# Exact solutions of the Harry Dym Equation using Lie group method

**Mobarek Awadh Assabaai and Omer Faraj. Mukherij**
Department of Mathematics, Faculty of Science, Hadhramout University, Yemen
Email: asaba973@gmail.com , Email: mukherij55@yahoo.com
DOI: https://doi.org/10.47372/uajnas.2020.n2.a15

## Abstract

In this paper, we apply Lie symmetry group analysis to the Harry Dym equation. Then, the similarity reduction will be found, and the invariant solutions of Harry Dym equation is obtained from the solutions of reduced ordinary differential equations.

**Keywords**: Harry Dym equation, Lie group, similarity reduction, invariant solutions.

## 1. Introduction:

The linear and nonlinear fractional partial differential equations appear in many branches of science and engineering, that are in fluid mechanics, acoustic, electro- magnetism, analytical chemistry, biology, signal processing and other physical applications, see for instance [10, 20]. The Harry Dym equation has a strong links to the Korteweg-de Vries equation (Kdv) and the Sturm-Liouville operator [5, 19].

The nonlinear fractional Harry Dym equation has the following form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = u^3(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} \quad (1)$$

Where $\alpha$ is a positive parameter describing the order of the fractional derivative and $u(x,t)$ is a function of the space $x$ and the time $t$.

In this paper, we consider the nonlinear fractional Harry Dym equation for $\alpha = 1$, which takes the form

$$\frac{\partial u(x,t)}{\partial t} = u^3(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} \quad (2)$$

Equation (2) is called classical nonlinear Harry Dym equation the time fractional Harry Dym equation.

There are many authors applied analytical and numerical methods to solve the classical nonlinear Harry Dym equation. Al-Khaled and Alquran [1], applied Adomian decomposition method (ADM) to obtain an approximate solution of time fractional generalized Harry Dym equation. Fonseca [4] solved the nonlinear Harry Dym equation using Lattice-Boltzmannn and a solitary wave methods. Ghiasi and Saleh [6] employed the homotopy analysis method (HAM) to obtain the approximate analytical solution of the nonlinear Harry-Dym (HD) equation.in[9] , Iyiola and Gaba used q-homotopy analysis method (q-HAM) to obtain analytical solutions of the to time-fractional Harry Dym equation. Kumar et al. [11]used homotopy perturbation Sumudu transform method (HPSTM) and Adomian decomposition method (ADM) to solve nonlinear fractional Harry Dym equation. Kumar et al [12]. applied the homotopy perturbation method (HPM) to obtain approximate solution of the time fractional Harry Dym equation. Maitama and Abdullahi [13] presented a new analytical method called the natural homotopy perturbation method (NHPM) for solving linear and nonlinear fractional partial differential equations, which is combined of natural transform (NTM) and homotopy perturbation methods (HPM). Mokhtari [14]applied Adomian decomposition method, He's variational iteration, direct integration, and power series methods to generate exact travelling wave solutions of the Harry Dym equation. Rawashdeh [16] employed a new approach to solve fractional Harry Dym namely, fractional reduced differential transform method (FRDTM) and he found approximate solutions of the nonlinear Harry Dym equation. Shunmugarajan [17], the homotopy analysis method (HAM) is applied to get approximate solution of time fractional generalized Harry Dym equation. Soltani and Khorshidi[18]usedthe reconstruction of variational iteration (RVIM) and homotopy perturbation methods (HPM)to obtain an analytical solution of Harry-Dym equation.

