)}{2}} \psi(2^{-s}x)$, then $\|\psi_n\|_{L_\alpha^2(\mathbb{R}_+^d)} = \|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)}$, supp $\psi_n(x) \subset B_+(0, 2^{-s+1})/B_+(0, 2^s)$, and

$$\mathcal{F}_W(\psi_n)(\xi) = 2^{\frac{2\alpha s + ds + s}{2}} \mathcal{F}_W(\psi)(2^s \xi).$$

Therefore, $\{\psi_n\}_{n=1}^{\infty}$ is an orthonormal sequence in $L_\alpha^2(\mathbb{R}_+^d)$ such that

$$\||x|\psi_n\|_{L_\alpha^2(\mathbb{R}_+^d)} = 2^s \||x|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)}, \qquad \||\xi|\mathcal{F}_W(\psi_n)\|_{L_\alpha^2(\mathbb{R}_+^d)} = 2^{-s} \||\xi|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)}.$$

Hence,

$$\||x|\psi_n\|_{L_\alpha^2(\mathbb{R}_+^d)} \||\xi|\mathcal{F}_W(\psi_n)\|_{L_\alpha^2(\mathbb{R}_+^d)} = \||x|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)} \||\xi|\psi\|_{L_\alpha^2(\mathbb{R}_+^d)} < \infty.$$

We construct orthogonal sequence and $\left\|\psi_{k,s}(x)\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \leq 1$. We use Weinstein transform in $L_\alpha^2(\mathbb{R}_+^d)$, our obtained results are complement to N. Ben salem and A. Rashed [3]. Let's construct an orthogonal sequence on $\mathbb{R}_+^d$ as the following . For, a positive integer s , define

$$\psi_{k,s}(x) = 2^{\frac{-(2\alpha s + ds + s)}{2}} \psi(2^{-s}x)\overline{\Psi(x, 2^{-s}k)}, \tag{3.3}$$

where

$$\overline{\Psi(x, 2^{-s}k)} = e^{i\langle x', 2^{-s}k'\rangle} j_\alpha(2^{-s}k_d x_d), \qquad k = (k_1, \cdots, k_d), \qquad k = (k', k_d) \in 4\mathbb{Z}^d,$$
$$|k_l| \leq 2^s \quad .$$

Observe that $\Psi_{k,s}(x)$ is orthogonal sequence.

Lemma 3.2. Let $\psi_{k,s}(x)$ as (3.3) Then , supp $(\psi_{k,s}(x)) \subset \{x = (x_1, \cdots, x_d) ⍰ 2^{s-1} < x_m < 2^s, m = 1, \cdots, d\}$, the sequence $\{\psi_{k,s}(x)\}_{k,s}$ is orthogonal, and for $p = 2$ there exists

$$\mathcal{F}_W(\psi_{k,s}) \leq 2^{\frac{2\alpha s + ds + s}{2}} \mathcal{F}_W(\psi)(2^s\xi' - k', 2^s\xi_d + k_d).$$
$$\left\||\xi - 2^{-s}k|\mathcal{F}_W(\psi_{k,s})(\xi)\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \leq 2^{-2s} D.$$

Proof. From (2,1),

$$\mathcal{F}_W(\psi_{k,s})(\xi) = 2^{\frac{-(2\alpha s + ds + s)}{2}} \int_{\mathbb{R}_+^d} \psi(2^{-s}x)\overline{\Psi(x, 2^{-s}k)}\, \Psi(x, \xi) d\mu_\alpha(x)$$
$$= 2^{\frac{-(2\alpha s + ds + s)}{2}} \int_{\mathbb{R}_+^d} \psi(2^{-s}x) e^{-i\langle x', \xi' - 2^{-s}k'\rangle} j_\alpha(2^{-s}k_d x_d) j_\alpha(\xi_d x_d)\, d\mu_\alpha(x),$$

from (2.2), we get

$j_\alpha(2^{-s}k_d x_d) j_\alpha(\xi_d x_d)$

$$= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}\int_0^\pi j_\alpha\left(x_d\sqrt{(2^{-s}k_d)^2+(\xi_d)^2+22^{-s}k_d\xi_d\cos\theta}\right)(sin\theta)^{2\alpha}d\theta ,$$

so that

$$j_\alpha(2^{-s}k_dx_d)j_\alpha(\xi_dx_d) \le \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}j_\alpha(x_d(\xi_d+2^{-s}k_d))\int_0^\pi (sin\theta)^{2\alpha}d\theta$$
$$\le j_\alpha\big(x_d(\xi_d+2^{-s}k_d)\big).$$

