On Maximal $\alpha$-Continuous Maps in Topological spaces

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Abstract

In this paper, we introduce new types of maps called maximal $\alpha$-continuous, maximal $\alpha$-irresolute, minimal-maximal $\alpha$-continuous and strongly maximal $\alpha$-continuous maps in topological spaces, studying some of their fundamental properties and their relations with others. Also, we introduce a new class of topological spaces called $\alpha T_{\max}$ studying some of their fundamental properties.

Keywords: Minimal open set, maximal open set, maximal $\alpha$-open and minimal $\alpha$-closed sets.

1-Introduction

The concepts of minimal open sets and maximal open sets in topological spaces are introduced and considered by F. Nakaoka and N. Oda in [5], [6] and [7]. More precisely, in 2001, Nakaoka and Oda [5] characterized minimal open sets and proved that any subset of a minimal open set is pre-open. By the dual concepts of minimal open sets and maximal open sets, Nakaoka & Oda [7] introduced the concepts of minimal closed sets and maximal closed sets. Family of minimal open (minimal closed) sets and maximal open (maximal closed) sets are denoted by $M_{i}O(X)$ ($M_{i}C(X)$) and $M_{a}O(X)$ ($M_{a}C(X)$) respectively.

Bechallia et al. [1] introduced the class of maps called minimal continuous, maximal continuous, minimal irresolute, maximal irresolute, minimal-maximal continuous and maximal-minimal continuous maps in topological spaces and studied their relations with various types of continuous maps.

2-Preliminaries

**Definition 2.1.** [8]. A subset $A$ of a space $X$ is said to be $\alpha$-open set if $A \subseteq \text{Int(Cl(Int}(A)))$. The complement of $\alpha$-open set is said to be $\alpha$-closed. Family of $\alpha$-open sets is denoted by $\alpha O(X)$.

**Definition 2.2.** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:
(i) $\alpha$-continuous map [4] if the inverse image of every open set in $Y$ is $\alpha$-open set in $X$.
(ii) $\alpha$-irresolute map [3] (briefly $\alpha$-irresolute) if the inverse image of every $\alpha$-open set in $Y$ is $\alpha$-open set in $X$.

**Definition 2.3.** [5]. A proper nonempty open set $U$ of $X$ is said to be a minimal open set if any open set which contained in $U$ is $\emptyset$ or $U$.

**Definition 2.4.** [6]. A proper nonempty open set $U$ of $X$ is said to be a maximal open set if any open set which contains $U$ is $X$ or $U$.

**Definition 2.5.** [7]. A proper nonempty closed subset $F$ of $X$ is said to be a maximal closed set if any closed set which contains $F$ is $X$ or $F$.

**Definition 2.6.** [7]. A proper nonempty closed subset $F$ of $X$ is said to be a minimal closed set if any closed set which contained in $F$ is $\emptyset$ or $F$.

**Definition 2.7.** [1]. A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:
(i) maximal continuous map (briefly max-continuous) if the inverse image of every maximal open (or minimal closed) set in $Y$ is an open (or closed) set in $X$.
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(ii) maximal irresolute (briefly max-irresolute) if $f^{-1}$ (M) is maximal open set in $X$ for every maximal open set $M$ in $Y$.

(iii) minimal-maximal continuous (briefly min-max continuous) if $f^{-1}$ (M) is maximal open set in $X$ for every minimal open set $M$ in $Y$.

(iv) maximal-minimal continuous (briefly max-min continuous) if $f^{-1}$ (M) is minimal open set in $X$ for every maximal open set $M$ in $Y$.

(v) strongly maximal open map if the image of every maximal open (resp. maximal closed) set in $X$ is maximal open in $Y$.

**Definition 2.8.**[2] A proper nonempty $\alpha$-open subset $U$ of a topological space $X$ is said to be a maximal $\alpha$-open set if any $\alpha$-open set which contains $U$ is $X$ or $U$.

**Definition 2.9.**[2] A proper nonempty $\alpha$-closed subset $F$ of a topological space $X$ is said to be a minimal $\alpha$-closed set if any $\alpha$-closed set which is contained in $F$ is $\emptyset$ or $F$.

The family of all maximal $\alpha$-open (resp. minimal $\alpha$-closed) sets will be denoted by $M_\alpha O(X)$ (resp. $M_\alpha C(X)$).

**Theorem 2.10.**[2] Let $A$ be a proper nonempty subset of $X$. Then $A$ is a maximal $\alpha$-open set if $X \setminus A$ is a maximal $\alpha$-closed set.