## 2. Method description

We apply Lie symmetry group analysis [2, 3, 7, 8, 15] for the nonlinear Harry-Dym (HD) equation (2) to derive symmetry generators of it. We consider one-parameter $\varepsilon$ -Lie group point of transformations, which makes equation (2) invariant. These transformations are given by

$$\bar{x} = \bar{x}(x,t,u;\varepsilon) = x + \varepsilon\xi_1(x,t,u) + O(\varepsilon^2),$$
$$\bar{t} = \bar{t}(x,t,u;\varepsilon) = t + \varepsilon\xi_2(x,t,u) + O(\varepsilon^2),$$
$$\bar{u} = \bar{u}(x,t,u;\varepsilon) = u + \varepsilon\varphi(x,t,u) + O(\varepsilon^2) \tag{3}$$

The infinitesimal generator associated with (3) is given by

$$X = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \varphi(x,t,u)\frac{\partial}{\partial u} \tag{4}$$

and the third prolongation of the infinitesimal generator (4) is given by

$$\mathrm{Pr}^{(3)}\, X = X + \varphi^{t}\frac{\partial}{\partial u_t} + \varphi^{xxx}\frac{\partial}{\partial u_{xxx}} \tag{5}$$

We consider

$$\Delta = \frac{\partial u(x,t)}{\partial t} - u^3(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} \tag{6}$$

To calculate the infinitesimals $\xi_1, \xi_2$ and $\varphi$ , applying (5) to (6), that is,

$$\mathrm{Pr}^{(3)}\, X(\Delta) = 0, \quad when \;\; \Delta = 0 \tag{7}$$

Eq. (7) is called condition of the invariant, and it leads to

$$\varphi^{t} + \alpha\varphi^{x} - \varepsilon\varphi^{xx} = 0 \tag{8}$$

where

$$\varphi^{J\chi_i} = D_{\chi_i}\phi^{J} - (D_{\chi_i}\xi_i)u_{J\chi_i} - (D_{\chi_i}\xi_j)u_{J\chi_j}\;,\; i \neq j \tag{9}$$

where $\chi_i$ refers to the independent variables. Conditions on the infinitesimals $\xi_1, \xi_2$ and $\varphi$ are determined by equating coefficients of like derivatives of monomials in $u_x$, $u_t$ and higher derivatives by zero. This leads to a system of partial differential equations from which we can determine $\xi_1, \xi_2$ and $\varphi$ . These equations are solved to get the infinitesimals solutions $\xi_1, \xi_2$ and $\varphi$ in the following forms:

$$\xi_1(x,t,u) = \frac{1}{2}c_3x^2 + c_4x + c_5\,,$$
$$\xi_2(x,t,u) = c_1t + c_2\,, \tag{10}$$
$$\varphi(x,t,u) = -\frac{1}{3}(c_1 - 3c_3x - 3c_4)u$$

where $c_i, i = 1,2,\ldots,5$ are arbitrary constants, Eqs.(10) shows that the nonlinear Harry-Dym (HD) equation (2) has the following generators symmetry group:

$$\begin{aligned} X_1 &= t\partial_t - \frac{1}{3}u\partial_u, \\ X_2 &= \partial_t, \\ X_3 &= \frac{1}{2}x^2\partial_x + xu\partial_u, \\ X_4 &= x\partial_x + u\partial_u, \\ X_5 &= \partial_x \end{aligned} \tag{11}$$

**2.1 Similarity Reduction and Similarity Solutions**

We consider the generators $(X_2 - cX_5), (X_1 + X_4)$ and $(X_2 + X_4)$.

**1.** For the generator $X_2 - cX_5$:

we get the similarity transformations

$$\eta = x + ct, \; u(x,t) = f(\eta) \tag{12}$$

Substituting Eq. (12) in to Eq. (2), we obtain the following ordinary differential equation:

$$f^3(\eta)\, f'''(\eta) - c\, f'(\eta) = 0 \tag{13}$$

Eq. (13) can be written as

$$f'''(\eta) = c\, f^{-3}(\eta)\, f'(\eta) \tag{14}$$

Integrating Eq. (14) with respect to $\eta$, yields

$$f''(\eta) + \frac{c}{2f^2(\eta)} = C_1 \tag{15}$$

Multiplying Eq. (15) by $f'(\eta)$ and integrating with respect to $\eta$, yields

$$(f'(\eta))^2 = \frac{c}{f(\eta)} + C_1 f(\eta) + C_2 \tag{16}$$

Eq. (16) give us

$$\eta = \pm \int \sqrt{\frac{f(\eta)}{C_1 f^2(\eta) + C_2 f(\eta) + c}}\, df(\eta) + C_3 \tag{17}$$

where $C_1, C_2$ and $C_3$ are integration constants.

