On Maximal α-Continuous Maps in Topological spaces

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Abstract

In this paper, we introduce new types of maps called maximal α-continuous, maximal α-irresolute, minimal-maximal α-continuous and strongly maximal α-continuous maps in topological spaces, studying some of their fundamental properties and their relations with others. Also, we introduce a new class of topological spaces called αT_{max} studying some of their fundamental properties.

Keywords: Minimal open set, maximal open set, maximal α-open and minimal α-closed sets.

1-Introduction

The concepts of minimal open sets and maximal open sets in topological spaces are introduced and considered by F. Nakaoka and N. Oda in [5], [6] and [7]. More precisely, in 2001, Nakaoka and Oda [5] characterized minimal open sets and proved that any subset of a minimal open set is pre-open . By the dual concepts of minimal ,open sets and maximal open sets, Nakaoka & Oda [7] introduced the concepts of minimal closed sets and maximal closed sets. Family of minimal open (minimal closed) sets and maximal open (maximal closed) sets are denoted by $M_iO(X)$ ($M_iC(X)$) and $M_aO(X)$ ($M_aC(X)$) respectively.

Bechalli et al [1] introduced the class of maps called minimal continuous, maximal continuous, minimal irresolute, maximal irresolute, minimal-maximal continuous and maximal-minimal continuous maps in topological spaces and studied their relations with various types of continuous maps.

2-Preliminaries

Definition 2.1.[8]**.** A subset A of a space X is said to be α-open set if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The complement of α-open set is said to be α-closed. Family of α-open sets is denoted by αO(X) .

Definition 2.2. A mapping f : (X, τ) → (Y, σ) is said to be:
(i)α-continuous map[4] if the inverse image of every open set in Y is α-open set in X.
(ii)α-irresolute map[3](briefly α-irresolute)if the inverse image of every α-open set in Y is α-open set in X.

Definition 2.3.[5]**.** A proper nonempty open set U of X is said to be a minimal open set if any open set which contained in U is Ø or U.

Definition 2.4.[6]. A proper nonempty open set U of X is said to be a maximal open set if any open set which contains U is X or U.

Definition 2.5.[7]. A proper nonempty closed subset F of X is said to be a maximal closed set if any closed set which contains F is X or F.

Definition 2.6.[7]. A proper nonempty closed subset F of X is said to be a minimal closed set if any closed set which contained in F is Ø or F.

Definition 2.7.[1]. A mapping f: (X ,τ) → (Y, σ) is said to be
(i) maximal continuous map (briefly max-continuous) if the inverse image of every maximal open (or minimal closed) set in Y is an open (or closed) set in X.

(ii) maximal irresolute (briefly max-irresolute) if f^{-1} (M) is maximal open set in X for every maximal open set M in Y .
(iii) minimal-maximal continuous (briefly min-max continuous) if f^{-1} (M) is maximal open set in X for every minimal open set M in Y .
(iv) maximal-minimal continuous (briefly max-min continuous) if f^{-1} (M) is minimal open set in X for every maximal open set M in Y .
(v) strongly maximal open map if the image of every maximal open (resp. maximal closed) set in X is maximal open set in Y.
Definition 2.8.[2]. A proper nonempty α-open subset U of a topological space X is said to be a maximal α-open set if any α-open set which contains U is X or U .
Definition 2.9.[2]. A proper nonempty α-closed subset F of a topological space X is said to be a minimal α-closed set if any α-closed set which is contained in F is Ø or F.
The family of all maximal α-open (resp. minimal α-closed) sets will be denoted by M_aαO(X) (resp. M_iαC(X)).
Theorem 2.10.[2]. Let A be a proper nonempty subset of X. Then A is a maximal α-open set if $X\setminus A$ is a minimal α-closed set.

3. Maximal α-continuous maps

Definition 3.1. A mapping f : (X, τ) → (Y, σ) is said to be
(i) maximal α-continuous (briefly max α-continuous) map if the inverse image of every maximal open set in Y is α-open set in X.
(ii) maximal α-irresolute (briefly max α-irresolute) if $f^{-1}(A)$ is a maximal α-open set in X for every maximal α-open set A in Y.
(iii) Minimal-maximal α-continuous (briefly min-max α-continuous) if $f^{-1}(A)$ is a maximal α-open set in X for every minimal open set A in Y.
(v) strongly maximal α-continuous (briefly strongly max α-continuous) if the inverse image of every maximal open set in Y is maximal α-open set in X.
Theorem 3.2. Every continuous map is maximal α-continuous map.
Proof: Let f : X →Y be continuous map and let A be maximal open in Y. As maximal open imply open set, A is open set in Y. Then X contains $f^{-1}(A)$ as open set. Since every open imply α-open set. Then $f^{-1}(A)$ is α-open set in X every maximal open set A in Y. Hence f is maximal α-continuous.
Remark 3.3. The converse of above theorem is not true.
Example 3.4. Let X=Y={a, b, c} and f : (X,□)□(Y,□) is the identity map, where □□={ □□, {a}, {a, b}, X} and □□={ □□,{b}, {a, b},Y} then f is maximal α-continuous but f is not continuous since $f^{-1}(\{b\}) = \{b\}$ is not open set.
Theorem 3.5. Let X and Y be topological spaces, if f :X□Y is an α-continuous, then f is maximal α-continuous and not conversely.
Proof: Take U be a maximal open subset of Y. Then , U is open set in Y ,Since f is α-continuous so $f^{-1}(U)$ is α-open subset of X. Thus g is maximal α-continuous.
Example 3.6. From examples 3.4, we find f is maximal α-continuous since, but f is not α-continuous since $f^{-1}(\{b\}) = \{b\}$ is not α-open set.
Theorem 3.7. Let X and Y be the topological spaces. A map f : X → Y is maximal α-continuous if and only if the inverse image of each a minimal closed set in Y is α-closed set in X.
Proof: The proof follows from the definition and fact that the complement of α-open set is α-closed set, and the complement of maximal open set is minimal closed set.
Theorem 3.8. Every strongly maximal α-continuous map is maximal α-continuous.