3. Maximal $\alpha$-continuous maps

**Definition 3.1.** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

(i) maximal $\alpha$-continuous (briefly max $\alpha$-continuous) map if the inverse image of every maximal open set in $Y$ is $\alpha$-open set in $X$.

(ii) maximal $\alpha$-irresolute (briefly max $\alpha$-irresolute) if $f^{-1}$ ($A$) is a maximal $\alpha$-open set in $X$ for every maximal $\alpha$-open set $A$ in $Y$.

(iii) Minimal-maximal $\alpha$-continuous (briefly min-max $\alpha$-continuous) if $f^{-1}$ ($A$) is a maximal $\alpha$-open set in $X$ for every minimal $\alpha$-open set $A$ in $Y$.

(v) strongly maximal $\alpha$-continuous (briefly strongly max $\alpha$-continuous) if the inverse image of every maximal open set in $Y$ is maximal $\alpha$-open set in $X$.

**Theorem 2.2.** Every continuous map is maximal $\alpha$-continuous map.

**Proof:** Let $f : X \rightarrow Y$ be continuous map and let $A$ be maximal open in $Y$. As maximal open imply open set, $A$ is open set in $Y$. Then $X$ contains $f^{-1}$ ($A$) as open set. Since every open imply $\alpha$-open set. Then $f^{-1}$ ($A$) is $\alpha$-open set in $X$ every maximal open set $A$ in $Y$. Hence $f$ is maximal $\alpha$-continuous.

**Remark 3.3.** The converse of above theorem is not true.

**Example 3.4.** Let $X=Y=$ $\{a, b, c\}$ and $f : (X, \alpha) \rightarrow (Y, \beta)$ is the identity map, where $\alpha = \{a, b, c\}$, $\{a\}, \{a, b\}, X$ and $\beta = \{\emptyset, \{b\}, \{a, b\}, Y\}$. Then $f$ is maximal $\alpha$-continuous but $f$ is not continuous since $f^{-1}$ ($\{b\}$) = $\{b\}$ is not open set.

**Theorem 3.5.** Let $X$ and $Y$ be topological spaces, if $f : X \rightarrow Y$ is an $\alpha$-continuous, then $f$ is maximal $\alpha$-continuous and not conversely.

**Proof:** Take $U$ be a maximal open subset of $Y$. Then, $U$ is open set in $Y$, since $f$ is $\alpha$-continuous so $f^{-1}$ ($U$) is $\alpha$-open subset of $X$. Thus $g$ is maximal $\alpha$-continuous.

**Example 3.6.** From examples 3.4, we find $f$ is maximal $\alpha$-continuous since, but $f$ is not $\alpha$-continuous since $f^{-1}$ ($\{b\}$) = $\{b\}$ is not $\alpha$-open set.

**Theorem 3.7.** Let $X$ and $Y$ be the topological spaces. A map $f : X \rightarrow Y$ is maximal $\alpha$-continuous if and only if the inverse image of each a minimal closed set in $Y$ is $\alpha$-closed set in $X$.

**Proof:** The proof follows from the definition and fact that the complement of $\alpha$-open set is $\alpha$-closed, and the complement of maximal open set is minimal closed set.

**Theorem 3.8.** Every strongly maximal $\alpha$-continuous map is maximal $\alpha$-continuous.
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**Proof**: Let $X$ and $Y$ be topological spaces and the map $f : X \to Y$ is strongly maximal $\alpha$-continuous, to show that $f$ is maximal $\alpha$-continuous. Let $U$ be a maximal open subset of $Y$, thus $f^{-1}(U)$ is maximal $\alpha$-open subset of $X$. Since maximal $\alpha$-open set implies $\alpha$-open set, then $f^{-1}(U)$ is $\alpha$-open set in $X$. Hence $f$ is maximal $\alpha$-continuous.

**Remark 3.9.** The converse of above theorem is not true in general as in the following examples.

**Example 3.10.** Let $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, X\}$ and $\sigma = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, Y\}$. Let $f$ be an identity map which is maximal $\alpha$-continuous but not strongly maximal $\alpha$-continuous as $\{1, 2\}$ is maximal open in $Y$. Then $f^{-1}(\{1, 2\}) = \{1, 2\}$ is not maximal $\alpha$-open set in $X$.

**Theorem 3.11.** If $f : X \to Y$ is $\alpha$-continuous map and $g : Y \to Z$ is maximal continuous map, then $g \circ f : X \to Z$ is maximal $\alpha$-continuous.