Therefore

$$\mathcal{F}_W(\psi_{k,s})(\xi) \le 2^{\frac{-(2\alpha s+ds+s)}{2}}\int_{\mathbb{R}_+^d}\psi(2^{-s}x)e^{-i\langle x',\xi'-2^{-s}k'\rangle}j_\alpha(x_d(\xi_d+2^{-s}k_d))d\mu_\alpha(x)$$

$$\mathcal{F}_W(\psi_{k,s})(\xi) \le 2^{\frac{(2\alpha s+ds+s)}{2}}\int_{\mathbb{R}_+^d}\psi(\eta)\,e^{-i\langle \eta',2^s\xi'-k'\rangle}j_\alpha(\eta_d(2^s\xi_d+k_d))d\mu_\alpha(\eta).$$

Hence,

$$\mathcal{F}_W(\psi_{k,s})(\xi) \le 2^{\frac{(2\alpha s+ds+s)}{2}}\mathcal{F}_W(\psi)(2^s\xi'-k',2^s\xi_d+k_d).$$

Now,

$$\big\||\xi-2^{-s}k|\mathcal{F}_W(\psi_{k,s})(\xi)\big\|^2_{L^2_\alpha(\mathbb{R}_+^d)} = \int_{\mathbb{R}_+^d}|\xi-2^{-s}k|^2\big|\mathcal{F}_W(\psi_{k,s})(\xi)\big|^2d\mu_\alpha(\xi),$$

$$\le 2^{(2\alpha s+ds+s)}\int_{\mathbb{R}_+^d}|\xi-2^{-s}k|^2\left|\mathcal{F}_W(\psi)\left(2^s\xi'-k',2^s\xi_d+k_d\right)\right|^2d\mu_\alpha(\xi)$$

$$\le 2^{(2\alpha s+ds+s)}\int_{\mathbb{R}_+^d}\sum_{i=1}^{d-1}|\xi_i-2^{-s}k_i|^2+|\xi_d-2^{-s}k_d|^2|\mathcal{F}_W(\psi)(2^s\xi'-k',2^s\xi_d+k_d)|^2d\mu_\alpha(\xi)$$

$$\le 2^{(2\alpha s+ds+s)}\int_{\mathbb{R}_+^d}\sum_{i=1}^{d-1}(|\xi_i-2^{-s}k_i|^2+|\xi_d+2^{-s}k_d|^2)|\mathcal{F}_W(\psi)(2^s\xi'-k',2^s\xi_d+k_d)|^2d\mu_\alpha(\xi)$$

$$\le 2^{-2s}\int_{\mathbb{R}_+^d}\left(|\eta'|^2+|\eta_d|^2\right)\left|\mathcal{F}_W(\psi)(\eta',\eta_d)\right|^2d\mu_\alpha(\eta',\eta_d)$$

$$\le 2^{-2s}\int_{\mathbb{R}_+^d}|\eta|^2|\mathcal{F}_W(\psi)(\eta)|^2\,d\mu_\alpha(\eta)$$

$$\le 2^{-2s}\||\eta|\mathcal{F}_W(\psi)(\eta)\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \le 2^{-2s}D.$$

Remark 3.3. Enumerate k for fixed $s \ge 2$ as $\{k(n,s)\}_{n=1}^{K_s}$, and write $\psi_{n,s}(x) = \psi_{K(n,s),s}(x)$ for $n = 1,\cdots,K_s$

Lemma 3.4. Let $\psi_{k,s}(x)$ be defined as Lemma3.2.The supp $(\psi_{k,s}(x)) \subset \{x=(x_1,\cdots,x_d)$ 2^{s-1} < x_m < 2^s, m=1,\cdots,d\}$. Then

$$\left\||\xi-2^{-s}k_l|\sum_{n=1}^{l-1}\sigma_n\mathcal{F}_w\left(\psi_{n,s}\right)\right\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \le B(\psi,d),$$

where $B(\psi,d)$ constant depends on ψ,d.