Proof: Let X and Y be topological spaces and the map f : X□Y is strongly maximal α-continuous, to show that f is maximal α-continuous. Let U be a maximal open subset of Y, thus f^{-1} (U) is maximal α-open subset of X. Since maximal α-open set implies α-open set, then f^{-1} (U) is α-open set in X. Hence f is maximal α-continuous.

Remark 3.9. The converse of above theorem is not true in general as in the following examples.

Example 3.10. Let X = Y = {1, 2, 3, 4} with topologies τ = {Ø, {1}, {1, 2}, {1, 2, 3}, X} and σ = {Ø, {1}, {2}, {1, 2}, Y}. Let f be an identity map which is maximal α-continuous but not strongly maximal α-continuous as {1, 2} is maximal open in Y, Then $f^{-1}(\{1,2\}) = \{1,2\}$ is not maximal α-open set in X.

Theorem 3.11. If f : X → Y is α-continuous map and g : Y → Z is maximal continuous map, then $g \circ f : X \to Z$ is maximal α-continuous.

Proof: Let N be any maximal open set in Z. Since g is maximal continuous, $g^{-1}(N)$ is an open set in Y . Again since f is α-continuous, $f^{-1}(g^{-1}(N)) = (g \circ f)^{-1}(N)$ is α-open set in X. Hence g o f is an α-continuous.

Theorem 3.12. If f : X → Y is strongly maximal α-continuous map and g : Y → Z strongly maximal continuous map, then g o f: X → Z is strongly maximal α-continuous.

Proof: Similar to that of Theorem 3.11.

Remark 3.13. Composition of maximal α-continuous is not maximal α-continuous, which is shown below.

Example 3.14. Let X = Y = Z = {1, 2, 3} with τ = {Ø, {3}, {2, 3}, {1, 3}, X}, σ = {Ø, {2},{3}, {2, 3}, {1, 3} ,Y}, and η = {Ø, {2}, {1, 3}, Z}. If g : X → Y and h : Y → Z be identity maps. Then g and h are maximal α-continuous maps but not h o g is maximal α-continuous. Now {2} is Maximal open in Z, then $(h \circ g)^{-1}(\{2\})=\{2\} \notin \alpha O(X)$.

Remark 3.15. Composition of strongly maximal α-continuous is not strongly maximal α-continuous, which is shown below.

Example 3.16. By Example 3.14, we have g and h are strongly maximal α-continuous functions but not h o g is strongly maximal α-continuous. Now $\{2\} \in M_aO(Z)$, then $(h \circ g)^{-1}(\{2\}) = \{2\} \notin M_a\alpha O(X)$.

Theorem 3.17. If g and h are maximal α-irresolute. Then, hog is maximal α-irresolute.

Theorem 3.18. If g is maximal α-continuous, then restriction map g_A : A→ Y is maximal α-continuous.

Proof: Consider a maximal α-continuous map g and non-empty subset A of X. Let $M \in M_aO(Y)$. By hypothesis, $g^{-1}(M) \in \alpha O(X)$. Therefore, by definition of g_A: A→ Y it is evident that ${g_A}^{-1}(M) = A \cap g^{-1}(M)$. Therefore, $A \cap {g_A}^{-1}(M)$ is α-open set in A. Therefore, by definition g_A : A→ Y is maximal α-continuous.

Theorem 3.19. A mapping g is maximal α-continuous iff for any p ∈ X and $M \in M_aO(Y)$ containing g (p), ∃ N∈ αO(X) ∋ p ∈ N and g (N) ⊂ M.

Proof: Let $M \in M_aO(Y)$ containing g (p) for p ∈ N, where N∈ αO(X). As g is maximal α-continuous, we have $g^{-1}(M) \in \alpha O(X)$. Take $N = g^{-1}(M)$ which implies $g(N) = g(g^{-1}(M)) \subset M$. Therefore, g (N) ⊂ M.