**Proof**: Let $N$ be any maximal open set in $Z$. Since $g$ is maximal continuous, $g^{-1}(N)$ is an open set in $Y$. Again since $f$ is $\alpha$-continuous, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is $\alpha$-open set in $X$. Hence $g \circ f$ is an $\alpha$-continuous.

**Theorem 3.12.** If $f : X \to Y$ is strongly maximal $\alpha$-continuous map and $g : Y \to Z$ strongly maximal continuous map, then $g \circ f : X \to Z$ is strongly maximal $\alpha$-continuous.

**Proof**: Similar to that of Theorem 3.11.

**Remark 3.13.** Composition of maximal $\alpha$-continuous is not maximal $\alpha$-continuous, which is shown below.

**Example 3.14.** Let $X = Y = Z = \{1, 2, 3\}$ with $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 3\}, X\}$, $\sigma = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{1, 3\}, Y\}$, and $\eta = \{\emptyset, \{2\}, \{1, 3\}, Z\}$. If $g : X \to Y$ and $h : Y \to Z$ be identity maps. Then $g$ and $h$ are maximal $\alpha$-continuous maps but not $h \circ g$ is maximal $\alpha$-continuous. Now $\{2\}$ is Maximal open in $Z$, then $(h \circ g)^{-1}(\{2\}) = \{2\} \notin \alpha\eta O(X)$.

**Remark 3.15.** Composition of strongly maximal $\alpha$-continuous is not strongly maximal $\alpha$-continuous, which is shown below.

**Example 3.16.** By Example 3.14, we have $g$ and $h$ are strongly maximal $\alpha$-continuous functions but not $h \circ g$ is strongly maximal $\alpha$-continuous. Now $\{2\} \notin \alpha\eta M_o(Z)$, then $(h \circ g)^{-1}(\{2\}) = \{2\} \notin \alpha\eta M_o(X)$.

**Theorem 3.17.** If $g$ and $h$ are maximal $\alpha$-irresolute. Then, $h \circ g$ is maximal $\alpha$-irresolute.

**Theorem 3.18.** If $g$ is maximal $\alpha$-continuous, then restriction map $g_A : A \to Y$ is maximal $\alpha$-continuous.

**Proof**: Consider a maximal $\alpha$-continuous map $g$ and non-empty subset $A$ of $X$. Let $M \in \alpha_o M_o(Y)$. By hypothesis, $g^{-1}(M) \in \alpha\sigma O(X)$. Therefore, by definition of $g_A : A \to Y$ it is evident that $g_A^{-1}(M) = A \cap g^{-1}(M)$. Therefore, $A \cap g_A^{-1}(M)$ is $\alpha$-open set in $A$. Therefore, by definition $g_A : A \to Y$ is maximal $\alpha$-continuous.

**Theorem 3.19.** A mapping $g$ is maximal $\alpha$-continuous iff for any $p \in X$ and $M \in \alpha_o M_o(Y)$ containing $g(p)$, $\exists \ N \in \alpha\sigma O(X) \ni p \in N$ and $g(N) \subset M$.

**Proof**: Let $M \in \alpha_o M_o(Y)$ containing $g(p)$ for $p \in N$, where $\subset \alpha\sigma O(X)$. As $g$ is maximal $\alpha$-continuous, we have $g^{-1}(M) \in \alpha\sigma O(X)$. Take $N = g^{-1}(M)$ which implies $g(N) = g(g^{-1}(M)) \subset M$. Therefore, $g(N) \subset M$.

Conversely, let $M \in \alpha_o M_o(Y)$. By hypothesis, $N \in \alpha\sigma O(X)$, $p \in N$ which implies $g(p) \in g(N) \subset M$ which implies $p \in g^{-1}(g(N)) \in g^{-1}(M)$. Thus $g^{-1}(M) \in \alpha\sigma O(X), M \in \alpha_o M_o(Y)$. Therefore, $g$ is maximal $\alpha$-continuous.

4. $\alpha T_{\text{max}}$ space
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**Definition 4.1.** A topological space $X$ is said to be $\alpha T_{\text{max}}$ space if every nonempty proper $\alpha$-open subset of $X$ is maximal $\alpha$-open set.

**Theorem 4.2.** A topological space $X$ is $\alpha T_{\text{max}}$ space if and only if any pair of two proper nonempty $\alpha$-open sets are disjoint.