Proof : we have

$$\left\| |\xi - 2^{-s}k_l| \sum_{n=1}^{l-1} \sigma_n \mathcal{F}_w\left(\psi_{n,s}\right) \right\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$
$$= \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left| \sum_{n=1}^{l-1} \sigma_n \mathcal{F}_w\left(\psi_{n,s}\right) \right|^2 d\mu_\alpha(\xi)$$
$$\leq 2^2 \int_{\mathbb{R}^d_+} (1 + |\xi|^2) \left| \sum_{n=1}^{l-1} \sigma_n \mathcal{F}_w\left(\psi_{n,s}\right) \right|^2 d\mu_\alpha(\xi)$$
$$\leq 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \|\psi_{n,s}\|^2_{L^2_\alpha(\mathbb{R}^d_+)} + 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}^d_+} |\xi|^2 \left|\mathcal{F}_w(\psi_{n,s})\right|^2 d\mu_\alpha(\xi).$$

Put I= $I_1 + I_2$,

$$I_1 = 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \|\psi_{n,s}\|^2_{L^2_\alpha(\mathbb{R}^d_+)} \leq \frac{2^2}{K_s},$$
$$I_2 = 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}^d_+} |\xi|^2 \left|\mathcal{F}_w(\psi_{n,s})\right|^2 d\mu_\alpha(\xi)$$
$$= 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}^d_+} \left|\mathcal{F}_w(-\Delta_w^{\frac{1}{2}} \psi_{n,s})\right|^2 d\mu_\alpha(\xi)$$
$$= 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}^d_+} \left|-\Delta_w^{\frac{1}{2}} \psi_{n,s}\right|^2 d\mu_\alpha(x).$$
$$= 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}^d_+} \left|-\Delta_w \psi_{n,s}\right| d\mu_\alpha(x).$$

Since

$$\Delta_w\left(\psi(\eta)\overline{\Psi(\eta,k)}\right) = \psi(\eta)\Delta_w\overline{\Psi(\eta,k)} + \overline{\Psi(\eta,k)}\Delta_w\psi(\eta) + 2\sum_{j=1}^{d} \frac{\partial\psi(\eta)}{\partial\eta_j}\frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_j},$$

let supp $\psi \subset [-1,1]^{d-1} \times [0,1]$,

$$\psi(\eta)\Delta_w\overline{\Psi(\eta,k)} = \psi(\eta)\left(\sum_{j=1}^{d-1} \frac{\partial^2\overline{\Psi(\eta,k)}}{\partial\eta_j^2} + \frac{\partial^2\overline{\Psi(\eta,k)}}{\partial\eta_d^2} + \frac{2\alpha+1}{\eta_d}\frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_d}\right).$$

Since

$$\frac{\partial^2 j_\alpha(\eta_d k_d)}{\partial\eta_d^2} + \frac{2\alpha+1}{\eta_d}\frac{\partial j_\alpha(\eta_d k_d)}{\partial\eta_d} = -{k_d}^2 j_\alpha(\eta_d k_d).$$
$$\psi(\eta)\Delta_w\overline{\Psi(\eta,k)} = \psi(\eta)\left(|k'|^2\overline{\Psi(\eta,k)} + {k_d}^2\overline{\Psi(\eta,k)}\right)$$
$$= |k|^2\psi(\eta)\overline{\Psi(\eta,k)}.$$
$$2\sum_{j=1}^{d} \frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_j}\frac{\partial\psi(\eta)}{\partial\eta_j} = 2\sum_{j=1}^{d-1} \frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_j}\frac{\partial\psi(\eta)}{\partial\eta_j} + 2\frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_d}\frac{\partial\psi(\eta)}{\partial\eta_d}$$
$$= 2\sum_{j=1}^{d-1} ik_j\,\overline{\Psi(\eta,k)}\frac{\partial\psi(\eta)}{\partial\eta_j} + 2\frac{\partial\overline{\Psi(\eta,k)}}{\partial\eta_d}\frac{\partial\psi(\eta)}{\partial\eta_d},$$

