On Generalized R^h-Trirecurrent Space

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Abstract

In the present paper, a Finsler space F_n whose Cartan's fourth curvature tensor R^i_{jkh} satisfies $R^i_{(jkh|\ell|m|n)} = c_{\ell mn} R^i_{jkh} + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, R_jkh^i≠0, where $c_{\ell mn}$ and $d_{\ell mn}$ are non-zero covariant tensor fields, of third order is introduced and such space is called as *generalized R^h-trirecurrent Finsler space* and denote it briefly by GR^h-TRF_n , we obtained some generalized trirecurrent spaces. Also we introduced Ricci generalized trirecurrent space.

Keywords: Ricci tensor R_{jk}, generalized trirecurrent tensors.

1. Introduction

Ruse [12] introduced and studied a three dimensional space as space of recurrent curvature. The recurrent of an n-dimensional space was extended to Finsler space by Moor [5-7] for the first time. Due to different connections of Finsler space, the recurrence of different curvature tensors have has been discussed by Mishra and Pande [4] and Pandey [8]. Dikshit [2] discussed Finsler space in which Cartan's third curvature tensor R^i_{jkh} is birecurrent. Qasem [9] discussed a Finsler space for which Cartan's third curvature tensor R^i_{jkh} is generalized and special generalized birecurrent of the first and second kind.

Qasem and Saleem [10] discussed a Finsler space h-curvature tensor U^i_{jkh} and Wely's projective curvature tensor W^i_{jkh} are generalized birecurrent.

Al-Qashbari [1] introduced the R^h-recurrent space which is characterized by
$R^i_{jkh|\ell} = \lambda_\ell R^i_{jkh} + \mu_\ell(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where λ_ℓ is non-zero covariant vector field known by the recurrence vector field. Hadi [13] discussed the R^h-birecurrent space which is characterized by $R^i_{jkh|\ell|m} = a_{\ell m} R^i_{jkh} + b_{\ell m}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where $a_{\ell m}$ is non-zero covariant tensor field of second order known by the birecurrence tensor field.
The metric tensor g_{ij} and the associate metric tensor g^{ij} are covariant constant with respect to h-covariant derivative [11] i.e.

(1.1) $g_{ij|k} = 0$, where

(1.2) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 & \text{if } i = k , \\ 0 & \text{if } i \neq k . \end{cases}$

The contra covariant derivative of the vector y^i , vanishs identically[11] i.e.

(1.3) $y^i_{|k} = 0$, where

(1.4) $y_i\, y^i = F^2$

The vectors y_i and δ_k^i also satisfy the following relations [11]

(1.5) a) $\delta_k^i\, y^k = y^i$, b) $\delta_j^i\, g^{jk} = g^{ik}$ and c) $\delta_k^i\, g_{ji} = g_{jk}$.

By using Euler's theorem, the C_{ijk} and C_{jk}^i tensors satisfy, the following identities [11]

(1.6) a) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ and b) $C_{jk}^i\, y^j = C_{kj}^i\, y^j = 0$,

where

(1.7) $C_{ijk} = g_{hj}\, C_{ik}^h$.

The associate curvature tensor R_{ijkh} of the curvature tensor R_{jkh}^i is given by [11]

(1.8) a) $R_{ijkh} = g_{rj}\, R_{ikh}^r$ and b) $R_{jrkh}\, g^{ir} = R_{jkh}^i$.

The R-Ricci tensor R_{jk} , the curvature scalar R and the deviation tensor R_j^i related by [11]

(1.9) a) $R_{jki}^i = R_{jk}$, and b) $R_{jk} g^{jk} = R$.

The curvature tensor R_{jkh}^i satisfies the relations [11]

(1.10) $R_{jkh}^i\, y^j = H_{kh}^i$.

The associate tensor R_h^r of the curvature tensor R_{jkh}^i is given by [11]

(1.11) $R_h^r = g^{ik}\, R_{ikh}^r$.

Also, we have [11]

(1.12) a) $H_k = H_{ki}^i$ and b) $H = \frac{1}{(n-1)}\, H_i^i$,

where H_{hk}^i and H_k^i are called H-Ricci tensor and the curvature scalar, respectively and defined by [11]

(1.13) $H_{hk}^i\, y^h = H_k^i$, also [11]

(1.14) $R_{jkh}^i = K_{jkh}^i + C_{jm}^i\, H_{kh}^m$.

The curvature tensor R_{jkh}^i and its associate tensor R_{ijhk} satisfies the following identities known as *Bianchi identities* [11]

(1.15) $R_{hjk}^i + R_{jkh}^i + R_{khj}^i - \left(C_{hr}^i H_{jk}^r + C_{jr}^i H_{kh}^r + C_{kr}^i H_{jh}^r\right) = 0$.