**Proof:** Let $U_1$ and $U_2$ be two proper nonempty $\alpha$-open sets in $X$. Assume $U_1 \cap U_2 \neq \emptyset$, then $\emptyset \neq U_1 \cap U_2 \subseteq U_1 \subseteq X$, that is, $U_1 \cap U_2$ is a proper nonempty $\alpha$-open set and so $U_1 \cap U_2$ is maximal $\alpha$-open. Since $U_1 \cap U_2 \subseteq U_1 \subseteq X$ and $U_1$ is $\alpha$-open, then $U_1 = U_1 \cap U_2$. Similarly, $U_2 = U_1 \cap U_2$. This implies that $U_1 = U_2$, which is a contradiction. Therefore, $U_1 \cap U_2 = \emptyset$.

$\Rightarrow$ Let $U$ be a proper nonempty $\alpha$-open set and $W$ a nonempty $\alpha$-open set such that $W \subseteq U$, so $W = U$. [Otherwise, $W \neq U$ and $W \cap U = \emptyset$.] Hence $U$ is a maximal $\alpha$-open, that is, $X$ is $\alpha T_{\text{max}}$ space.

**Theorem 4.3.** A topological space $X$ is $\alpha T_{\text{max}}$ space if and only if every nonempty proper $\alpha$-closed subset of $X$ is minimal $\alpha$-closed set in $X$.

**Proof:** Let $F$ be a proper $\alpha$-closed subset of $X$, suppose $F$ is not minimal $\alpha$-closed in $X$. So there is a proper $\alpha$-closed subset of $X$ such that $K \subset F$. Thus $X-F \subset X-K$ but $X-K$ is proper $\alpha$-open in $X$ so $X-F$ is not maximal $\alpha$-open in $X$. Contradiction to the fact $X-F$ is maximal $\alpha$-open.

$\Leftarrow$ Let $U$ be a proper $\alpha$-open subset of $X$, then $X-U$ is a proper $\alpha$-closed subset of $X$ and so it is minimal $\alpha$-closed set by the fact that the complement of maximal $\alpha$-open set is minimal $\alpha$-closed set, hence we get that $U$ is maximal $\alpha$-open.

**Theorem 4.4.** Union of every pair of different maximal $\alpha$-open sets in $\alpha T_{\text{max}}$ space is $X$.

**Proof:** Let $U$ and $V$ be maximal $\alpha$-open subsets of $\alpha T_{\text{max}}$ space $X$ such that $U \neq V$ to show that $U \cup V = X$ suppose not i.e. $U \cup V \neq X$. So $U \subset U \cup V$ and $V \subset U \cup V$. Since $U \subset U \cup V$ and $U$ is maximal $\alpha$-open, then $U \cup V = U$ or $U \cup V = X$.

Thus $U \cup V = U$… (1). Now since $V \subset U \cup V$ and $V$ is maximal $\alpha$-open then $U \cup V = V$ or $U \cup V = X$. Thus $U \cup V = V$… (2). Hence, from (1) and (2), we get that $U = V$ this result contradicts the fact that $U$ and $V$ are different. Therefore, $U \cup V = X$.

**Theorem 4.5.** Let $X$ and $Y$ be topological spaces, if $f: X \rightarrow Y$ is a maximal $\alpha$-continuous onto map and $X$ is $\alpha T_{\text{max}}$ space then $f$ is strongly maximal $\alpha$-continuous.

**Proof:** It is clear that the inverse image of $\emptyset$ and $\emptyset$ are $\alpha$-open subsets of $X$. So let $U$ be a maximal open subset of $Y$. Since $f$ is maximal $\alpha$-continuous so $f^{-1}(U)$ is proper $\alpha$-open subset of $X$, but $X$ is $\alpha T_{\text{max}}$ so $f^{-1}(U)$ maximal $\alpha$-open. Therefore, $f$ is strongly maximal $\alpha$-continuous.

**Remark 4.6.** The converse is not true, in general, as in the following example.

**Example 4.7.** Let $X=\{a, b, c\}$ and $f: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is the identity map, where $\mathcal{A} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{B} = \emptyset, \{a, c\}, Y\}$, then $f$ is strongly maximal $\alpha$-continuous since the only maximal open subset of $Y$ is $\{a, c\}$ and $f^{-1}(\{a, c\}) = \{a, c\}$ is maximal $\alpha$-open in $X$. But $X$ is not $\alpha T_{\text{max}}$.
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References
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max $\alpha$-Continuous, max $\alpha$-irresolute, in this paper we introduce and study some new types of maximal strongly $\alpha$-Continuous and minimal maximal-$\alpha$-Continuous maps in topological spaces, and investigate some of their basic properties and relationships.

Key words: the smallest open sets, the largest open sets, the smallest maximal-$\alpha$-open sets, and the largest strongly $\alpha$-open sets.