Therefore

$$I_2 \le 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 |k|^2 \int_{\mathbb{R}_+^d} |\psi(\eta)| \left|\overline{\Psi(\eta,k)}\right| d\mu_\alpha(\eta)$$

$$+2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \int_{\mathbb{R}_+^d} |\Delta_w \psi(\eta)| \left|\overline{\Psi(\eta,k)}\right| d\mu_\alpha(\eta) + 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 \sum_{j=1}^{d-1} |k_j|\, 2 \int_{\mathbb{R}_+^d} \left|\frac{\partial \psi(\eta)}{\partial \eta_j}\right| \left|\overline{\Psi(\eta,k)}\right| d\mu_\alpha(\eta)$$

$$+2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 2 \int_{\mathbb{R}_+^d} \left|\frac{\partial \psi(\eta)}{\partial \eta_d}\right| \left|\frac{\partial \overline{\Psi(\eta,k)}}{\partial \eta_d}\right| d\mu_\alpha(\eta).$$

Since,

$$\left|\overline{\Psi(\eta,k)}\right| \le 1, \qquad \left|\frac{\partial j_\alpha(\eta_d k_d)}{\partial \eta_d}\right| \le k_d.$$

$$I_2 \le 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 |k|^2 + 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 C(\Psi,d) + 2^2 \sum_{n=1}^{l-1} |\sigma_n|^2 C(\Psi,d).$$

Hence,

$$\left\| |\xi - 2^{-s} k_l| \sum_{n=1}^{l-1} \sigma_n \mathcal{F}_W(\psi_{n,s}) \right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \le B(d,\psi),$$

where $B(d,\psi)$ constant depends on ψ, d.

Theorem 3.5. There exists an orthonormal basis $\{\phi_n\}_{n=1}^{\infty}$ for $L_\alpha^2(\mathbb{R}_+^d)$ and bounded sequences $\{e_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^d$ and $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_+^d$, such that

$$\sup_n \left(\| |x - e_n| \phi_n \|_{L_\alpha^2(\mathbb{R}_+^d)} \| |\xi - a_n| \mathcal{F}_W(\phi_n) \|_{L_\alpha^2(\mathbb{R}_+^d)} \right) < \infty.$$

Proof. Let the sequence $\{f_k\}_{k=1}^{\infty}$ of smooth functions with compact supports be dense on the unite sphere in $L_\alpha^2(\mathbb{R}_+^d)$. There we construct orthonormal basis of the form $\cup_{k=1}^{\infty} A_k$, where each $A_k$is a finite orthonormal system with compact support is already obtained. let G_{k-1}be the linear span of these functions, we also put $G_0 = \{0\}$. Define

$$f = f_k - P_{G_{k-1}} f_k.$$

Note that $\|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \le 1$, because $f \perp P_{G_{k-1}} f_k$, and $\|f_k\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 = 1$.

Take s large enough such that the supports of $\Psi_{j,s}$do not intersect the supports of f and of functions in G_{k-1}, so that

$$\text{supp}\, f \subseteq [-2^{s-2}, 2^{s-2}]^{d-1} \times [0, 2^{s-2}]$$
$$\text{supp}\, b \subseteq [-2^{s-2}, 2^{s-2}]^{d-1} \times [0, 2^{s-2}]$$

and we have used the fact that $\mathcal{F}_W(f)$ in the Schwartz space

$$\int_{\mathbb{R}_+^d} (|\xi| + 1)^2 |\mathcal{F}_W(f)|^2 d\mu_\alpha(\xi) < C.$$

. Now, define

$$b_1 = \frac{\theta}{\sqrt{K_s}} f + \gamma_1 \psi_{1,s},$$

$$b_2 = \frac{\theta}{\sqrt{K_s}} f + \sigma_1 \psi_{1,s} + \gamma_2 \psi_{2,s},$$

$$b_3 = \frac{\theta}{\sqrt{K_s}} f + \sigma_1 \psi_{1,s} + \sigma_2 \psi_{2,s} + \gamma_3 \psi_{3,s},$$

$$b_{K_s} = \frac{\theta}{\sqrt{K_s}} f + \sigma_1 \psi_{1,s} + \sigma_2 \psi_{2,s} + \cdots + \sigma_{K_s-1} \psi_{K_s-1,s} + \gamma_{K_s} \psi_{K_s,s}.$$