Setting $C_1 = C_2 = 0$, Eq. (17) becomes

$$\eta = \pm \int \sqrt{\frac{f(\eta)}{c}}\, df(\eta) + C_3 \tag{18}$$

Eq. (18) leads to the following similarity solution

$$f(\eta) = \left[ \pm \frac{3\sqrt{c}}{2}(C_3 - \eta) \right]^{2/3} \tag{19}$$

which is equivalent to

$$f(\eta)=\left[a\pm\frac{3\sqrt{c}}{2}\eta\right]^{2/3} \tag{20}$$

Hence, the invariant solution of Eq. (2) is

$$u(x,t)=\left[a\pm\frac{3\sqrt{c}}{2}(x+ct)\right]^{2/3} \tag{21}$$

where $a=\pm\frac{3\sqrt{c}}{2}C_3$.

**2. For the generator** $X_1+X_4$**:**

we obtain the similarity transformations

$$\eta=x/t,\; u(x,t)=(x+t)^{2/3}f(\eta) \tag{22}$$

Substituting Eq. (22) in to Eq. (2), we get

$$(\eta+1)^3 f^3 f'''(\eta)+2(\eta+1)^2 f''-(\eta+1)(\frac{2}{3}f^3-\eta)f'(\eta)+\frac{2}{3}(\frac{4}{9}f^3-1)f(\eta)=0 \tag{23}$$

Solution of Eq. (23) is

$$f(\eta)=\frac{\chi e^{\int H(\chi)d\chi+c_1}}{\left[e^{\int H(\chi)d\chi+c_1}+1\right]^{2/3}} \tag{24}$$

where

$$\chi=\frac{(\eta+1)^{2/3}f(\eta)}{\eta},\qquad H(\chi)=\frac{3\eta(\eta+1)^{1/3}}{3\eta(\eta+1)f'(\eta)-(\eta+3)f(\eta)} \tag{25}$$

**3.** For the generator $X_2+X_4$:

Here we have the similarity reduction

$$\eta=\ln x-t,\quad u(x,t)=x f(\eta) \tag{26}$$

By using Eq. (26), Eq. (2) becomes

$$f^3(\eta)f'''(\eta)-(f^3(\eta)-1)f'(\eta)=0 \tag{27}$$

Solution of Eq. (27) as the form

$$\eta=\pm\int\sqrt{\frac{f(\eta)}{2C_2 f(\eta)+f^3(\eta)-2C_1 f^2(\eta)-1}}\,df(\eta)+C_3 \tag{28}$$

Setting $C_1=C_2=0$, Eq. (28) becomes

$$\eta=\pm\int\sqrt{\frac{f(\eta)}{f^3(\eta)-1}}\,df(\eta)+C_3 \tag{29}$$

The integrating in Eq. (29) leads to

$$\eta = \pm\frac{2}{3}\ln\left|\sqrt{f^3(\eta)} + \sqrt{f^3(\eta) - 1}\right| + C_3 \tag{30}$$

## Conclusion

In this study, Lie symmetry group method for calculating the solutions of the Harry Dym equation (2) is presented. Therefore, we have obtained the explicit solution (21) of the studied equation. The impliait solution of the Harry Dymequation (2) has been found in equations (24) and (30).