2. A Generalized R^h −Trirecurrent Tensor

Let us consider a Finsler space F_n for which Cartan's third curvature tensor R_{jkh}^i satisfied the generalized recurrence condition [1]

(2.1) $R_{jkh|\ell}^i = \lambda_\ell\, R_{jkh}^i + \mu_\ell(\delta_k^i g_{jh} - \delta_h^i g_{jk})$, $R_{jkh}^i \neq 0$

and called it *generalized R^h-recurrent space.*

Let us consider a Finsler space F_n for which Cartan's third curvature tensor R_{jkh}^i satisfied the generalized birecurrence condition [13]

(2.2) $R_{jkh|\ell|m}^i = a_{\ell m}\, R_{jkh}^i + b_{\ell m}(\delta_k^i g_{jh} - \delta_h^i g_{jk})$, $R_{jkh}^i \neq 0$

and called it *generalized R^h-birecurrent space.*

Taking h-covariant derivative of (2.2) with respect to x^n and using (1.1), we get

(2.3) $R_{jkh|\ell|m|n}^i = c_{\ell mn} R_{jkh}^i + d_{\ell mn}\left(\delta_k^i g_{jh} - \delta_h^i g_{jk}\right)$, $R_{jkh}^i \neq 0$,

where $_{|\ell|m|n}$ is h-covariant derivative of third order with respect to x^ℓ, x^m and x^n successfully, $c_{\ell mn} = a_{\ell|mn} + a_{\ell m}\, \lambda_n$ and $d_{\ell mn} = a_{\ell m}\mu_n + b_{\ell|m|n}$ are non-zero covariant tensors fields of third order, called *recurrence tensors field.*

Remark 2.1. The space which is characterized by the condition (2.3) called the *generalized* $R^h -$ *trirecurrent* space and denoted by *G R - TR* F_n.
Theorem 2.1. *In generalized R^h-recurrent space, the generalized R^h-birecurrent space is GR^h-TRF_n.*
Transvecting the condition (2.3) by g_{ir} , using (1.1),(1.8a) and (1.5c), we get
(2.4) $R_{jrkh|\ell|m|n} = c_{\ell mn} R_{jrkh} + d_{\ell mn}(g_{kr} g_{jh} - g_{hr} g_{jk})$, $R_{jrkh} \neq 0$.
Conversely, the transvection of the condition (2.4) by g^{ir}, by using (1.8), (1.1) and (1.2), yields the condition (2.4).
Thus, we may conclude
Theorem 2.2. *The GR^h-TR F_n , may characterized by the condition (2.4).*

3. Certain Generalized h-Tensors of Third Order in GR^h-TR F_n

Let us consider a GR^h-TR F_n .
Transvecting the condition (2.3) by y^j , using (1.10) and (1.3), we get
(3.1) $H^i_{kh|\ell|m|n} = c_{\ell mn} H^i_{kh} + d_{\ell mn}(\delta^i_k y_h - \delta^i_h y_k)$.
Transvecting (3.1) by y^k , using (1.3), (1.13), (1.5) and (1.4), we get
(3.2) $H^i_{h|\ell|m|n} = c_{\ell mn} H^i_h + d_{\ell mn}(y^i y_h - \delta^i_h F^2)$.
Thus, we may conclude
Theorem 3.1. *In GR^h-TR F_n , the h-covariant derivative of third order for the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h given by (3.1) and (3.2), respectively.*
Contracting the indices i and h in (3.1), using (1.5a) and in view of (1.2), we get
(3.3) $H_{k|\ell|m|n} = c_{\ell mn} H_k + (1-n)\, d_{\ell mn}\, y_k$.
Contracting the indices i and h in (3.2), using (1.4) and in view of (1.2), we get
(3.4) $H_{|\ell|m|n} = c_{\ell mn} H_k - F^2 d_{\ell mn}$.
The equations (3.3) and (3.4) show the curvature vector H_k and the scalar curvature H can't vanish because the vanishing of any one of them would imply $d_{\ell mn} = 0$, acontradiction.
Thus, we may conclude
Theorem 3.2. *In GR^h-TR F_n , the curvature vector H_k and the scalar curvature H are non-vanishing*.
Contracting the indices i and h in (2.3), using (1.9a), (1.5c) and in view of (1.2), we get
(3.5) $R_{jk|\ell|m|n} = c_{\ell mn} R_{jk} + (1-n)\, d_{\ell mn}\, g_{jk}$.
Transvecting (3.5) by g^{jk} , using (1.9b) and in view of (1.1) and (1.2), we get
(3.6) $R_{|\ell|m|n} = c_{\ell mn} R + (1-n)\, d_{\ell mn}$.
equations (3.5) and (3.6) show that the R-Ricci tensor R_{jk} and the scalar curvature R, can't vanish because the vanishing of any one of them would imply $d_{\ell mn} = 0$, a contradiction.
Thus, we may conclude
Theorem 3.3. *In GR^h-TR F_n , the R-Ricci tensor R_{jk} and the scalar curvature R are non-vanishing.*
Transvecting the condition (2.3) by g^{jk} , using (1.11) and in view of (1.1), we get

$$R^i_{h|\ell|m|n} = c_{\ell mn}\, R^i_h.$$

Thus, we may conclude

Theorem 3.4. *In GR^h-TR F_n , the deviation tensorR^i_hbehaves as trirecurrent.*