Let $0< \theta < 1/4$ be sufficiently small. Clearly, b_l are orthogonal to G_{k-1}. The constants $\sigma_1, \cdots, \sigma_{K_s-1}$ and $\gamma_1, \cdots, \gamma_{K_s}$ are chosen for $\{b_l\}_{l=1}^{K_s}$ an orthonormal sequence. Thus

$$\int_{\mathbb{R}_+^d} b_l b_k d\mu_\alpha(x) = \int_{\mathbb{R}_+^d} \left(\frac{\theta}{\sqrt{K_s}} f + \sum_{n=1}^{l-1} \sigma_n \psi_{n,s} + \gamma_l \psi_{l,s}\right)\left(\frac{\theta}{\sqrt{K_s}} f + \sum_{n=1}^{k-1} \sigma_n \psi_{n,s} + \gamma_k \psi_{k,s}\right) d\mu_\alpha(x),$$

when $(k \neq l)$,

$$\int_{\mathbb{R}_+^d} b_l b_k d\mu_\alpha(x) = 0,$$

we obtain (when $1 \leq l < k$),

$$\int_{\mathbb{R}_+^d} \frac{\theta^2}{\mathrm{K}_s} |f|^2 \, d\mu_\alpha(x) + \int_{\mathbb{R}_+^d} \sum_{n=1}^{l-1} \sigma_n^2 \left|\psi_{n,s}\right|^2 d\mu_\alpha(x) + \int_{\mathbb{R}_+^d} \sigma_l \gamma_l \left|\psi_{n,s}\right|^2 d\mu_\alpha(x) = 0,$$

$$\frac{\theta^2}{K_s} \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \sum_{n=1}^{l-1} \sigma_n^2 + \sigma_l \gamma_l = 0.$$

Hence

$$\sigma_l \gamma_l = -\frac{\theta^2}{K_s} \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 - \sum_{n=1}^{l-1} \sigma_n^2. \tag{3.7}$$

When $(l = k)$

$$\int_{\mathbb{R}_+^d} |b_l|^2 d\mu_\alpha(x) = 1,$$

$$\int_{\mathbb{R}_+^d} |b_l|^2 d\mu_\alpha(x) = \int_{\mathbb{R}_+^d} \frac{\theta^2}{K_s} |f|^2 \, d\mu_\alpha(x) + \int_{\mathbb{R}_+^d} \sum_{n=1}^{l-1} \sigma_n^2 \left|\psi_{n,s}\right|^2 d\mu_\alpha(x) + \int_{\mathbb{R}_+^d} \gamma_l^2 \left|\psi_{n,s}\right|^2 d\mu_\alpha(x) = 1.$$

Then

$$\frac{\theta^2}{K_s} \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \sum_{n=1}^{l-1} \sigma_n^2 + \gamma_l^2 = 1.$$

Hence

$$\gamma_l^2 = 1 - \frac{\theta^2}{K_s} \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 - \sum_{n=1}^{l-1} \sigma_n^2. \tag{3.8}$$

Clearly, from inequalities (3.7), (3.8), we obtain

$$|\gamma_l| \geq |\gamma_l|^2 \geq 1 - \frac{2\theta^2}{K_s},$$

$$|\sigma_l| \leq \frac{\theta}{K_s}.$$

By the last two inequalities, we know that σ_l is close to zero and γ_l is close to one. Thus we expect to have b_l close to $\Psi_{l,s}$. In fact

$$\left\|b_l - \psi_{l,s}\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \leq \frac{3\theta}{K_s}. \tag{3.9}$$

To see this,

$$\left\|b_l - \psi_{l,s}\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 \leq \left\|b_l - \gamma_l \psi_{l,s}\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + |1 - \gamma_l|^2$$

$$\leq \left\|b_l - \gamma_l \psi_{l,s}\right\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \frac{\theta^2}{{K_s}^2}$$

$$\leq \frac{\theta^2}{K_s} \|f\|_{L_\alpha^2(\mathbb{R}_+^d)}^2 + \sum_{n=1}^{l-1} |\sigma_n|^2 + \frac{\theta}{K_s}$$

$$\leq \frac{\theta^2}{K_s} + K_s \frac{\theta^2}{{K_s}^2} + \frac{\theta}{K_s}$$
$$\leq \frac{2\theta^2}{K_s} + \frac{\theta}{K_s}$$
$$\leq \frac{2\theta}{K_s} + \frac{\theta}{K_s}$$
$$\leq \frac{3\theta}{K_s}.$$