Conversely, let $M \in M_aO(Y)$. By hypothesis, N ∈ αO(X), p ∈ N which implies g(p) ∈ g(N) ⊂ M which implies $p \in g^{-1}(g(N)) \in g^{-1}(M)$. Thus $g^{-1}(M) \in \alpha O(X)$, $M \in M_aO(Y)$. Therefore , g is maximal α-continuous.

4. αT_{max} space

Definition 4.1. A topological space X is said to be αT_{max} space if every nonempty proper α-open subset of X is maximal α-open set.

Theorem 4.2. A topological space X is αT_{max} space if and only if any pair of two proper nonempty α-open sets are disjoint.

Proof: ⟹ Let U_1 and U_2 be two proper nonempty α-open sets in X. Assume $U_1 \cap U_2 \neq \phi$, then $\phi \neq U_1 \cap U_2 \subseteq U_1 \subset X$, that is, $U_1 \cap U_2$ is a proper nonempty α-open set and so $U_1 \cap U_2$ is maximal α-open. Since $U_1 \cap U_2 \subseteq U_1 \subset X$ and U_1 is α-open, then $U_1 = U_1 \cap U_2$. Similarly, $U_2 = U_1 \cap U_2$. This implies that $U_1 = U_2$, which is a contradiction. Therefore, $U_1 \cap U_2 = \phi$.

⟸ Let U be a proper nonempty α-open set and W a nonempty α-open set such that W ⊆ U, so W = U. [Otherwise, $W \neq U$ and $W \cap U = \phi$]. Hence U is a maximal α-open ,that is: X is αT_{max} space.

Theorem 4.3. A topological space X is αT_{max} space if and only if every nonempty proper α-closed subset of X is minimal α-closed set in X.

Proof: ⟹Let F be a proper α-closed subset of X, suppose F is not minimal α-closed in X, So there is a proper α-closed subset of X such that K ⊂ F. Thus X−F ⊂ X−K but X−K is proper α-open in X so X−F is not maximal α-open in X. Contradiction to the fact X−F is maximal α-open.

⟸Let U be a proper α-open subset of X, then X−U is a proper α-closed subset of X and so it is minimal α-closed set by the fact that the complement of maximal α-open set is minimal α-closed set, hence we get that U is maximal α-open.

Theorem 4.4. Union of every pair of different maximal α-open sets in αT_{max} space is X.

Proof: Let U and V be maximal α-open subsets of αT_{max} space X such that U ≠V to show that $U \cup V = X$ suppose not i.e. $U \cup V \neq X$. So $U \subset U \cup V$ and $V \subset U \cup V$. Since $U \subset U \cup V$ and U is maximal α-open, then $U \cup V = U$ or $U \cup V = X$.
thus $U \cup V = U$…(1). Now since $V \subset U \cup V$ and V is maximal α-open then $U \cup V = V$ or $U \cup V = X$. thus $U \cup V = V$… (2). Hence, from (1) and (2), we get that U=V this result contradicts the fact that U and V are different. Therefore, $U \cup V = X$.

Theorem 4.5. Let X and Y be topological spaces, if *f* :X□Y is a maximal α-continuous onto map and X is $\boldsymbol{\alpha T_{max}}$ space then f is strongly maximal α-continuous.

Proof: It is clear that the inverse image of □□and Y are α-open subsets of X. So let U be a maximal open subset of Y. Since f is maximal α-continuous so f^{-1} (U) is proper α-open subset of X, but X is $\boldsymbol{\alpha T_{max}}$ so f^{-1} (U) maximal α-open. Therefore, f is strongly maximal α-continuous.

Remark 4.6. the converse is not true, in general, as in the following example.

Example 4.7. Let X=Y={a, b, c}and f : (X,□)□(Y,□) is the identity map, where □□={ □□, {a}, {c},{a, c}, X} and □□={ □□, {a, c},Y}, then f is strongly maximal α-continuous since the only maximal open subset of Y is {a, c} and f^{-1} ({a, c}) □□{a, c} is maximal α-open in X. but X is not $\boldsymbol{\alpha T_{max}}$

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حول الدوال الأعظمية آلفا المستمرة في الفراغات الطوبولوجية

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الملخص

في هذه الورقة البحثية نعرض أنواعًا جديدة من الدوال تسمى max α-Continuous, max α-irresolute, minimal maximal-α-Continuous و maximal strongly α-Continuous في الفراغات الطوبولوجية، وندرس بعض خصائصها الأساسية وعلاقاتها ببعضها. بالإضافة الى دراسة نوع جديد من الفراغات الطوبولوجية يسمى αT_{max} ودراسة بعض خصائصه الأساسية والمميزة.

الكلمات المفتاحية: المجموعة الأصغرية المفتوحة، المجموعة الاعظمية المفتوحة، المجموعات الاعظمية آلفا المفتوحة، الأصغرية الفا المغلقة.