## References


1. Al-Khaled, K. and M. Alquran (2014). An approximate solution for a fractional model of generalized Harry Dym equation. Mathematical Sciences 8:125–130.
2. Assabaai, M. A. and S. F. Bamsaoud (2017). Exact solutions of the one-dimensional convection-diffusion equation using Lie group method. Assiut University. Journal of Mathematics and Computer Science 46 (1): 1-12.
3. Bluman, G. W. and S. C. Anco (2002). Symmetry and Integration Methods for Differential Equations. Springer, New York.
4. Fonseca, F. (2017). A Solution of the Harry-Dym Equation Using Lattice-Boltzmannn and a Solitary Wave Methods. Applied Mathematical Sciences 11(52): 2579 – 2586.
5. Gesztesy, F. and K. Unterkofler (1992). Isospectral deformations for Strum Liuville and Dirac-type operators and associated nonlinear evolution equations. Reports on Mathematical Physics 31(2): 113–137.
6. Ghiasi, E. K. and R. Saleh (2017). A Mathematical Approach Based on the Homotopy Analysis Method: Application to Solve the Nonlinear Harry-Dym (HD) Equation. Applied Mathematics 8: 1546-1562.
7. Hizel, E. and N.C. Turgay (2007). Symmetry group analysis and similarity solutions for nonlinear Reaction-Diffusion system of Gray-Scott type. Int. Math. Forum. 2: 2847-2858.
8. Ibragimov, N. H. (1994). CRC Handbook of Lie groups analysis of Differential Equations. II. Application in engineering and physical sciences, CRC Press, Boca Raton.
9. Iyiola, O. S. and Y. U. G. Gaba (2016). An Analytical Approach to Time- Fractional Harry Dym Equation. Applied Mathematics & Information Sciences 2: 409-412.
10. Kruskal, M.D. J. Moser (1975). Dynamical systems theory and applications lecturer notes physics. Berlin, Springer.
11. Kumar, D., J. Singh, and A. Kiliman (2013). An Efficient Approach for Fractional Harry Dym Equation by Using Sumudu Transform. Abstract and Applied Analysis, Article ID 608943, 8 pages.
12. Kumar, S., M.P. Tripathi, and O.P. Singh (2013). A fractional model of Harry Dym equation and its approximate solution. Ain Shams Engineering Journal 4: 111–115.
13. Maitama, S. and I. Abdullahi (2016). A New Analytical Method for Solving Linear and Nonlinear Fractional Partial Differential Equations. Progress in Fractional Differentiation and Applications. Progr. Fract. Differ. Appl. 2( 4): 247-256.
14. Mokhtari, R. (2011). Exact Solutions of the Harry-Dym Equation. Communications in Theoretical Physics 55: 204-208.
15. Olver, P. J. (1993). Applications of Lie Groups to Differential Equations. Grad. Texts in Math., vol. 107, Springer-Verlag, New York.
16. Rawashdeh, M.S. (2014). A New Approach to Solve the Fractional Harry Dym Equation Using the FRDTM. International Journal of Pure and Applied Mathematics 95: 553-566.

17. Shunmugarajan, B. (2016). An Efficient Approach for Fractional Harry Dym Equation by Using Homotopy Analysis Method. International Journal of Engineering Research & Technology (IJERT) 5: 561-566.
18. Soltani, D. and M. A. Khorshidi (2013). Application of Homotopy Perturbation and Reconstruction of Variational Iteration Methods for Harry Dym Equation and Compared with Exact Solution. International Journal of Multidisciplinary and current Research (IJMCR), 196-201.
19. Vosconcelos, G. L. and L. P. Kaclanoff (1991). Stationary solution for the Saffman- Taylor problem with surface tension. Physical Review A, 44(10): 6490–6495.
20. West, B. J., M. Bologna and P. Grigolini (2003). Physics of fractional operators, New York: Springer.

# الحلول المضبوطة لمعادلة هاري ديم باستخدام طريقة مجموعة لي

**مبارك عوض السباعي وعمر فرج مخرج**
قسم الرياضيات، كلية العلوم، جامعة حضرموت، اليمن
Email: asaba973@gmail.com , Email: mukherij55@yahoo.com
DOI: https://doi.org/10.47372/uajnas.2020.n2.a15

## الملخص

في هذا البحث طبقنا طريقة تناظر لي لمعادلة هاري ديم. باستخدام التحويلات المتماثلة تم تحويل هذه المعادلة إلى معادلة تفاضلية عادية ومن ثم حصلنا على الحل اللا تغيري لمعادلة هاري ديم من حلول المعادلة التفاضلية العادية.

**الكلمات المفتاحية:** معادلة هاري ديم، مجموعة لي، التحويلات المتماثلة، حلول لا تغيرية.