Now, we prove that

$$\||x - e_n|\Phi_n\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

It is sufficient to prove that

$$\||x|b_l(x)\|_{L^2_\alpha(\mathbb{R}^d_+)} < \infty.$$

Using inequality (3.9), we have

$$\||x|b_l(x)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \int_{\mathbb{R}^d_+} |x|^2 |b_l(x)|^2 d\mu_\alpha(x)$$
$$\leq \int_{\mathbb{R}^d_+} |x|^2 \left|b_l(x) - \psi_{l,s}(x)\right|^2 d\mu_\alpha(x) + \int_{\mathbb{R}^d_+} |x|^2 \left|\psi_{l,s}(x)\right|^2 d\mu_\alpha(x)$$
$$\leq 2^{2s} \left\|b_l(x) - \psi_{l,s}(x)\right\|^2_{L^2_\alpha(\mathbb{R}^d_+)} + \left\||x|\psi_{l,s}(x)\right\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$$
$$\leq 2^{2s} \left\|b_l(x) - \psi_{l,s}(x)\right\|^2_{L^2_\alpha(\mathbb{R}^d_+)} + 2^{2s} D$$
$$\leq 3\theta 2^{2s} + 2^{2s} D \leq C_s.$$

For the Weinstein transform we estimate $\||\xi - 2^{-s}k|\mathcal{F}_w(\mathcal{V}_l)\|^2_{L^2_\alpha(\mathbb{R}^d_+)}$, where

$$\||\xi - 2^{-s}k_l|\mathcal{F}_w(b_l)\|^2_{L^2_\alpha(\mathbb{R}^d_+)} = \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 |\mathcal{F}_w(b_l)|^2 d\mu_\alpha(\xi)$$
$$= \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left|\mathcal{F}_w\left(\frac{\theta}{\sqrt{K_s}} f + \sum_{n=1}^{l-1} \sigma_n \psi_{n,s} + \gamma_l \psi_{l,s}\right)\right|^2 d\mu_\alpha(\xi)$$
$$\leq 3 \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left|\mathcal{F}_w\left(\frac{\theta}{\sqrt{K_s}} f\right)\right|^2 d\mu_\alpha(\xi)$$
$$+3 \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left|\sum_{n=1}^{l-1} \sigma_n \mathcal{F}_w(\psi_{n,s})\right|^2 d\mu_\alpha(\xi)$$
$$+3 \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left|\mathcal{F}_w(\gamma_l \psi_{l,s})\right|^2 d\mu_\alpha(\xi).$$

To estimate the first term, we put $\theta = 1/4$

$$3 \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 \left|\mathcal{F}_w\left(\frac{\theta}{\sqrt{K_s}} f\right)\right|^2 d\mu_\alpha(\xi) = \frac{3}{16K_s} \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 |\mathcal{F}_w(f)|^2 d\mu_\alpha(\xi)$$
$$= \frac{3}{16K_s} \int_{\mathbb{R}^d_+} |\xi - 2^{-s}k_l|^2 |\mathcal{F}_w(f)|^2 d\mu_\alpha(\xi)$$
$$= \frac{3}{16K_s} \int_{\mathbb{R}^d_+} (|\xi| + 1)^2 |\mathcal{F}_w(f)|^2 d\mu_\alpha(\xi)$$
$$< \frac{3C}{16K_s} < 3C.$$

For the second term, using Lemma 3.4 we see that

$$3\int_{\mathbb{R}_+^d} |\xi - 2^{-s}k_l|^2 \left|\sum_{n=1}^{l-1} \sigma_n \mathcal{F}_w(\psi_{n,s})\right|^2 d\mu_\alpha(\xi) \le B(\psi, d).$$

The third term proved in Lemma 3.2

$$\int_{\mathbb{R}_+^d} |\xi - 2^{-s}k_l|^2 \left|\mathcal{F}_w(\gamma_l \psi_{l,s})\right|^2 d\mu_\alpha(\xi) \le D_{-s}.$$

Then

$$\| |\xi - 2^{-s}k_l| \mathcal{F}_w(b_l)\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \le D_{-s} + B(\psi, d) + 3C \le C_{-s}.$$

Thus

$$\| |x - 2^{-s}k_l| b_l\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \| |\xi - 2^{-s}k_l| \mathcal{F}_w(b_l)\|^2_{L^2_\alpha(\mathbb{R}_+^d)} < C.$$