Taking the h-covariant derivative for (1.14) three times with respect to x^ℓ, x^m and x^n, successively, we get

$$(C^i_{jr}H^r_{kh})_{|\ell|m|n} = R^i_{jkh|\ell|m|n} - K^i_{jkh|\ell|m|n}.$$

Using the condition (2.3) and suppose that, the curvature tensor K^i_{jkh} is generalized h – trirecurrent tensor, the above equation can be written as

(3.7) $(C^i_{jr}H^r_{kh})_{|\ell|m|n} = c_{\ell mn}\big(C^i_{jr}H^r_{kh}\big).$

Transvecting (3.7) by g_{ip} , using (1.1) and (1.7), we get

(3.8) $(C_{jpr}H^r_{kh})_{|\ell|m|n} = c_{\ell mn}(C_{jpr}H^r_{kh}).$

Transvecting (3.8) by y^k , using (1.3) and (1.13), we get

$$(C^i_{jr}H^r_h)_{|\ell|m|n} = c_{\ell mn}(C^i_{jr}H^r_h).$$

Thus, we may conclude

Theorem 3.5. *In GR^h-TRF_n, the tensors $\big(C^i_{jr}H^r_{kh}\big)$, $(C_{jpr}H^r_{kh})$ and $\big(C^i_{jr}H^r_h\big)$ are trirecurrent.*

Provided that the curvature tensor K^i_{jkh} is generalized h- trirecurrent tensor.

Taking the h-covariant derivative for (1.15) three times with respect to x^ℓ, x^m and x^n successively, we get $(R^i_{hjk} + R^i_{jkh} + R^i_{khj})_{|\ell|m|n} = (C^i_{hs}H^s_{jk} + C^i_{js}H^s_{hk} + C^i_{ks}H^s_{jh})_{|\ell|m|n}$.

In view of the condition (2.3), the above eguation can be written as

(3.9) $(C^i_{hs}H^s_{jk} + C^i_{js}H^s_{hk} + C^i_{ks}H^s_{jh})_{|\ell|m|n} = c_{\ell mn}(C^i_{hs}H^s_{jk} + C^i_{js}H^s_{hk} + C^i_{ks}H^s_{jh})$.

Transvecting (3.9) by g_{ip} , using (1.1) and (1.7), we get

(3.10) $(C_{hps}H^s_{jk} + C_{jps}H^s_{hk} + C_{kps}H^s_{jh})_{|\ell|m|n} = c_{\ell mn}(C_{hps}H^s_{jk} + C_{jps}H^s_{hk} + C_{kps}H^s_{jh})$.

Transvecting (3.10) by y^j , using (1.3), (1.13) and in view of (1.7), we get

$$(C^i_{hs}H^s_k + C^i_{ks}H^s_h)_{|\ell|m|n} = c_{\ell mn}(C^i_{hs}H^s_k + C^i_{ks}H^s_h).$$

Transvecting (3.10**)** by y^j , using (1.3), (1.13) and in view of (1.7), we get

$$(C_{hps}H^s_k + C_{kps}H^s_h)_{|\ell|m|n} = c_{\ell mn}(C_{hps}H^s_k + C_{kps}H^s_h).$$

Thus, we may conclude

Theorem 3.6. *In GR^h-TRF_n, the tensors $(C^i_{hs}H^s_{jk} + C^i_{js}H^s_{hk} + C^i_{ks}H^s_{jh})$, $(C_{hps}H^s_{jk} + C_{jps}H^s_{hk} + C_{kps}H^s_{jh})$, $(C^i_{hs}H^s_k + C^i_{ks}H^s_h)$ and $(C_{hps}H^s_k + C_{kps}H^s_h)$ behave as trirecurrent.*

4.Conclusions

(4.1) The space whose defined by (2.1) is called R^h-generalized trirecurrent Finsler space.

(4.2) In generalized R^h-recurrent space, the generalized R^h-birecurrent space is GR^h-TRF_n.

(4.3) AG R^h-TRF_n the curvature vector H_k , the scalar curvature H are non-vanishing.

(4.4) In GR^h-TRF_n, the deviation tensor R^i_h, the R-Ricci tensor R_{jk} and the scalar curvature R are non-vanishing

5.Recommendations

Authors recommend the need for the continuing research and development in Finsler space due to its vital applying importance in other fields.

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حول فضاء تعميم R^h – ثلاثي المعاودة

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الملخص

في هذه الورقة قدمنا فضاء فنسلر F_n الذي فيه الموترالتقوسي الرابع لكارتان R^i_{jkh} يحقق

$$R^i_{jkh|\ell|m|n} = c_{\ell mn}R^i_{jkh} + d_{\ell mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}), \quad R^i_{jkh} \neq 0,$$

حيث $c_{\ell mn}$، $d_{\ell mn}$ هي حقول موترات متحدة الاختلاف غير صفرية وهذا الفضاء يدعى تعميم فضاء R^h – ثلاثي المعاودة ويرمز له بـ GR^h- $TR\ F_n$، وتحصلنا على بعض فضاءات تعميم ثلاثي المعاودة وأيضا قدمنا ريتشي تعميم فضاء ثلاثي المعاودة.

الكلمات المفتاحية: موتر ريتشي R_{jk}، موترات تعميم ثلاثي المعاودة.