We set $A_K = \{b_1, \dots \dots \dots, b_k\}$ and continue the procedure. We want to check that the resulting orthonormal sequence is complete, once again. First,

$$P_{G_k} f_k = P_{G_{k-1}} f_k + \sum_{l=1}^{K_s} \langle f, b_l \rangle b_l,$$

$$\begin{aligned}
\left\|P_{G_k} f_k\right\|^2_{L^2_\alpha(\mathbb{R}_+^d)} &= \left\|P_{G_{k-1}} f_k\right\|^2_{L^2_\alpha(\mathbb{R}_+^d)} + \sum_{l=1}^{K_s} |\langle f, b_l \rangle|^2 \\
&= \|f_k - f\|^2_{L^2_\alpha(\mathbb{R}_+^d)} + \sum_{l=1}^{K_s} |\langle f, b_l \rangle|^2 \\
&= 1 - \|f\|^2_{L^2_\alpha(\mathbb{R}_+^d)} + K_s \left(\frac{\theta}{\sqrt{K_s}} \|f\|^2_{L^2_\alpha(\mathbb{R}_+^d)}\right)^2 \\
&= 1 - \|f\|^2_{L^2_\alpha(\mathbb{R}_+^d)} + \theta^2 \|f\|^4_{L^2_\alpha(\mathbb{R}_+^d)} \\
&\ge \theta^2.
\end{aligned}$$

When $\theta = \frac{1}{4}$, we have

$$\left\|P_{G_k} f_k\right\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \ge \frac{1}{16}.$$

Suppose that the orthonormal sequence $\cup_{k=1}^{\infty} A_K$ is not complete. Let G be closed span. Then there exists $h \in L^2_\alpha(\mathbb{R}_+)$ such that $\|h\|_{L^2_\alpha(\mathbb{R}_+^d)} = 1$ and h is orthogonal to G. For some k we have $\|h - f_k\|_{L^2_\alpha(\mathbb{R}_+)} < 1/4$ since $\{f_k\}$ is a dens sequence on the unite sphere of $L^2_\alpha(\mathbb{R}_+^d)$. Then we obtain the contradiction

$$\frac{1}{16} \le \left\|P_{G_k}(f_k)\right\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \le \|P_G(f_k)\|^2_{L^2_\alpha(\mathbb{R}_+^d)} = \|P_G(f_k - h)\|^2_{L^2_\alpha(\mathbb{R}_+^d)} \le \|f_k - h\|^2_{L^2_\alpha(\mathbb{R}_+^d)} < \frac{1}{16}.$$

Then the orthonormal sequence $\cup_{k=1}^{\infty} A_K$ is complete. The proof is finished**.**

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الفراغ الشكلي للمتتاليات المتعامدة معياريا في الفراغ $L_{\alpha}^{2}(\mathbb{R}_{+}^{d})$

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الملخص

كثيرون اهتموا بدراسة مبدأ عدم اليقين بالنسبة لمتتاليات متعامدة معياريًا كبورجين ولوبرسكيو ومالينكوفا للإجابة عن السؤال الآتي:

هل يوجد متتاليات متعامدة معياريًا في الفراغ $L^2(\mathbb{R})$ بحيث يكون حاصل ضرب التشتت وتحويل فورية ببعضهما محدودًا؟

وكل باحث أجاب على هذا السؤال في الفراغ خاصته.

وفي عملنا هذا قمنا بالتحقق وإثبات نتائج مالينكوفا بالنسبة لمحول وينشتاين في الفراغ $L_{\alpha}^{2}(\mathbb{R}_{+}^{d})$ أي أن:

نفرض $\{\phi_n\}_{n=1}^{\infty}$ أساس متعامدة معياريًا في الفراغ $L_{\alpha}^{2}(\mathbb{R}_{+}^{d})$ وإذا كان $\{e_n\}_{n=1}^{\infty} \subset \mathbb{R}_{+}^{d}$ و $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}_{+}^{d}$ متتاليات محدودة فإن:

$$\sup_n \left(\| |x - e_n| \phi_n \|_{L_{\alpha}^{2}(\mathbb{R}_{+}^{d})} \| |\xi - a_n| \mathcal{F}_W(\phi_n) \|_{L_{\alpha}^{2}(\mathbb{R}_{+}^{d})} \right) < \infty.$$

الكلمات المفتاحية: محّول وينشتاين، مبدأ عدم اليقين، أساس متعامد معيارياً، تركيز وقت التردد.